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# Hypersurfaces satisfying some curvature conditions on quasi-conformal curvature tensor in the semi-euclidean space

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**Abstract** We consider some curvature conditions on the quasi-conformal curvature tensor $\widetilde{C}$ on a hypersurface in the semi-Euclidean space $E_s^{n+1}$. We prove that every quasi-conformally Ricci-semisymmetric hypersurface $M$ satisfying the condition $\widetilde{C} \cdot R = 0$ is pseudosymmetric. We also consider the condition $\widetilde{C} \cdot S = 0$ on hypersurfaces of the semi-Euclidean space $E_s^{n+1}$.

**Keywords** Quasi-conformal curvature tensor · Pseudosymmetric manifold · Ricci-semisymmetric manifold

**Mathematics Subject Classification (2010)** 53A05 · 53A10 · 53B20 · 53C20

## 1 Introduction

Let $(M, g)$ be an $n$-dimensional, $n \geq 3$, differentiable manifold of class $C^\infty$. The quasi-conformal curvature tensor $\widetilde{C}$ was introduced by Yano and Sawaki in [14]. According to them, a quasi-conformal curvature tensor is defined by

$$\widetilde{C}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] - \frac{\kappa}{n}\left[\frac{a}{n-1} + 2b\right](g(Y,Z)X - g(X,Z)Y),$$

where $a$ and $b$ are constants, $S$ is the Ricci tensor, $Q$ is the Ricci operator and $\kappa$ is the scalar curvature of the manifold $M$.

In [8], Dabrowska, Defever, Deszcz and Kowalczyk studied semisymmetry and Ricci-semisymmetry for hypersurfaces of semi-Euclidean space.
Recently in [10], Özgür studied hypersurfaces satisfying some curvature conditions in the semi-Euclidean space. In [12], Özgür, Arslan and Murathan studied conharmonically semiparallel hypersurfaces in Euclidean space. In [11], Özgür and Arslan studied pseudosymmetric hypersurfaces satisfying Chen's equality in Euclidean

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space. In the present study, our aim is to study hypersurfaces of dimension $n \geq 4$, in $(n+1)$-dimensional semi-Euclidean space $E_s^{n+1}$ whose shape operator $A$ satisfies the condition

$$A^3 = tr(A)A^2 + \beta A + \gamma I_d \tag{1.1}$$

at every point $x \in M$ for some $\beta$ and $\gamma \in \mathbb{R}$. We show that if a quasi-conformally Ricci-semisymmetric hypersurface $M$ satisfies the condition $\widetilde{C} \cdot R = 0$, where $R$ denotes the curvature tensor of $M$, then $M$ is pseudosymmetric.

The paper is organized as follows: In Section 2, we give a brief account of quasi-conformal curvature tensor, pseudosymmetric manifolds and Kulkarni-Nomizu product. In Section 3, we give some information about hypersurfaces of semi-Euclidean space $E_s^{n+1}$ and the main results of the study are presented.

## 2 Preliminaries

We denote by $\nabla$, $R$, $\widetilde{C}$, $S$ and $\kappa$ are the Levi-Civita connection, the Riemannian-Christoffel curvature tensor, the quasi-conformal curvature tensor, the Ricci tensor and the scalar curvature of $(M, g)$, respectively. The Ricci operator $Q$ is defined by $g(QX, Y) = S(X, Y)$, where $X, Y \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields on $M$. Furthermore, the $(0, 2)$-tensor $S^2$ is defined by

$$S^2(X, Y) = S(QX, Y). \tag{2.1}$$

Next, we define the endomorphisms $\mathcal{R}(X, Y)$ and $\widetilde{\mathcal{C}}(X, Y)$ of $\chi(M)$ by

$$\mathcal{R}(X, Y)Z = [\nabla_X, \nabla_Y] Z - \nabla_{[X,Y]} Z$$

and

$$\begin{aligned} \widetilde{\mathcal{C}}(X, Y)Z &= a\mathcal{R}(X, Y)Z + b[(X \wedge QY + QX \wedge Y)\, Z] \\ &\text{-}\frac{\kappa}{n}\left[\frac{a}{n-1} + 2b\right](X \wedge Y)\, Z, \end{aligned} \tag{2.2}$$

respectively, where $(X \wedge Y)\, Z$ is the tensor, defined by

$$(X \wedge Y)\, Z = g(Y, Z)X - g(X, Z)Y,$$

and $Z \in \chi(M)$.

