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Primary decomposition of ideals (submodule) in BCK-algebras

R.A. Borzooei · S. Saidi Goraghani

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Abstract In this paper, we investigate the ideal theory in BCK-algebras and we define the notions of primary and P-primary ideals. Then we show that in bounded implicative BCK-algebras, if an ideal has a primary decomposition, then it has a reduced primary decomposition. In the follow, we extend the above concepts to X^E-modules (Extended BCK-modules) and we introduce the notions of primary and P-primary submodules. Finally, we prove that if X is a lower BCK-semilattice and M is a Noetherian X^E-module, then every proper submodule of M has a reduced primary decomposition.

Keywords BCK-algebra · Extended BCK-module · Radical, primary and P-primary ideals · Primary decomposition

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1 Introduction

The notion of BCK-algebra was formulated first in 1966 by Imai and Iséki [6]. This notion is originated from two different ways. One of the motivations is based on set theory. Another motivation is from classical and non-classical propositional calculus. As is well known, there is close relationship between the notion of the set difference in set theory and the implication functor in logical systems. Then the following problems arise from this relationship. What is the most essential and fundamental common properties? In general, can we establish a new and good theory of algebra? To give an answer this problems, Imai and Iséki introduced a notion of a new class of algebras, which are called BCK-algebras. This name is taken from BCK-system of

R.A. Borzooei
Department of Mathematics
Shahid Beheshti University
Tehran, IRAN
E-mail: borzooei@sbu.ac.ir

S. Saidi Goraghani
Department of Mathematics
Islamic Azad University of Central Tehran Branch
Tehran, IRAN
E-mail: SiminSaidi@yahoo.com

C. A. Meredith. BCK-algebras have been applied to many branches of mathematics, such as group theory, functional analysis, probability theory and topology. Since the notion of ideal in BCK-algebras is important, for completion of study of ideals in BCK-algebras, in this paper, we present definitions of radical of an ideal and primary decomposition of an ideal. The simplification of an ideal helps us for better studying it. Hence, the decomposition of an ideal can be useful and important. The notion of BCK-module was introduced in [1] as an action of a BCK-algebra over a commutative group by Aslam, Thaheem and Abujabal. The idea was further explored by Perveen, Aslam and Thaheem in [8]. The concept of BCK-module was extended by Borzooei, Shohani and Jafari in [4]. Later, this concept was extended in different way by Borzooei and Goraghani in [3]. Since submodules in BCK-modules are extension of ideals in BCK-algebras, in continuing of study of primary decomposition of an ideal, we introduce and investigate it on a submodule in extended BCK-modules and we obtain some results as mentioned in the abstract.

2 Preliminaries

Definition 2.1 ([7]) *A BCK-algebra is a structure $X = (X, *, 0)$ of type $(2, 0)$ such that:*

$(BCK1)$ $((x * y) * (x * z)) * (z * y) = 0$,
$(BCK2)$ $(x * (x * y)) * y = 0$,
$(BCK3)$ $x * x = 0$,
$(BCK4)$ $0 * x = 0$,
$(BCK5)$ $x * y = y * x = 0$ *implies that* $x = y$, *for all* $x, y, z \in X$.

The relation $x \leq y$ which is defined by $x * y = 0$ is a partial order with 0 as the least element which is called BCK-order. In BCK-algebra X, for every $x, y, z \in X$, we have:

$(BCK6)$ $(x * y) * z = (x * z) * y$,
$(BCK7)$ $x \leq y$ *implies* $z * y \leq z * x$,
$(BCK8)$ $x \leq y$ *implies* $x * z \leq y * z$.

Let $(X, *, 0)$ be a BCK-algebra. Then $S \subseteq X$ is said to be a *subalgebra* of X, if for every $x, y \in S$, $x * y \in S$, i.e., S is closed under the binary operation "$*$" of X. $\emptyset \neq I \subseteq X$ is called an *ideal* of X, if $0 \in I$ and for $x, y \in X$, $x * y \in I$ and $y \in I$ imply that $x \in I$. The set of all ideals of BCK-algebra X is denoted by $\mathcal{I}(X)$. $I \in \mathcal{I}(X)$ is called a *maximal* ideal of X if I is a proper ideal of X and it is not a proper subset of any proper ideal of X. X is called *bounded*, if there exists $1 \in X$ such that $x \leq 1$, for every $x \in X$ and in this case, we let $Nx = 1 * x$. X is said to be a *lower BCK-semilattice* if X is lower semilattice with respect to BCK-order "$\leq$". X is said to be *positive implicative* if $(x * z) * (y * z) = (x * y) * z$, for every $x, y, z \in X$. Moreover, X is positive implicative if and only if $x * y = (x * y) * y$, for every $x, y \in X$. X is said to be *commutative*, if $y * (y * x) = x * (x * y)$, for all $x, y \in X$. Moreover, every commutative BCK-algebra is a lower BCK-semilattice. X is said to be *implicative* if $x * (y * x) = x$, for all $x, y \in X$. In a lower BCK-semilattice X, a proper ideal I, is called a *prime ideal* of X, if $x \wedge y \in I$ implies that $x \in I$ or $y \in I$, for $x, y \in X$. In a lower BCK-semilattice, every maximal ideal is a prime ideal. $\emptyset \neq I \subseteq X$ is said to be a *positive implicative ideal* if $0 \in I$ and $(x*y)*z$, $y*z \in I$ implies that $x*z \in I$, for $x, y, z \in X$. $\emptyset \neq I \subseteq X$ is said to be an *implicative ideal* if $0 \in I$ and $(x*(y*x))*z$, $z \in I$ implies that $x \in I$, for $x, y, z \in X$. Furthermore, any positive implicative ideal (implicative ideal) is an ideal. In a BCK-algebra X, we let $x \wedge y = y*(y*x)$ and in a bounded BCK-algebra X, we let $x \vee y = N(Nx \wedge Ny)$, for all $x, y \in X$. In a bounded commutative BCK-algebra, $x \wedge y$ is the greatest lower bound and $x \vee y$ is the least upper bound and so $(L, \vee, \wedge)$ is a bounded lattice.

Theorem 2.2 **([7])** *A BCK-algebra is implicative if and only if it is both commutative and positive implicative.*

Lemma 2.3 **([5])** *Let X be a bounded implicative BCK-algebra . Then for all $x, y, z \in X$:*

(i) $x \wedge y = x * Ny$, $x*y = Ny \wedge x$, $NNx = x$,
(ii) $x*(x \wedge y) = x*y$,
(iii) $x \wedge (y*z) = (x \wedge y)*(x \wedge z)$,
(iv) $(x*y)+(y*x) = x+y$, *where* $x+y = (x*y) \vee (y*x)$,
(v) $(x+y) \wedge z = (x \wedge z)+(y \wedge z)$,
(vi) $x+x = 0$,
(vii) $x+0 = 0+x = x$.