The Riemannian-Christoffel curvature tensor $R$ and the quasi-conformal curvature tensor $\widetilde{C}$ are defined by

$$\begin{aligned} R\,(X, Y, Z, W) &= g(\mathcal{R}\,(X, Y)Z, W)\,, \\ \widetilde{C}\,(X, Y, Z, W) &= g(\widetilde{\mathcal{C}}\,(X, Y)Z, W)\,, \end{aligned}$$

respectively, where $W \in \chi(M)$. The (0,4)-tensor $G$ is defined by $G(X, Y, Z, W) = g((X \wedge Y)Z, W)$.

For a $(0,k)$-tensor field $T$, $k \geq 1$, and a $(0,2)$-tensor field $E$ on $(M,g)$ we define the tensors $R \cdot T$, $\widetilde{C} \cdot T$, and $Q(E,T)$ by

$$(R(X,Y)\cdot T)(X_1,...,X_k) = -T(\mathcal{R}(X,Y)X_1, X_2,...,X_k) - ... - T(X_1,...,X_{k-1},\mathcal{R}(X,Y)X_k). \quad (2.3)$$

$$(\widetilde{C}(X,Y)\cdot T)(X_1,...,X_k) = -T(\widetilde{\mathcal{C}}(X,Y)X_1, X_2,...,X_k) - ... - T(X_1,...,X_{k-1},\widetilde{\mathcal{C}}(X,Y)X_k). \quad (2.4)$$

$$Q(E,T)(X_1,...,X_k;X,Y) = -T((X\wedge_E Y)X_1, X_2,...,X_k) - ... - T(X_1,...,X_{k-1},(X\wedge_E Y)X_k), \quad (2.5)$$

respectively, where the tensor $X \wedge_E Y$ is defined by

$$(X\wedge_E Y)Z = E(Y,Z)X - E(X,Z)Y$$

If $E = g$ then we simply denote it by $X \wedge Y$.

If the tensors $R\cdot R$ and $Q(g,R)$ are linearly dependent, then $M$ is called *pseudosymmetric*. This is equivalent to

$$R\cdot R = L_R Q(g,R) \quad (2.6)$$

holding on the set $U_R = \{x \in M^n |\, Q(g,R) \neq 0 \text{ at } x\}$, where $L_R$ is some function on $U_R$ (see [6]). If $R \cdot R = 0$, then $M$ is called semi-symmetric (see [13]).

If the tensors $R\cdot S$ and $Q(g,S)$ are linearly dependent, then $M$ is called *Ricci-pseudosymmetric*. This is equivalent to

$$R\cdot S = L_S Q(g,S) \quad (2.7)$$

holding on the set $U_S = \{x \in M^n |\, S - \frac{\kappa}{n} g \neq 0 \text{ at } x\}$, where $L_S$ is some function on $U_S$ (see [6]).

The Kulkarni-Nomizu product $E \,\widetilde{\wedge}\, B$ is given by

$$(E\widetilde{\wedge}B)(X_1,X_2,X_3,X_4) = E(X_1,X_4)B(X_2,X_3) + E(X_2,X_3)B(X_1,X_4) - E(X_1,X_3)B(X_2,X_4) - E(X_2,X_4)B(X_1,X_3). \quad (2.8)$$