Definition 2.4 **([7])** *Let X be a BCK-algebra and $A \subseteq X$. The ideal I generated by A is the intersection of all ideals of X which contain A and it is denoted by $I = (A]$. Moreover, if A is a finite subset of X, then I is called finitely generated.*

Theorem 2.5 **([7])** *Let X be a BCK-algebra and $\emptyset \neq A \subseteq X$. Then $(A] = \{x \in X : \exists a_1, \cdots, a_n \in A$ such that $(...(x*a_1)*...)*a_n = 0\}$, $(a] = \{x : x*^n a = 0$, for some $n \in \mathbb{N}\}$. Moreover, if X is positive implicative, then $(a] = \{x : x*a = 0\}$, for every $a \in X$.*

Corollary 2.6 **([7])** *Let X be a commutative BCK-algebra and I be an ideal of X. If $x \wedge y \in I$, then $(I \cup \{x\}] \cap (y \cup \{y\}] = I$, for any $x, y \in X$.*

Definition 2.7 **([7])** *A commutative BCK-algebra X is said to be cancellative if $x \wedge y = 0$ implies that $x = 0$ or $y = 0$, for any $x, y \in X$.*

Definition 2.8 **([7])** *Let X be a BCK-algebra and $\emptyset \neq S \subseteq X$. We say that S is $\wedge$-closed, if $a \wedge b \in S$, for all $a, b \in S$.*

Theorem 2.9 **([7])** *Let X be a lower BCK-semilattice, $I \in \mathcal{I}(X)$, $S \subseteq X$ be $\wedge$-closed and $S \cap I = \emptyset$. Then there exists a maximal ideal P of X such that $P \supseteq I$ and $P \cap S = \emptyset$. Furthermore, P is a prime ideal of X.*

Definition 2.10 **([7])** *A BCK-algebra X is said to be Noetherian if each ideal of X is finitely generated.*

Definition 2.11 **([3])** *Let X be a BCK-algebra, M be an Abelian group and the operation $. : X \times M \longrightarrow M$ satisfies the following axioms:*

$(XM1)$ $(x \wedge y).m = x.(y.m)$,
$(XM2)$ $x.(m+n) = x.m + x.n$,

$(XM3)$ $0.m = 0$,
$(XM4)$ $(x * y).m = x.m - y.m$,

where $x * y \neq 0$, for $x \neq y$, for all $x, y \in X$ and $m, n \in M$. Then M is called an extended BCK-module or briefly X^E-module. If X is bounded and $1.m = m$, for every $m \in M$, then M is called a unitary X^E-module.

Definition 2.12 ([3]) *A subgroup N of X^E-module M is a submodule of M if for every $x \in X$ and $n \in N$, $x.n \in N$.*

Definition 2.13 ([3]) *Let M be an X^E-module and N be a submodule of M. Then N is called a prime submodule of M, if $N \neq M$ and for any $x \in X$ and $m \in M$, $x.m \in N$ implies that $m \in N$ or $x \in (N : M)$, where $(N : M) = \{x : x.M \subseteq N\}$.*

Note From now on, in this paper, we let X be a BCK-algebra.

3 Some results on prime ideals

In this section, we get some results on prime ideals in BCK-algebras, which is used in the next sections.

Proposition 3.1 (i) *If X is a finite lower BCK-semilattice and I is an ideal of X such that $|I| = |X| - 1$, then I is a prime ideal of X.*
(ii) *If X is cancellative, then $I = \{0\}$ is a prime ideal of X.*

Proof. (i) Let $x \wedge y \in I$ and $x \notin I$, for $x, y \in X$. Since $|I| = |X| - 1$, $y \in I$.
(ii) Let $x \wedge y \in I = \{0\}$, for $x, y \in X$. Then $x \wedge y = 0$ and so by hypothesis, $x = 0$ or $y = 0$. It means that $x \in I$ or $y \in I$. □

Definition 3.2 *Let X be bounded and $c \in X$. Then c is called a coatom of X if $c * x = 0$ implies that $x = c$ or $x = 1$, for every $x \in X$.*

Theorem 3.3 *Let X be bounded implicative which is a chain. If c is a coatom of X, then $(c]$ is a maximal ideal of X and so it is a prime ideal of X.*

Proof. By Theorem 2.2, X is positive implicative. Let $c \in X$ be a coatom of X. If $(c] = X$, then $1 \in (c]$ and so by Theorem 2.5, $1 * c = 0$. Hence $1 = c$, which is a contradiction. Now, let there exists an ideal $K \neq X$ such that $(c] \subset K$. Then there exists $x \in K$ such that $x \notin (c]$. By Theorem 2.5, it results that $x * c \neq 0$ and so $c * x = 0$. It means that $c < x < 1$, which is a contradiction. Because, c is a coatom. Hence, $(c]$ is a maximal ideal of X and so it is a prime ideal of X. □

Theorem 3.4 *Let X be implicative and P be a proper ideal of X. Then P is a prime ideal of X if and only if $x * y \in P$ or $y * x \in P$, for every $x, y \in X$.*

Proof. $(\Rightarrow)$ Let P be a prime ideal of X and $x, y \in X$. Then by $(BCK6)$, we have $(x*y)\wedge(y*x) = (y*x)*((y*x)*(x*y)) = (y*x)*((y*(x*y))*x) = (y*x)*(y*x) = 0 \in P$. Hence, $x * y \in P$ or $y * x \in P$.
$(\Leftarrow)$ Let $x \wedge y \in P$, for $x, y \in X$. If $y * x \in P$, since $y * (y * x) = x \wedge y \in P$ and P is an ideal of X, then $y \in P$. If $x * y \in P$, since $x * (x * y) = y \wedge x \in P$, then $x \in P$. Therefore, P is a prime ideal of X. □

Theorem 3.5 *Let X be implicative. Then X is a chain if and only if all proper ideals of X are prime.*

Proof. ($\Rightarrow$) Let X be a chain, P be a proper ideal of X and $a \wedge b \in P$, for $a, b \in X$. If $a \leq b$, then $a * b = 0$ and so $a = a * 0 = a * (a * b) = a \wedge b \in P$. If $b \leq a$, then by the similar way, we have $b \in P$. Hence, P is a prime ideal of X.

($\Leftarrow$) Let all ideals of X be prime and $a, b \in X$. We show that $a \leq b$ or $b \leq a$. Let $A = \{x : x \leq (a \wedge b)\}$. It is clear that $0 \in A$. Now, let $x * y, y \in A$. Then $x * y \leq (a \wedge b)$ and $y \leq (a \wedge b)$ and so $(x * y) * (a \wedge b) = y * (a \wedge b) = 0$. Since X is implicative, by Theorem 2.2, X is positive implicative. Hence

$$x * (a \wedge b) = (x * (a \wedge b)) * 0 = (x * (a \wedge b)) * (y * (a \wedge b)) = (x * y) * (a \wedge b) = 0.$$

It means that $x \leq (a \wedge b)$ and so $x \in A$. Hence, A is an ideal of X and so A is a prime ideal of X. Now, since $a \wedge b \in A$, we get $a \in A$ or $b \in A$. It follows that $a \leq a \wedge b \leq b$ or $b \leq a \wedge b \leq a$ and so $a \leq b$ or $b \leq a$. Therefore, X is a chain. □