We note that if $E = B$, then we have $\bar{E}= \frac{1}{2}E \,\widetilde{\wedge}\, E$, where the (0,4)-tensor $\bar{E}$ is defined by

$$\bar{E}(X_1,X_2,X_3,X_4) = E(X_1,X_4)E(X_2,X_3) - E(X_1,X_3)E(X_2,X_4).$$

Further, for a symmetric $(0,2)$-tensor $E$ and a $(0,k)$-tensor $T$, $k \geq 2$, we define their Kulkarni-Nomizu product $E \,\widetilde{\wedge}\, T$ by

$$\begin{aligned}(E\widetilde{\wedge}T)(X_1,X_2,X_3,X_4;Y_3,...,Y_k) &= E(X_1,X_4)T(X_2,X_3;Y_3,...,Y_k) \\ &+ E(X_2,X_3)T(X_1,X_4;Y_3,...,Y_k) \\ &- E(X_1,X_3)T(X_2,X_4;Y_3,...,Y_k) \\ &- E(X_2,X_4)T(X_1,X_3;Y_3,...,Y_k)\end{aligned} \quad (2.9)$$

(see [5]). For symmetric $(0,2)$-tensor field $E$ and $B$, we have the following identity ([5]):

$$E\widetilde{\wedge}Q(B,E) = Q(B,\bar{E}). \quad (2.10)$$

Note that $\bar{g} = G$.

## 3 Hypersurfaces

Let $M$, $n = dimM \geq 3$, be a connected hypersurface immersed isometrically in a semi-Riemannian manifold $(N, \widetilde{g})$. We denote by $g$ the metric tensor of $M$ induced from the metric tensor $\widetilde{g}$. Further, we denote by $\tilde{\nabla}$ and $\nabla$ the Levi-Civita connections corresponding to the metric tensors $\widetilde{g}$ and $g$, respectively. Let $\xi$ be a local unit vector field on $M$ in $N$ and let $\varepsilon = \widetilde{g}(\xi, \xi) = \pm 1$. We can present the Gauss formula and Weingarten formula of $M$ in $N$ in the following form:

$$\tilde{\nabla}_X Y = \nabla_X Y + \varepsilon H(X,Y)\xi, \qquad \tilde{\nabla}_X \xi = -A(X)$$

respectively, where $X, Y$ are vector fields tangent to $M$, $H$ is the second fundamental tensor and $A$ is the shape operator of $M$ in $N$ and $g(A(X), Y) = H(X,Y)$. Furthermore, for $k > 1$, we also have that $H^k(X,Y) = g(A^k(X), Y)$, $\mathrm{tr}(H^k) = tr(A^k)$, $k \geq 1$, $H^1 = H$ and $A^1 = A$. We denote, $R$ and $\tilde{R}$ the Riemannian-Christoffel curvature tensors of $M$ and $N$, respectively.

The Gauss equation of $M$ in $N$ has the following form:

$$R(X_1, X_2, X_3, X_4) = \tilde{R}(X_1, X_2, X_3, X_4) + \varepsilon \overline{H}(X_1, X_2, X_3, X_4). \tag{3.1}$$

From now on we will assume that $M$ is a hypersurface in a semi-Euclidean space $E_s^{n+1}$. So, Eq.(3.1) turns into

$$R(X_1, X_2, X_3, X_4) = \varepsilon \overline{H}(X_1, X_2, X_3, X_4), \tag{3.2}$$

where $X_1, X_2, X_3, X_4$ are vector fields tangent to $M$ and $\overline{H} = \frac{1}{2} H \overline{\wedge} H$. From (3.2), by contraction, we get easily

$$S(X_1, X_4) = \varepsilon(\mathrm{tr}(H)H(X_1, X_4) - H^2(X_1, X_4)). \tag{3.3}$$

Moreover, by contracting (3.3), we obtain

$$\kappa = \varepsilon(\mathrm{tr}(H)^2 - \mathrm{tr}(H^2)). \tag{3.4}$$

Now we give the following Lemmas which will be used in the main results.