Theorem 3.6 *Let X be a lower BCK-semilattice and $\mathcal{FGI}(X)$ be the set of all finitely generated ideals of X. If P is a maximal element of $\mathcal{I}(X) \setminus \mathcal{FGI}(X)$, then P is a prime ideal of X.*

Proof. Let P be a maximal element of $\mathcal{I}(X) \setminus \mathcal{FGI}(X)$, $a \wedge b \in P$ and suppose that $a \notin P$ and $b \notin P$, for $a, b \in X$. Consider $(P \cup \{a\}]$ and $(P \cup \{b\}]$. By the maximality of P, $(P \cup \{a\}] \in \mathcal{FGI}(X)$ and $(P \cup \{b\}] \in \mathcal{FGI}(X)$. On the other hand, by Corollary 2.6, $(P \cup \{a\}] \cap (P \cup \{b\}] = P$ and so $P \in \mathcal{FGI}(X)$, which is a contradiction. Hence, $a \in P$ or $b \in P$ and so P is a prime ideal of X. □

Theorem 3.7 *Let X be a lower BCK-semilattice. Then X is Noetherian if and only if every prime ideal of X is finitely generated.*

Proof. ($\Rightarrow$) Let X be Noetherian. Then every ideal of X is finitely generated and so every prime ideal of X is finitely generated.

($\Leftarrow$) Let every prime ideal of X is finitely generated. We should prove that $\mathcal{I}(X) = \mathcal{FGI}(X)$. Let $S = \{I : I \in \mathcal{I}(X) \setminus \mathcal{FGI}(X)\}$. We will prove that $S = \emptyset$. If $S \neq \emptyset$, then by Zorn Lemma, S has a maximal element P. By Theorem 3.6, P is a prime ideal of X and so $P \in \mathcal{FGI}(X)$, which is a contradiction. □

Note From now on, in this paper, let $\mathcal{PI}(X)$ be the set of all prime ideals of X.

4 Primary decomposition of ideals

In this section, we define the notions of primary and P-primary ideals in BCK-algebras and we show that in bounded implicative BCK-algebras, if an ideal has a primary decomposition, then it has a reduced primary decomposition.

Definition 4.1 *Let X be a lower BCK-semilattice and $I \in \mathcal{I}(X)$. Then the intersection of all prime ideals of X, including I, is called radical of I and it is denoted by $rad_X(I)$ or briefly $rad(I)$. If there is not any prime ideal of X including I, then we let $rad(I) = X$.*

Example 4.1 (i) Let $X = \{0, 1, 2, 3\}$ and operation $*$ be defined by

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

Then X is a commutative BCK-algebra and $I = \{0, 3\}$ is an ideal of X. It is clear that $P_1 = \{0, 2, 3\}$ and $P_2 = \{0, 1, 3\}$ are prime ideals of X containing I. Then $rad(I) = P_1 \cap P_2 = I$.

(ii) Let $X = \{0, 1, 2, 3, 4\}$ and operation $*$ be defined by

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

Then X is a lower BCK-semilattice. It is easy to see that $I = \{0, 1\}$ is an ideal of X and $P = \{0, 1, 2\}$ is only prime ideal of X including I. Hence, $rad(I) = P$.

Theorem 4.2 *Let X be an implicative chain. Then $rad(I) = I$, for every $I \in \mathcal{I}(X)$.*

Proof. By Theorem 3.5, the proof is clear. □

Lemma 4.3 *Let X be bounded implicative and $a, b, c \in X$. Then $a \wedge (b * c) = (a \wedge b) * c$.*

Proof. By Lemma 2.3 (i) and $(BCK6)$, $a \wedge (b * c) = (b * c) * Na = (b * Na) * c = (a \wedge b) * c$. □

Notation The set of all prime ideals of X that contain $J \in \mathcal{I}(X)$ will be denoted by $\mathcal{PI}_J(X)$.

Theorem 4.4 *Let X be bounded implicative and $I \in \mathcal{I}(X)$. Then $rad(I) = \{x \in X : \forall P \in \mathcal{PI}_I(X), \exists c \in X \setminus P$ such that $c \wedge x \in I\}$.*

Proof. Let $T = \{x \in X : \forall P \in \mathcal{PI}_I(X), \exists c \in X \setminus P \text{ such that } c \wedge x \in I\}$ and $x \in rad(I)$. Then $x \in P$, for every $P \in \mathcal{PI}_I(X)$. If $x \in I$, then by considering $c = 1$, we have $x \in T$. Now, let $x \notin I$. If $x \notin T$, then there exists $P_1 \in \mathcal{PI}_I(X)$ such that $c \wedge x \notin I$, for every $c \in X \setminus P_1$. Let $S = \{(c \wedge x) * y : y \in I \text{ and } c \in X \setminus P_1\}$. First, we show that S is $\wedge$-closed. Let $(c_1 \wedge x) * y_1, (c_2 \wedge x) * y_2 \in S$, where $c_1, c_2 \in X \setminus P_1$ and $y_1, y_2 \in I$. By Lemmas 2.3 (i) and 4.3,

$$\begin{aligned} &((c_1 \wedge x) * y_1) \wedge ((c_2 \wedge x) * y_2) \\ &= ((c_1 \wedge x) * y_1) \wedge (c_2 \wedge x)) * y_2 = ((c_2 \wedge x) \wedge ((c_1 \wedge x) * y_1)) * y_2, \\ &= (((c_2 \wedge x) \wedge (c_1 \wedge x)) * y_1) * y_2 = Ny_2 \wedge (((c_1 \wedge c_2) \wedge x) * y_1), \\ &= (Ny_2 \wedge ((c_1 \wedge c_2) \wedge x)) * y_1 = ((Ny_2 \wedge c_1 \wedge c_2) \wedge x) * y_1. \end{aligned}$$

Now, we show that $Ny_2 \wedge c_1 \wedge c_2 \in X \setminus P_1$. Let $Ny_2 \wedge c_1 \wedge c_2 \in P_1$. Since $c_1 \wedge c_2 \notin P_1$, $Ny_2 \in P_1$ and so $1 \in P_1$. Since $x \leq 1 \in P_1$, $x \in P_1$, for every $x \in X$ and so $P_1 = X$, which is a contradiction. Hence, $Ny_2 \wedge c_1 \wedge c_2 \in X \setminus P_1$ and so $((Ny_2 \wedge c_1 \wedge c_2) \wedge x) * y_1 \in S$. It means that$((c_1 \wedge x) * y_1) \wedge ((c_2 \wedge x) * y_2) \in S$ and so S is $\wedge$-closed. Now, we prove that $S \cap I = \emptyset$. If $S \cap I \neq \emptyset$, then there exist $c' \in X \setminus P_1$ and $y' \in I$ such that $(c' \wedge x) * y' \in I$. It results that $c' \wedge x \in I$. But, by definition of S, $c \wedge x \notin I$, for every $c \in X \setminus P_1$, which is a contradiction. Then $S \cap I = \emptyset$ and so by Theorem 2.9, there exists $P_2 \in \mathcal{PI}_I(X)$ such that $P_2 \cap S = \emptyset$. Since $(c \wedge x) * x = 0 \in P$ and $x \in P$, $c \wedge x \in P$, for every $c \in X \setminus P$ and for every $P \in \mathcal{PI}_I(X)$. Then $(c \wedge x) \in P_2$. On the other hand, $c \wedge x = (c \wedge x) * 0 \in S$. Hence, $c \wedge x \in P_2 \cap S$, which is a contradiction. It implies that $x \in T$. Therefore, $rad(I) \subseteq T$.