**Lemma 3.1** **([7])** *Let $E$ and $D$ be two symmetric $(0,2)-$tensors at point $x$ of a semi-Riemannian manifold $(M, g)$. If the condition*

$$\alpha Q(g,E) + \gamma Q(E,D) + \beta Q(g,D) = 0; \qquad \alpha, \beta, \gamma \in \mathbb{R}, \gamma \neq 0$$

*is satisfied at $x$, then the tensors $E - \frac{1}{n} tr(E) g$ and $D - \frac{1}{n} tr(D) g$ are linearly dependent.*

**Lemma 3.2** **([7])** *Any hypersurface $M$, immersed isometrically in an $(n+1)$- dimensional semi-Euclidean space $E_s^{n+1}, n \geq 4$, satisfies the condition*

$$R \cdot R = Q(S, R). \tag{3.5}$$

**Proposition 3.3** *Let $M$ be a hypersurface in a semi-Euclidean space $E_s^{n+1}$, $n \geq 4$, then we have*

$$\widetilde{C} \cdot R = -(a+b) R \cdot R + \left( \frac{\kappa}{n} \left[ \frac{a}{n-1} + 2b \right] + b\varepsilon \right) Q(g,R). \tag{3.6}$$

*Proof.* Using the definition of the second fundamental tensor, Eq. (1.1) can be written as

$$H^3 = \mathrm{tr}(H)H^2 + \beta H + \gamma g. \tag{3.7}$$

Let $X_h, X_i, X_j, X_k, X_l, X_m \in \chi(M)$. So using (2.4) we have

$$\begin{aligned}(\widetilde{C}(X_h, X_i) \cdot R)(X_j, X_k, X_l, X_m) &= -R(\widetilde{\mathcal{C}}(X_h, X_i)X_j, X_k, X_l, X_m)\\ &\quad - R(X_j, \widetilde{\mathcal{C}}(X_h, X_i)X_k, X_l, X_m)\\ &\quad - R(X_j, X_k, \widetilde{\mathcal{C}}(X_h, X_i)X_l, X_m)\\ &\quad - R(X_j, X_k, X_l, \widetilde{\mathcal{C}}(X_h, X_i)X_m).\end{aligned} \tag{3.8}$$

Then using (2.2), (1.1), (2.5) and (2.8), we have

$$\begin{aligned}\widetilde{C} \cdot R &= aH\widetilde{\wedge}Q(H^2, H) - bQ(S, R) + b\lambda H\widetilde{\wedge}Q(g, H)\\ &\quad + \frac{\kappa\varepsilon}{n}\left[\frac{a}{n-1} + 2b\right](H\widetilde{\wedge}Q(g, H).\end{aligned} \tag{3.9}$$

Thus, by (2.10), Eq. (3.9) turns into

$$\begin{aligned}\widetilde{C} \cdot R &= aQ(H^2, \overline{H}) - bQ(S, R) + b\lambda Q(g, \overline{H})\\ &\quad + \frac{\kappa\varepsilon}{n}\left[\frac{a}{n-1} + 2b\right]Q(g, \overline{H}).\end{aligned} \tag{3.10}$$

From (3.3), since

$$H^2 = \mathrm{tr}(H)H - \varepsilon S,$$

using (3.2) and Lemma 3.2, the Eq. (3.10) can be rewritten as

$$\widetilde{C} \cdot R = -(a+b)R \cdot R + (\frac{\kappa}{n}\left[\frac{a}{n-1} + 2b\right] + b\varepsilon)Q(g, R). \tag{3.11}$$

This completes the proof of the proposition. □

As an immediate consequence of Proposition 3.3, we have the following theorem:

**Theorem 3.4** *Let $M$ be a hypersurface in a semi-Euclidean space $E_s^{n+1}$, $n \geq 4$. If the shape operator $A$ of $M$ satisfies* (1.1) *and the condition $\widetilde{C} \cdot R = 0$ holds on $M$, then $M$ is pseudosymmetric.*

**Remark 3.1** It is well known that a pseudosymmetric manifold is Ricci-pseudosymmetric, i.e. satisfies $R \cdot S = L_s Q(g, S)$. In [1] it was shown that on every Ricci-pseudosymmetric manifold the Ricci tensor is subjected to the condition $\mathrm{R}_{im}\mathrm{R}^m_{jkl} + \mathrm{R}_{jm}\mathrm{R}^m_{kil} + \mathrm{R}_{km}\mathrm{R}^m_{ijl} = 0$ and the manifold is called Riemann compatible (see [9]).