Now, let $x \in T$. Hence, for every $P \in \mathcal{PI}_I(X)$ there exists $c \in X \setminus P$ such that $c \wedge x \in I \subseteq P$. Since $c \notin P$, $x \in P$, for every $P \in \mathcal{PI}_I(X)$. It means that $x \in rad(I)$ and so $T \subseteq rad(I)$. Therefore, $T = rad(I)$. □

Proposition 4.5 *Let X be bounded implicative and $I \in \mathcal{I}(X)$. If for every $P \in \mathcal{PI}(X)$, $P \cap I \neq \{0\}$ implies that $I \subseteq P$, then*

$$rad(I) = \{x \in X : \forall\ P \in \mathcal{PI}(X) \textit{ with } P \cap I \neq \{0\}, \\ \exists c \in X \setminus P \textit{ such that } c \wedge x \in I\}.$$

Proof. By Theorem 4.4, the proof is clear. □

Theorem 4.6 *Let X be a lower BCK-semilattice and $I, J, I_1, \cdots, I_n$ be ideals of X. Then:*

(i) $I \subseteq rad(I)$,
(ii) $I \subseteq J$ *implies* $rad(I) \subseteq rad(J)$,
(iii) $rad(I) \cup rad(J) \subseteq rad(I \cup J)$.

Moreover, if X is bounded implicative and $P \cap I_k \neq \{0\}$ implies that $I_k \subseteq P$, for every $P \in \mathcal{PI}(X)$ and $1 \leq k \leq n$, then:

(iv) $rad(rad(I)) = rad(I)$,
(v) $rad(\bigcap_{k=1}^n I_k) = \bigcap_{k=1}^n rad(I_k)$.

Proof. The proofs of (i), (ii) and (iii) are easy.

(iv) By (i), $rad(I) \subseteq rad(rad(I))$. Now, let $x \in rad(rad(I))$ and $P \in \mathcal{PI}(X)$ with $P \cap I \neq \{0\}$. Then by (i), $P \cap rad(I) \neq \{0\}$. Since $x \in rad(rad(I))$, by Proposition 4.5, there exists $c_1 \in X \setminus P$ such that $c_1 \wedge x \in rad(I)$. Since $c_1 \wedge x \in rad(I)$ and $P \cap I \neq \{0\}$, by Proposition 4.5, there exists $c_2 \in X \setminus P$ such that $(c_2 \wedge c_1) \wedge x = c_2 \wedge (c_1 \wedge x) \in I$. It is clear that $c = c_1 \wedge c_2 \in X \setminus P$. Similarly, for every $P \in \mathcal{PI}(X)$ with $P \cap I \neq \{0\}$ there is $c \in X \setminus P$ such that $c \wedge x \in I$. Hence, by Proposition 4.5, $x \in rad(I)$. Therefore, $rad(rad(I)) \subseteq rad(I)$.

(v) Let $x \in rad(\bigcap_{k=1}^n I_k)$ and $P \in \mathcal{PI}_{I_t}(X)$, for $1 \leq t \leq n$. Since $I_t \subseteq P$, $\bigcap_{k=1}^n I_k \subseteq I_t \subseteq P$. Since $x \in rad(\bigcap_{k=1}^n I_k)$, by Theorem 4.4, there exists $c \in X \setminus P$ such that $c \wedge x \in \bigcap_{k=1}^n I_k \subseteq I_t$ and so $c \wedge x \in I_t$. Hence, $x \in rad(I_t)$. Similarly, $x \in rad(I_k)$, for every $1 \leq k \leq n$ and so $x \in \bigcap_{k=1}^n rad(I_k)$. Hence, $rad(\bigcap_{k=1}^n I_k) \subseteq \bigcap_{k=1}^n rad(I_k)$.

Now, let $x \in \bigcap_{k=1}^n rad(I_k)$ and $P \in \mathcal{PI}(X)$ with $P \cap (\bigcap_{k=1}^n I_k) \neq \{0\}$. Then $P \cap I_k \neq \{0\}$, for every $1 \leq k \leq n$. Since $x \in rad(I_k)$, by Proposition 4.5, there is $c_k \in X \setminus P$ such that $c_k \wedge x \in I_k$, for every $1 \leq k \leq n$. Let $c = c_1 \wedge \cdots \wedge c_n$. It is clear that $c \notin P$. On the other hand, since $(c \wedge x) \leq (c_k \wedge x) \in I_k$, $c \wedge x \in I_k$, for every $1 \leq k \leq n$. Then $c \wedge x \in \bigcap_{k=1}^n I_k$. Therefore, by Proposition 4.5, $x \in rad(\bigcap_{k=1}^n I_k)$ and so $\bigcap_{k=1}^n rad(I_k) \subseteq rad(\bigcap_{k=1}^n I_k)$ □

Definition 4.7 *Let X be a lower BCK-semilattice and Q be a proper ideal of X. Then Q is called a primary ideal of X if $a \wedge b \in Q$, then there exists $c \in X \setminus P$ such that $c \wedge b \in Q$ or $a \wedge c \in Q$, for every $P \in \mathcal{PI}_Q(X)$ and $a, b \in X$.*

Example 4.2 Let $X = \{0, 1, 2, 3, 4\}$ and operation $*$ be defined on X by

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	2	1	0	0
4	4	4	4	4	0

Then $(X, *, 0)$ is a lower BCK-semilattice. Let $I = \{0, 1\}$ and $J = \{0, 2\}$. It is easy to show that I and J are primary ideals of X.

Proposition 4.8 *Let X be bounded implicative and Q be an ideal of X. Then Q is a primary ideal of X if and only if $a \wedge b \in Q$ implies that $a \in rad(Q)$ or $b \in rad(Q)$, for any $a, b \in X$.*

Proof. ($\Rightarrow$) Let Q be a primary ideal of X and $a \wedge b \in Q$, for $a, b \in X$. If $a \in Q$, then $a \in rad(Q)$. Let $a \notin Q$. Then there exits $c \in X \setminus P$ such that $c \wedge b \in Q$ or $a \wedge c \in Q$, for every $P \in \mathcal{PI}_Q(X)$. If $c \wedge b \in Q$, then $c \wedge b \in P$, for every $P \in \mathcal{PI}_Q(X)$. Since $c \notin P$, $b \in P$, for every $P \in \mathcal{PI}_Q(X)$. It results that $b \in \bigcap_{Q \subseteq P} P = rad(Q)$. Similarly, if $a \wedge c \in Q$, then $a \in rad(Q)$.