**Proposition 3.5** *Let $M$ be a hypersurface in a semi-Euclidean space $E_s^{n+1}$, $n\geq 4$, then we have*

$$\widetilde{C}\cdot H=(a+b)\varepsilon Q(H,H^2)+b\varepsilon tr(H)Q(g,H^2)$$
$$-b\varepsilon Q(g,H^3)-\frac{\kappa}{n}\left[\frac{a}{n-1}+2b\right]Q(g,H) \tag{3.12}$$

*and*

$$\widetilde{C}\cdot H^2=a\varepsilon Q(H,H^3)+b\varepsilon tr(H)Q(H,H^2)+b\varepsilon tr(H)Q(g,H^3)$$
$$-b\varepsilon Q(g,H^4)-\frac{\kappa}{n}\left[\frac{a}{n-1}+2b\right]Q(g,H^2). \tag{3.13}$$

*Proof.* Let $X_h,X_i,X_j,X_k\in\chi(M)$. So using (2.4), we have

$$(\widetilde{C}\cdot H)(X_h,X_i;X_j,X_k)=-H(\widetilde{\mathcal{C}}(X_j,X_k)X_h,X_i)$$
$$-H(X_h,\widetilde{\mathcal{C}}(X_j,X_k)X_i) \tag{3.14}$$

and, similarly,

$$(\widetilde{C}\cdot H^2)(X_h,X_i;X_j,X_k)=-H^2(\widetilde{\mathcal{C}}(X_j,X_k)X_h,X_i)$$
$$-H^2(X_h,\widetilde{\mathcal{C}}(X_j,X_k)X_i). \tag{3.15}$$

Then by making use of (2.2), (2.5) and (3.2), we get Eq. (3.12) and Eq. (3.13). □

**Lemma 3.6 ([4])** *Let $M$ be a hypersurface in a semi-Euclidean space $E_s^{n+1}$, $n\geq 3$. Then $M$ is pseudosymmetric if and only if $R\cdot R=0$ or the second fundamental tensor $H$ of $M$ satisfies the condition*

$$H^2=\alpha H+\beta g \quad \alpha,\beta\in\mathbb{R}.$$

**Definition 3.7** *Let $M$ be a hypersurface in a semi-Euclidean space $E_s^{n+1}$. If $\widetilde{C}\cdot H=0$ then $M$ is called quasi-conformally semiparallel.*

**Theorem 3.8** *Let $M$ be a hypersurface in a semi-Euclidean space $E_s^{n+1}$, $n\geq 4$. If $M$ is quasi-conformally semiparallel, then $M$ is pseudosymmetric.*

*Proof.* Since $M$ is quasi-conformally semiparallel, by using (3.12) we have

$$(a+b)\varepsilon Q(H,H^2)+b\varepsilon \mathrm{tr}(H)Q(g,H^2)-b\varepsilon Q(g,H^3)$$
$$-\frac{\kappa}{n}\left[\frac{a}{n-1}+2b\right]Q(g,H)=0. \tag{3.16}$$

Hence, from (3.16), by a contraction, we have

$$H^3=\frac{(a+(1+n)b)}{nb}\mathrm{tr}(H)H^2-\frac{(a+b)}{nb}\mathrm{tr}(H^2)H$$
$$-\frac{\kappa}{n}\left[\frac{a}{(n-1)b\varepsilon}+\frac{2}{\varepsilon}\right]H+\frac{1}{n}(-\mathrm{tr}(H)\mathrm{tr}(H^2) \tag{3.17}$$
$$+\mathrm{tr}(H^3)-\frac{\kappa}{n}\left[\frac{a}{(n-1)b\varepsilon}+\frac{2}{\varepsilon}\right]\mathrm{tr}(H))g.$$

So, substituting (3.17) into (3.16), we get

$$(a+b)\varepsilon Q(H,H^2)-\left(\frac{a+b}{n}\right)\varepsilon\mathrm{tr}(H)Q(g,H^2)$$
$$+\left(\frac{a+b}{n}\right)\varepsilon\mathrm{tr}(H^2)Q(g,H)=0. \tag{3.18}$$

Then, by Lemma 3.1, the tensors

$$H^2-\frac{1}{n}\mathrm{tr}(H^2)g \qquad \text{and} \quad H-\frac{1}{n}\mathrm{tr}(H)g$$

are linearly dependent. Thus, we get

$$H^2=\lambda H+\frac{1}{n}(\mathrm{tr}(H^2)-\lambda\mathrm{tr}(H))g.$$

So, by Lemma 3.6, $M$ is pseudosymmetric. □

**Definition 3.9** Let $M$ be a hypersurface in a semi-Euclidean space $E_s^{n+1}$,n$\geq$ 4. If $\widetilde{C}\cdot S=0$, then $M$ is called quasi-conformally Ricci-semisymmetric.

**Lemma 3.10** *Let $M$ be a hypersurface in a semi-Euclidean space $E_s^{n+1}, n\geq 4$. If $M$ is quasi-conformally Ricci-semisymmetric, then there is a real valued function $\lambda$ on $M$ such that*

$$H^3=tr(H)H^2-\lambda H+\frac{1}{n}(\lambda tr(H)-tr(H)tr(H^2)+tr(H^3))g. \tag{3.19}$$

*Proof.* Since $M$ is quasi-conformally Ricci-semisymmetric, by using (3.3), we have

$$\widetilde{C}\cdot S=\varepsilon(\mathrm{tr}(H)\widetilde{C}\cdot H-\widetilde{C}\cdot H^2)=0. \tag{3.20}$$

Thus, by substituting (3.12) and (3.13) into (3.20), we obtain

$$a\mathrm{tr}(H)Q(H,H^2)+(b\mathrm{tr}(H)^2+\frac{\kappa\varepsilon}{n}\left[\frac{a}{n-1}+2b\right])Q(g,H^2)$$
$$-2b\mathrm{tr}(H)Q(g,H^3)-\frac{\varepsilon\kappa}{n}\left[\frac{a}{n-1}+2b\right]\mathrm{tr}(H)Q(g,H) \tag{3.21}$$
$$-aQ(H,H^3)+bQ(g,H^4)=0.$$

Hence, from (3.22), by a contraction, we have

$$H^4=-\frac{(a+bn)}{bn}\mathrm{tr}(H)^2H^2+\varepsilon\kappa\left[\frac{a}{(n-1)\,bn}+\frac{2}{n}\right](-H^2+\mathrm{tr}(H)H)$$
$$+(2+\frac{a}{bn})\mathrm{tr}(H)H^3+\frac{a}{bn}\mathrm{tr}(H)\mathrm{tr}(H^2)H-\frac{a}{bn}\mathrm{tr}(H^3)H \tag{3.22}$$
$$+\left(\frac{\varepsilon\kappa}{n}\left[\frac{a}{n-1}+2b\right](\mathrm{tr}(H^2)-\mathrm{tr}(H)^2)+b\mathrm{tr}(H)^2\mathrm{tr}(H^2)\right.$$
$$\left.-2b\mathrm{tr}(H)\mathrm{tr}(H^3)+b\mathrm{tr}(H)^4\right)g.$$