($\Leftarrow$) Let $Q \in \mathcal{I}(X)$. If $a \wedge b \in Q$, then $a \in rad(Q)$ or $b \in rad(Q)$, for $a, b \in X$ and so by Theorem 4.4, there exists $c \in X \setminus P$ such that $c \wedge b \in Q$ or $a \wedge c \in Q$, for every $P \in \mathcal{PI}_Q(X)$. It means that Q is a primary ideal of X. $\square$

Theorem 4.9 *In a bounded lower BCK-semilattice, every prime ideal is a primary ideal.*

Proof. Let X be a bounded lower BCK-semilattice, Q be a prime ideal of X, $a \wedge b \in Q$ and $a \notin Q$, for $a, b \in X$. Then by considering $c = 1 \in X \setminus P$, for every $P \in \mathcal{PI}_Q(X)$, we have $c \wedge b = b \in Q$. Hence, P is a primary ideal of X. $\square$

Example 4.3 In Example 4.2, I is a primary ideal of X, but it is not a prime ideal of X.

Theorem 4.10 *Let X be bounded implicative and $I \cap P \neq \{0\}$ implies that $I \subseteq P$, for every $I \in \mathcal{I}(X)$ and $P \in \mathcal{PI}(X)$. Then the radical of every primary ideal of X is a prime ideal of X.*

Proof. Let Q be a primary ideal of X. If $rad(Q) = X$, then $1 \in rad(Q)$. Hence, by Theorem 4.4, for every $P \in \mathcal{PI}_Q(X)$, there exists $c \in X \setminus P$ such that $c \wedge 1 = c \in Q \subseteq P$ and so $c \in P$, which is a contradiction. Now, let $a \wedge b \in rad(Q)$, for $a, b \in X$. Then there exists $c \in X \setminus P$ such that $(c \wedge a) \wedge b = c \wedge (a \wedge b) \in Q$, for every $P \in \mathcal{PI}_Q(X)$. If $a \notin rad(Q)$, then by Theorem 4.4, there is $P \in \mathcal{PI}_Q(X)$ such that $c \wedge a \notin Q$, for every $c \in X \setminus P$. Since Q is a primary ideal of X and $(c \wedge a) \wedge b \in Q$, there is $c' \in X \setminus P$ such that $c' \wedge b \in Q$, for every $P \in \mathcal{PI}_Q(X)$ and so $b \in rad(Q)$. Therefore, $rad(Q)$ is a prime ideal of X. $\square$

Definition 4.11 *Let X be a lower BCK-semilattice and $Q, P \in \mathcal{I}(X)$. Then Q is called a P-primary ideal of X if Q is a primary ideal of X and $rad(Q) = P$.*

Example 4.4 In Example 4.2, I is a P-primary ideal of X, where $P = \{0, 1, 2, 3\}$.

Definition 4.12 *Let X be a lower BCK-semilattice, $I \in \mathcal{I}(X)$ and there exist primary ideals $Q_1, Q_2, \cdots, Q_n$ of M such that $I = Q_1 \cap Q_2 \cdots \cap Q_n$. Then we say $Q_1 \cap Q_2 \cdots \cap Q_n$ is a primary decomposition of I and I has a primary decomposition. This decomposition is reduced if:*

(i) $Q_j \not\supseteq \bigcap_{i \neq j} Q_i$, *for every* $1 \leq i, j \leq n$,

(ii) $rad(Q_i) \neq rad(Q_j)$, *for every* $1 \leq i, j \leq n$.

Example 4.5 (*i*) In Example 4.4, $I = I \cap P$ is a primary decomposition of I.

(*ii*) In Example 4.1 (*i*), $I = P_1 \cap P_2$ is a reduced primary decomposition of I.

Lemma 4.13 *Let X be bounded implicative and $Q_1, Q_2, \cdots, Q_n$ be P'-primary ideals of X such that $P \cap Q_i \neq \{0\}$ implies that $Q_i \subseteq P$, for every $P \in \mathcal{PI}(X)$, where $P' \in \mathcal{PI}(X)$. Then $\bigcap_{i=1}^n Q_i$ is P'-primary.*

Proof. It is clear that $\bigcap_{i=1}^n Q_i \neq X$. By Theorem 4.6 (*v*), $rad(\bigcap_{i=1}^n Q_i) = \bigcap_{i=1}^n rad(Q_i) = \bigcap_{i=1}^n P' = P'$. Now, let $a \wedge b \in \bigcap_{i=1}^n Q_i$ and $a \notin \bigcap_{i=1}^n Q_i$, for any $a, b \in X$. Then $a \wedge b \in Q_i$, for every $1 \leq i \leq n$ and there exists $1 \leq j \leq n$ such that $a \notin Q_j$. Since Q_j is a primary ideal of X, by Proposition 4.8, $b \in rad(Q_j)$ or $a \in rad(Q_j)$. Let $b \in rad(Q_j) = P'$. Then $b \in rad(Q_i)$, for every $1 \leq i \leq n$ and so by Proposition 4.5, there exists $c_i \in X \setminus P$ such that $c_i \wedge b \in Q_i$, for every $P \in \mathcal{PI}(X)$ with $P \cap Q_i \neq \{0\}$. Let $c = c_1 \wedge c_2 \wedge \cdots \wedge c_n$. If there exists $P \in \mathcal{PI}(X)$ with $P \cap \bigcap_{i=1}^n Q_i \neq \{0\}$ and $c \in P$, then there exists $1 \leq i \leq n$ such that $c_i \in P$. On the other hand, $P \cap \bigcap_{i=1}^n Q_i \neq \{0\}$ implies that $P \cap Q_i \neq \{0\}$. Hence, $c_i \notin P$, for every $1 \leq i \leq n$, which is a contradiction. It results that $c \in X \setminus P$, for every $P \in \mathcal{PI}(X)$ with $P \cap \bigcap_{i=1}^n Q_i \neq \{0\}$. It is easy to show that $c \wedge b \in Q_i$, for every $1 \leq i \leq n$ and so $c \wedge b \in \bigcap_{i=1}^n Q_i$. Similarly, if $a \in rad(Q_j)$, then there is a similar c such that $a \wedge c \in \bigcap_{i=1}^n Q_i$. Hence, by Proposition 4.5, $b \in rad(\bigcap_{i=1}^n Q_i)$ or $a \in rad(\bigcap_{i=1}^n Q_i)$ and so by Proposition 4.8, $\bigcap_{i=1}^n Q_i$ is a primary ideal of X. Therefore, $\bigcap_{i=1}^n Q_i$ is P'-primary. □

Theorem 4.14 *Let X be bounded implicative, $I = Q_1 \cap \cdots \cap Q_n$ be a primary decomposition of I and $P \cap Q_i \neq \{0\}$ implies that $Q_i \subseteq P$, for every $P \in \mathcal{PI}(X)$ and $1 \leq i \leq n$. Then I has a reduced primary decomposition.*