So, by substituting (3.23) into (3.22), we get

$$aQ(H, \mathrm{tr}(H)H^2 - H^3) - \frac{a}{n}\mathrm{tr}(H)Q(g, \mathrm{tr}(H)H^2 - H^3) + \frac{a}{n}(\mathrm{tr}(H)\mathrm{tr}(H^2) - \mathrm{tr}(H^3))Q(g, H) = 0. \tag{3.23}$$

Then by Lemma 3.1, the tensors

$$\mathrm{tr}(H)H^2 - H^3 - \frac{1}{n}(\mathrm{tr}(H)\mathrm{tr}(H^2) - \mathrm{tr}(H^3))g$$

and

$$H - \frac{1}{n}\mathrm{tr}(H)g$$

are linearly dependent, which proves the lemma. □

Hence, by combining Theorem 3.4 and Lemma 3.10, we have the following theorem:

**Theorem 3.11** *Let $M$ be a hypersurface in a semi-Euclidean space $E_s^{n+1}$, $n \geq 4$. If $M$ is quasi-conformally Ricci-semisymmetric and the condition $\widetilde{C} \cdot R = 0$ holds on $M$, then $M$ is pseudosymmetric.*

**Theorem 3.12** *Let $M$ be a hypersurface in a semi-Euclidean space $E_s^{n+1}$, $n \geq 4$. If $M$ is quasi-conformally Ricci-semisymmetric and Ricci curvature of $M$ satisfies the condition*

$$S^2 = \alpha S + \beta g, \qquad \alpha, \beta : M \to \mathbb{R}, \tag{3.24}$$

*then $M$ is Ricci-pseudosymmetric.*

*Proof.* By using (2.4), (2.1) and (2.5), we have

$$(\widetilde{C} \cdot S) = a(R \cdot S) + bQ(g, S^2) - \frac{\kappa}{n}\left[\frac{a}{n-1} + 2b\right]Q(g, S). \tag{3.25}$$

Since the condition $\widetilde{C} \cdot S = 0$ holds on $M$, by using (3.24) in Eq. (3.25) we get

$$R \cdot S = \left[-\frac{b}{a}\alpha + \frac{\kappa}{n}(\frac{a}{n-1} + 2b)\right]Q(g, S).$$

This completes the proof of the theorem. □

**Lemma 3.13 ([3])** *[3]Let $M$ be a hypersurface in a semi-Euclidean space $E_s^{n+1}, n \geq 4$. $M$ satisfies the condition*

$$R \cdot S = Q(H, tr(H)H^2 - H^3). \tag{3.26}$$

**Theorem 3.14** *Let $M$ be a hypersurface in a semi-Euclidean space $E_s^{n+1}$, $n \geq 4$. If the condition $H^2 = tr(H)H$ holds on $M$, then $M$ is quasi-conformally Ricci-semisymmetric.*

*Proof.* Since $H^2 = \mathrm{tr}(H)H$ and $g(A(X),Y) = H(X,Y)$, we have

$$H^3 = tr(H)H^2 \quad \text{and} \quad H^4 = tr(H)^2H^2. \tag{3.27}$$

So, by substituting (3.27) into (3.26), we get $R \cdot S = 0$. Thus, Eq.(3.25) turns into

$$(\widetilde{C} \cdot S) = bQ(g,S^2) - \frac{\kappa}{n}\left[\frac{a}{n-1} + 2b\right]Q(g,S).$$

Since $H^2 = \mathrm{tr}(H)H$, by using (3.27) and (3.3), we get $Q(g,S) = 0$ and $Q(g,S^2) = 0$, which proves $M$ is quasi-conformally Ricci-semisymmetric. □

*Example 3.1* Let $\mathbf{S}^2 = \left\{p \in \mathbb{R}^3 \text{ such that } |p| = 1\right\}$ be the standard unit sphere. First we consider

$$M^4 = \mathbf{S}_1^2 \times \mathbf{S}_2^2 = \left\{(p,q) \in \mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3 \text{ such that } |p| = |q| = 1\right\}.$$

Next, we take the cone

$$\mathbf{C}^5 = \left\{(tp,tq) \in \mathbb{R}^6 \text{ such that } |p| = |q| = 1,\ \mathrm{t} > 0,\ \mathrm{t} \in \mathbb{R}\right\}.$$

In [2], the authors show that the principal curvatures of $\mathbf{C}^5$ are $(0, \frac{1}{\sqrt{2}t}, \frac{1}{\sqrt{2}t}, -\frac{1}{\sqrt{2}t}, -\frac{1}{\sqrt{2}t})$ and the cone $\mathbf{C}^5$ is Ricci-semisymmetric, but not semi-symmetric. It can be easily seen that the cone $\mathbf{C}^5$ satisfies the conditions $\widetilde{C} \cdot S = 0$, $\widetilde{C} \cdot R = 0$ and it is pseudosymmetric.

## References


1. Arslan, K.; Celik, Y.; Deszcz, R.; Ezenta, R. – *On the equivalence of Ricci-semisymmetry and semisymmetry*, Colloq. Math., 76 (1998), 279–294.
2. Abdalla, B.E.; Dillen, F. – *A Ricci-semi-symmetric hypersurface of Euclidean space which is not semi-symmetric*, Proc. Amer. Math. Soc., 130 (2002), 1805–1808.
3. Defever, F.; Deszcz, R.; Dhooghe, P.; Verstraelen, L.; Yaprak, S. – *On Ricci-pseudosymmetric hypersurfaces in spaces of constant curvature*, Results Math., 27 (1995), 227–236.
4. Deszcz, R. – *On Certain Classes of Hypersurfaces in Spaces of Constant Curvature*, Geometry and topology of submanifolds, VIII (Brussels, 1995/Nordfjordeid, 1995), 101-110, World Sci. Publ., River Edge, NJ, 1996.
5. Deszcz, R.; Głogowska, M. – *Some nonsemisymmetric Ricci-semisymmetric warped product hypersurfaces*, Publ. Inst. Math. (Beograd) (N.S.), 72 (2002), 81–93.
6. Deszcz, R. – *On pseudosymmetric spaces*, Bull. Soc. Math. Belg. Sér., A, 44 (1992), 1–34.
7. Deszcz, R.; Verstraelen, L.; Yaprak, S. – *Hypersurfaces with pseudosymmetric Weyl tensor in conformally flat manifolds*, Geometry and topology of submanifolds, IX (Valenciennes/Lyon/Leuven, 1997), 108–117, World Sci. Publ., River Edge, NJ, 1999.
8. Dabrowska, M.; Defever, F.; Deszcz, R.; Kowalczyk, D. – *Semisymmetry and Ricci-semisymmetry for hypersurfaces of semi-Euclidean spaces*, Publ. Inst. Math. (Beograd) (N.S.), 67 (2000), 103–111.
9. Mantica, C.A.; Suh, Y.J. – *Pseudo-Z symmetric space-times*, J. Math. Phys., 55 (2014), 042502, 12 pp.
10. Özgür, C. – *Hypersurfaces satisfying some curvature conditions in the semi-Euclidean space*, Chaos Solitons Fractals, 39 (2009), 2457–2464.
11. Özgür, C.; Arslan, K. – *On some class of hypersurfaces in $\mathbb{E}^{n+1}$ satisfying Chen's equality*, Turkish J. Math., 26 (2002), 283–293.
12. Özgür, C.; Arslan, K.; Murathan, C. – *Conharmonically semi-parallel hypersurfaces*, Proceedings of the Centennial "G. Vranceanu", Part II (Bucharest, 2000), An. Univ. Bucure'sti Mat. Inform., 50 (2001), 121–132.
13. Szabá, Z.I. – *Structure theorems on Riemannian spaces satisfying $R(X,Y) \cdot R = 0$. I,* The local version. J. Differential Geom., 17 (1982), 531–582 (1983).
14. Yano, K.; Sawaki, S. – *Riemannian manifolds admitting a conformal transformation group*, J. Differential Geometry, 2 (1968), 161–184.