Proof. Let $I = Q_1 \cap \cdots \cap Q_n$, where Q_i is a primary ideal of X, for every $1 \leq i \leq n$. If $Q_j \supseteq \bigcap_{i=1}^n Q_i$, where $i \neq j$, then we set $I = Q_1 \cap \cdots \cap Q_{j-1} \cap Q_{j+1} \cap \cdots \cap Q_n$, for every $1 \leq j \leq n$ and so by renumbering, $I = \bigcap_{i=1}^k Q'_i$, where $k \leq n$ and $Q'_j \not\supseteq \bigcap_{i=1}^k Q'_i$, for every $1 \leq j \leq k$. Let $T = \{P_1, \cdots, P_m\}$, where $P_i \neq P_j$ and $m \leq k$, for every $1 \leq i, j \leq m$ and $rad(Q'_i) = P_i$, for some $1 \leq i \leq k$. Now, we resume $I = (Q'_{i_1} \cap \cdots \cap Q'_{i_t}) \cap (Q'_{j_1} \cap \cdots \cap Q'_{j_l}) \cap \cdots \cap (Q'_{s_1} \cap \cdots \cap Q'_{s_w})$, where by Lemma 4.13,

$$rad(\bigcap_{h=1}^{t} Q'_{i_h}) = \bigcap_{h=1}^{t} rad(Q'_{i_h})$$
$$= \bigcap_{h=1}^{t} p_1 = p_1, \cdots, rad(\bigcap_{h=1}^{w} Q'_{s_h}) = \bigcap_{h=1}^{w} rad(Q'_{s_h}) = \bigcap_{h=1}^{w} p_m = p_m.$$

Therefore, I has a reduced primary decomposition. □

5 Primary decomposition of submodules

In this section, we define the notions of primary and P-primary submodules of an X^E-module. As a fundamental result, we introduce an X^E-module that all its proper submodules have reduced primary decomposition.

Proposition 5.1 *Let M be an X^E-module and N be a submodule of M. Then $Q_N = \{x \in X : x.M \subseteq N\}$ is an ideal of X.*

Proof. Let $x * y, y \in Q_N$, for $x, y \in X$. Then $(x * y).m,\ y.m \in N$, for every $m \in M$. If $x * y = 0$, then $x.m = (x * (x * y)).m = (y \wedge x).m = (x \wedge y).m = x.(y.m) \in N$. If $x * y \neq 0$, then by $(XM4)$, $(x * y).m = x.m - y.m \in N$. Since $y.m \in N$, $x.m \in N$, for every $m \in M$. Hence $x.M \subseteq N$ and so Q_N is an ideal of X. □

Definition 5.2 *Let X be a lower BCK-semilattice, M be an X^E-module and N be a proper submodule of M. Then N is called a primary submodule of M, if for any $x \in X$ and $m \in M$, $x.m \in N$ implies that $m \in N$ or $\exists c \in X \setminus P$ such that $(c \wedge x).M \subseteq N$, for every $P \in \mathcal{PI}_{Q_N}(X)$.*

Example 5.1 (i) Let $X = \{0, 1, 2, 3, 4\}$ and operation $*$ be defined by

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	3	2	0

Then $(X, *, 0)$ is a bounded lower BCK-semilattice. Let $Y = \{0, 1, 4\}$ and $M = \{0, 2, 3, 4\}$. It is clear that Y is a subalgebra of X and so it is a BCK-algebra. It is easy to show that $(M, +)$ is an Abelian group, where $x + y = (x * y) \vee (y * x)$, for every $x, y \in M$. Now, we define operation $. : Y \times M \to M$ by $y.m = y \wedge m$, for every $y \in Y$ and $m \in M$. Then M is a Y^E-module. Let $N = \{0, 2\}$. It is easy to show that N is a submodule of M. Note that $1.3 = 1 \wedge 3 = 0 \in N$ and $3 \notin N$, but $(4 \wedge 1).M \subseteq N$, where $4 \in X \setminus P$ and $Q_N = \{0, 1\}$, for every $P \in \mathcal{PI}_{Q_N}(X)$. Similarly, we can show that $y \wedge m \in N$ implies that $m \in N$ or $(c \wedge y).M \subseteq N$, where $c \in Y \setminus P$, for every $P \in \mathcal{PI}_{Q_N}(X)$. Hence, N is a primary submodule of M.

(ii) Let $D = \{0, \frac{1}{2}, 1\}$, $X_1 = \{a, b\}$ and $0, f, I$ be functions from X_1 to D such that $0(x) = 0$, $f(x) = \frac{1}{2}$ and $I(x) = 1$, for every $x \in X_1$. We define operation $*$ on $X = \{0, f, I\}$ by $(g * h)(x) = g(x) - min\{g(x), h(x)\}$, for every $g, h \in X$. Then it is easy to show that $(X, *, 0)$ is a bounded lower BCK-semilattice. Consider the Abelian group $A = \{\frac{m}{2^n} : m \in \mathbb{Z}, n \in \mathbb{N} \cup \{0\}\}$ with ordinary addition operation. Let operation $. : X \times A \longrightarrow A$ be defined by $g.\frac{m}{2^n} = \frac{g(x)m}{2^n}$, for every $g \in X$ and $\frac{m}{2^n} \in A$. Then A is an X^E-module. Let $B = \frac{1}{2}A$. It is clear that B is a submodule of A. Let $g.\frac{m}{2^n} \in B$. If $g = f$, then by considering $c = I$, we have $(c \wedge g).A \subseteq B$, where $Q_B = \{0, f\}$. If $g = I$ or $g = 0$, then the proof is clear. Therefore, B is a primary submodule of A.

Proposition 5.3 *Let X be a bounded lower BCK-semilattice, M be an X^E-module and N be a prime submodule of M. Then N is a primary submodule of M.*

Proof. Let $x.m \in N$ and $m \notin N$, for $x \in X$ and $m \in M$. Then we consider $c = 1 \in X \setminus P$ and so $(c \wedge x).M = x.M \subseteq N$, for every $P \in \mathcal{PI}_{Q_N}(X)$. $\square$

Theorem 5.4 *Let X be a bounded lower BCK-semilattice, M be an X^E-module and N be a primary submodule of M. Then Q_N is a primary ideal of X.*

Proof. If $Q_N = X$, then $1 \in Q_N$ and so $M = N$, which is a contradiction. Let $a \wedge b \in Q_N$ and $a \notin Q_N$, for $a, b \in X$. Then by $(XM1)$, $b.(a.m) = (b \wedge a).m = (a \wedge b).m \in N$, for every $m \in M$. Since $a \notin Q_N$, there exists $m' \in M$ such that $a.m' \notin N$. Moreover, since $b.(a.m') \in N$ and $a.m' \notin N$, there exists $c \in X \setminus P$ such that $(c \wedge b).M \subseteq N$, for every $P \in \mathcal{PI}_{Q_N}(X)$. It results that $c \wedge b \in Q_N$. Therefore, Q_N is a primary ideal of X. $\square$

Note In Theorem 5.4, if X is bounded implicative such that $I \cap P \neq \{0\}$ implies that $I \subseteq P$, then by Theorem 4.10, $rad(Q_N)$ is a prime ideal of X and $rad(Q_N) = \{x \in X : \forall P \in \mathcal{PI}_{Q_N}(X), \exists c \in X \setminus P$ such that $(c \wedge x).M \subseteq N\}$.

Definition 5.5 *Let X be a lower BCK-semilattice, M be an X^E-module and N be a proper submodule of M. Then N is called a P-primary submodule of M, if N is a primary submodule of M and $rad(Q_N) = P$.*

Example 5.2 In Example 5.1 (i), $N = \{0, 2\}$ is a P-primary submodule of M, where $P = rad(Q_N) = rad(\{0, 1\}) = \{0, 1, 2\} \cap \{0, 1, 3\} = \{0, 1\}$.

Proposition 5.6 *Let X be bounded implicative, M be an X^E-module, $N_1, \cdots, N_k$ be P'-primary submodules of M such that $Q_{\bigcap_{i=1}^k N_i} \neq 0$. If $P \cap I \neq \{0\}$ implies that $I \subseteq P$, for every ideal $I \in \mathcal{I}(X)$ and $P \in \mathcal{PI}(X)$, then $\bigcap_{i=1}^k N_i$ is a P'-primary submodule of M.*

Proof. It is clear that $\bigcap_{i=1}^k N_i \neq M$. Let $x.m \in \bigcap_{i=1}^k N_i$ and $m \notin \bigcap_{i=1}^k N_i$, for $x \in X$ and $m \in M$. Then $x.m \in N_i$, for every $1 \leq i \leq k$ and there exists $1 \leq j \leq k$ such that $m \notin N_j$. Since $x.m \in N_j$ and $m \notin N_j$, there exists $c_j \in X \setminus P$ such that $(c_j \wedge x).M \subseteq N_j$, for every $P \in \mathcal{PI}_{Q_{N_j}}(X)$. It results that $x \in rad(Q_{N_j}) = P' = rad(Q_{N_i})$, for every $1 \leq i \leq k$. Hence, there exists $c_i \in X \setminus P$ such that $(c_i \wedge x).M \subseteq N$, for every $P \in \mathcal{PI}_{Q_{N_i}}(X)$. Now, we show that $rad(Q_{\bigcap_{i=1}^k N_i}) = P'$. For every $1 \leq i \leq k$,

$$x \in Q_{\bigcap_{i=1}^k N_i} \Leftrightarrow x.M \subseteq \bigcap_{i=1}^{k} N_i \Leftrightarrow x.M \subseteq N_i \Leftrightarrow x \in Q_{N_i} \Leftrightarrow x \in \bigcap_{i=1}^{k} Q_{N_i}.$$

Then $Q_{\bigcap_{i=1}^k N_i} = \bigcap_{i=1}^k Q_{N_i}$ and so by Theorem 4.6 (v),

$$rad(Q_{\bigcap_{i=1}^k N_i}) = rad(\bigcap_{i=1}^{k} Q_{N_i}) = \bigcap_{i=1}^{k} rad(Q_{N_i}) = \bigcap_{i=1}^{k} P' = P'.$$

Let $c = c_1 \wedge c_2 \cdots \wedge c_k$ and there exists $P \in \mathcal{PI}_{Q_{\bigcap_{i=1}^k N_i}}(X)$ such that $c \in P$. Then there is $1 \leq i \leq k$ such that $c_i \in P$. Since $\{0\} \neq Q_{\bigcap_{i=1}^k N_i} \subseteq Q_{N_i}$, $Q_{N_i} \cap P \neq \{0\}$ and so $Q_{N_i} \subseteq P$, for every $1 \leq i \leq k$. It results that $c_i \notin P$, for every $1 \leq i \leq k$, which is a contradiction. Hence, $c \in X \setminus P$, for every $P \in \mathcal{PI}_{Q_{N_{\bigcap_{i=1}^{k} N_i}}}(X)$. On the other hand, since $(c_i \wedge x).m \in N_i$, by $(XM1)$, $(c \wedge x).m = (c_1 \wedge \cdots \wedge c_{i-1} \wedge c_{i+1} \cdots \wedge c_k \wedge c_i \wedge x).m = (c_1 \wedge \cdots \wedge c_{i-1} \wedge c_{i+1} \cdots \wedge c_k).((c_i \wedge x).m) \in N_i$ and so $(c \wedge x).m \in \bigcap_{i=1}^{k} N_i$, for every $m \in M$. Therefore, $\bigcap_{i=1}^{k} N_i$ is a P'-primary submodule of M. □

Definition 5.7 *Let X be a lower BCK-semilattice, M be an X^E-module, N be a proper submodule of M and there exist proper submodules $A_1, A_2, \cdots, A_n$ of M such that A_i is a P_i-primary of M, for every $1 \leq i \leq n$ and $N = A_1 \cap A_2 \cap \cdots \cap A_n$. Then we say $A_1 \cap A_2 \cap \cdots \cap A_n$ is a primary decomposition of N and so N has a primary decomposition. Furthermore, this decomposition is reduced if*

(i) $A_i \not\supseteq \bigcap_{i \neq j} A_j$,
(ii) $rad(Q_{A_i}) \neq rad(Q_{A_j})$, *for every* $1 \leq i, j \leq n$.

Example 5.3 (i) In Example 5.1 (i), it is easy to show that $A_1 = \{0, 2\}$ and $A_2 = \{0, 1, 2\}$ are P-primary submodules of M, where $P = \{0, 1\}$. Then $A_1 \cap A_2$ is a primary decomposition of A_1.

(ii) Let $X = \{0, 1, 2\}$ and operation $*$ be defined by

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	2	0

Then $(X, *, 0)$ is a BCK-algebra. Now, let operation $. : X \times \mathbb{Z} \longrightarrow \mathbb{Z}$ be defined by $2.n = n$ and $1.n = 0.n = 0$, for every $n \in \mathbb{Z}$. Then $\mathbb{Z}$ is an X^E-module. It is easy to show that $2\mathbb{Z}$ and $3\mathbb{Z}$ are prime submodules of $\mathbb{Z}$ and so are primary submodules of $\mathbb{Z}$. We have $Q_{2\mathbb{Z}} = Q_{3\mathbb{Z}} = \{0, 1\}$ and so $rad(Q_{2\mathbb{Z}}) = rad(Q_{3\mathbb{Z}}) = \{0, 1\} = P$. It results that $2\mathbb{Z}$ and $3\mathbb{Z}$ are P-primary submodules of M. Since $6\mathbb{Z} = 2\mathbb{Z} \cap 3\mathbb{Z}$, $2\mathbb{Z} \cap 3\mathbb{Z}$ is a primary decomposition of $6\mathbb{Z}$.

(iii) Consider BCK-algebra X in (ii) and $M = \mathbb{Z}_6$. If operation $. : X \times M \longrightarrow M$ is defined by $2.n = n$ and $1.n = 0.n = 0$, for every $n \in M$, then M is an X^E-module. It is easy to show that $\prec \bar{2} \succ = \{\bar{0}, \bar{2}, \bar{4}\}$ and $\prec \bar{3} \succ = \{\bar{0}, \bar{3}\}$ are primary submodules of M. We have $Q_{\prec \bar{2} \succ} = Q_{\prec \bar{3} \succ} = \{0, 1\}$ and so $rad(Q_{\prec \bar{2} \succ}) = rad(Q_{\prec \bar{3} \succ}) = \{0, 1\} = P$. It results that $\prec \bar{2} \succ$ and $\prec \bar{3} \succ$ are P-primary submodules of M. Since $\{\bar{0}\} = \prec \bar{2} \succ \cap \prec \bar{3} \succ$, $\prec \bar{2} \succ \cap \prec \bar{3} \succ$ is a reduced primary decomposition of $\{\bar{0}\}$.

Theorem 5.8 *Let X be bounded implicative, M be an X^E-module, N be a submodule of M that has a primary decomposition and $I \cap P \neq \{0\}$ implies that $I \subseteq P$, for every $I \in \mathcal{I}(X)$ and $P \in \mathcal{PI}(X)$. Then N has a reduced primary decomposition.*

Proof. By Proposition 5.6, the proof is routine. □

Definition 5.9 *Let M be an X^E-module. Then:*

(i) M is called Noetherian if M satisfies the ascending chain condition (ACC): any chain $N_1 \subseteq N_2 \subseteq \cdots$ of submodules of M is stationary.
(ii) We say M satisfies the maximum condition, if every non-empty family of submodules of M has a maximum element.

Theorem 5.10 *Let M be an X^E-module. Then M is Noetherian if and only if M has maximum condition.*

Proof. The proof is routine. □

Theorem 5.11 *Let X be a lower BCK-semilattice and M be a Noetherian X^E-module. Then every proper submodule of M has a reduced primary decomposition.*

Proof. Let $T = \{N : N$ is a proper submodule of M such that N has no any reduced primary decomposition$\}$. We show that $T = \emptyset$. Let $T \neq \emptyset$. Since M is Noetherian, by Theorem 5.10, T has a maximum element G. It is clear that G is not primary submodule of M. So there exists $x \in X$ and $m \in M$ such that $x.m \in G$, $m \notin G$ and for every $c \in X \setminus P$, $(c \wedge x).M \nsubseteq G$, where $P \in \mathcal{PI}_{Q_G}(X)$. We give an index $i \geq 1$ to every $c \in X \setminus P$. Let $B_i = \{m \in M : (c_1 \wedge c_2 \cdots \wedge c_i \wedge x).m \in G\}$, for every $i \geq 1$ and $m \in B_i$. Then $(c_1 \wedge c_2 \wedge \cdots \wedge c_i \wedge c_{i+1} \wedge x).m = (c_{i+1} \wedge c_1 \wedge \cdots \wedge c_i \wedge x).m = c_{i+1}.((c_1 \wedge \cdots \wedge c_i \wedge x).m) \in G$ and so $B_i \subseteq B_{i+1}$, for every $i \geq 1$. Since M is Noetherian, there exists $k \in \mathbb{N}$ such that $B_k = B_n$, for every $n \geq k$. We show that B_k is a submodule of M. Let $m_1, m_2 \in B_k$. Then $(c_1 \wedge \cdots \wedge c_k \wedge x).m_1, (c_1 \wedge \cdots \wedge c_k \wedge x).m_2 \in G$. We have $(c_1 \wedge \cdots \wedge c_k \wedge x).(m_1 - m_2) = (c_1 \wedge \cdots \wedge c_k \wedge x).m_1 - (c_1 \wedge \cdots \wedge c_k \wedge x).m_2 \in G$. So $m_1 - m_2 \in B_K$. On the other hand, $(c_1 \wedge \cdots \wedge c_k \wedge x).(a.m) = (c_1 \wedge \cdots \wedge c_k \wedge x \wedge a).m = a.((c_1 \wedge \cdots \wedge c_k \wedge x).m) \in G$ and so $a.m \in B_k$, for every $a \in X$ and $m \in B_k$. Hence, B_k is a submodule of M.

Let $D = \{(c_1 \wedge \cdots \wedge c_k \wedge x).m' + g : m' \in M \text{ and } g \in G\}$. We show that D is a submodule of M. Let $d_1, d_2 \in D$. It is easy to show that $d_1 - d_2 \in D$. Let $d \in D$ and $a \in X$. So there exist $m' \in M$ and $g \in G$ such that

$$\begin{aligned} a.d &= a.((c_1 \wedge \cdots \wedge c_k \wedge x).m' + g) = a.((c_1 \wedge \cdots \wedge c_k \wedge x).m') + a.g \\ &= (a \wedge c_1 \wedge \cdots \wedge c_k \wedge x).m' + a.g = (c_1 \wedge \cdots \wedge c_k \wedge x).(a.m') + a.g \in D. \end{aligned}$$

Hence, D is a submodule of M. Now, we prove that $G = D \cap B_k$, $G \subsetneq D$ and $G \subsetneq B_k$. Let $g \in G$. Then $g = (c_1 \wedge \cdots \wedge c_k \wedge x).0 + g \in D$. On the other hand, $(c_1 \wedge \cdots \wedge c_k \wedge x).g \in G$. So $g \in B_k$ and so $G \subseteq D \cap B_k$. Let $m \in D \cap B_k$. Since $m \in B_k$, $(c_1 \wedge \cdots \wedge c_k \wedge x).m \in G$ and since $m \in D$, there exist $m' \in M$ and $g \in G$ such that $m = (c_1 \wedge \cdots \wedge c_k \wedge x).m' + g$. By $(XM1)$ and $(XM2)$,

$$\begin{aligned} &(c_1 \wedge \cdots \wedge c_k \wedge x).m' + (c_1 \wedge \cdots \wedge c_k \wedge x).g \\ &= (c_1 \wedge \cdots \wedge c_k \wedge x).((c_1 \wedge \cdots \wedge c_k \wedge x).m') + (c_1 \wedge \cdots \wedge c_k \wedge x).g \\ &= (c_1 \wedge \cdots \wedge c_k \wedge x).((c_1 \wedge \cdots \wedge c_k \wedge x).m' + g) \\ &= (c_1 \wedge \cdots \wedge c_k \wedge x).m \in G. \end{aligned}$$

Since $(c_1 \wedge \cdots \wedge c_k \wedge x).g \in G$, $(c_1 \wedge \cdots \wedge c_k \wedge x).m' \in G$ and so $m \in G$. Hence, $D \cap B_k \subseteq G$. It is enough to show that $G \subsetneq D$ and $G \subsetneq B_k$. We have $(c \wedge x).M \nsubseteq G$, for every $c \in X \setminus P$, where $P \in \mathcal{PI}_{Q_G}(X)$. Then there exists $t \in M$ such that $(c \wedge x).t \notin G$. But if $c = c_1 \wedge \cdots \wedge c_k$, then $(c \wedge x).t = (c \wedge x).t + 0 \in D$ and so $G \subsetneq D$. On the other hand, there existed $m \in M$ and $x \in X$ such that $x.m \in G$ and $m \notin G$, but $(c_1 \wedge \cdots \wedge c_k \wedge x).m = (c_1 \wedge \cdots \wedge c_k).(x.m) \in G$. It means that $m \in B_k$ and so $G \subsetneq B_k$. By the maximality G, D and B_k have primary decomposition. It results that G has primary decomposition, which is a contradiction. Therefore, $T = \emptyset$. □

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