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Biharmonic hypersurfaces in E_s^5

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Abstract In this paper, we prove that every biharmonic non-degenerate hypersurface with three distinct principal curvatures in 5-dimensional semi-Euclidean space with diagonal shape operator must be minimal.

Keywords Biharmonic submanifolds · Mean curvature vector · Chen's conjecture

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1 Introduction

In 1964, Eells and Sampson [9] introduced the notion of poly-harmonic maps as a natural generalization of the well-known harmonic maps. Thus, while harmonic maps between Riemannian manifolds $\phi : (M, g) \to (N, h)$ are critical points of the energy functional

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2 v_g,$$

the biharmonic maps are critical points of the bienergy functional

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 v_g,$$

where $\tau = \operatorname{trace} \nabla d\phi$ is the tension field of ϕ.

The study of biharmonic submanifolds in Euclidean spaces was initiated by B. Y. Chen in the middle of 1980's. In particular, he proved that biharmonic surfaces in Euclidean 3-spaces are minimal. By definition, any minimal submanifold is biharmonic.

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Therefore, to find non-minimal (or proper) biharmonic submanifolds is an interesting problem. There are many non-existence results in Euclidean spaces developed by Dimitri'c in his doctoral thesis [7, 8]. Based on these results, Chen [2] in 1991 posed the following well-known conjecture: The only biharmonic submanifolds of Euclidean spaces are the minimal ones. Although the conjecture was later proved by Hasanis and Vlachos [11] for hypersurfaces in Euclidean 4-spaces, it is still open in general so far. It was proved that the Chen's conjecture is true for $\delta(2)$-ideal and $\delta(3)$-ideal hypersurfaces of a Euclidean space of arbitrary dimension [6] and that every biharmonic hypersurface with three distinct principal curvatures in Euclidean space is minimal [10, 12].

In contrast to the submanifolds of Euclidean spaces, Chen's conjecture is not always true for the submanifolds of the pseudo-Euclidean spaces. For example, Chen and Ishikawa [4, 5] obtained some examples of proper biharmonic surfaces in 4-dimensional pseudo-Euclidean spaces for $s = 1, 2, 3$, (see also [3]). But for hypersurfaces in pseudo-Euclidean spaces, it is reasonable that Chen's conjecture is also right. This is supported by the following facts: Chen and Ishikawa proved in [4, 5] that biharmonic surfaces in pseudo-Euclidean 3-spaces are minimal, and Arvanitoyeorgosa et al. [1] proved that biharmonic Lorentzian hypersurfaces in Minkowski 4-spaces are minimal.

In this paper, we prove that Chen's conjecture is true for biharmonic hypersurfaces with three distinct principal curvatures in semi-Euclidean spaces E^5_s.

2 Preliminaries

Let (M^4_r, g), $r = 0, 1, 2, 3, 4$, be a 4-dimensional hypersurface isometrically immersed in a 5-dimensional semi-Euclidean space $(E^5_s, \overline{g})$, $s = 0, 1, 2, 3, 4, 5$ and $g = \overline{g}_{|M^4_r}$. We denote by the ξ unit normal vector to M^4_r with $\overline{g}(\xi, \xi) = \varepsilon$, where $\varepsilon = \pm 1$, according as M^4_r is pseudo-Riemannian or Riemannian hypersurface.

Let $\overline{\nabla}$ and ∇ denote linear connections on E^5_s and M^4_r, respectively. Then, the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \ \ \forall \ \ X, Y \in \Gamma(TM^4_r), \tag{2.1}$$

$$\overline{\nabla}_X \xi = -A_\xi X, \tag{2.2}$$

where h is the second fundamental form and A is the shape operator. It is well known that the second fundamental form h and shape operator A are related by

$$\overline{g}(h(X, Y), \xi) = g(A_\xi X, Y). \tag{2.3}$$

The mean curvature vector is given by

$$\varepsilon H = \frac{1}{4} \operatorname{trace} A. \tag{2.4}$$

The Gauss and Codazzi equations are given by

$$R(X, Y)Z = g(AY, Z)AX - g(AX, Z)AY, \tag{2.5}$$

$$(\nabla_X A)Y = (\nabla_Y A)X, \tag{2.6}$$

respectively, where R is the curvature tensor and

$$(\nabla_X A)Y = \nabla_X AY - A(\nabla_X Y), \tag{2.7}$$

for all $X, Y, Z \in \Gamma(TM_r^4)$.

A biharmonic submanifold in a semi-Euclidean space is called proper biharmonic if it is not minimal. The necessary and sufficient conditions for M_r^4 to be biharmonic in E_s^5 is

$$\triangle H + \varepsilon H \ \text{trace}\, A^2 = 0, \tag{2.8}$$

$$A \,\text{grad}\ H + 2\varepsilon H \ \text{grad}\ H = 0, \tag{2.9}$$

where H denotes the mean curvature. Also the Laplace operator $\triangle$ of a scalar valued function f is given by [3]

$$\triangle f = -\sum_{i=1}^{4} \epsilon_i (e_i e_i f - \nabla_{e_i} e_i f), \tag{2.10}$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal local tangent frame on M_r^4 and $g(e_i, e_i) = \epsilon_i$.

A vector X in E_s^5 is called spacelike, timelike or lightlike according as $\overline{g}(X, X) > 0$, $\overline{g}(X, X) < 0$ or $\overline{g}(X, X) = 0$, respectively. A non-degenerate hypersurface M_r^4 of E_s^5 is called Riemannian or pseudo-Riemannian according as the induced metric on M_r^4 from the indefinite metric on E_s^5 is definite or indefinite. The shape operator of pseudo-Riemannian hypersurfaces is not always diagonalizable unlike the Riemannian hypersurfaces.

3 Biharmonic non-degenerate hypersurfaces with three distinct principal curvatures

In this section we study biharmonic non-degenerate hypersurfaces Riemannian or pseudo-Riemannian M_r^4 with shape operator diagonal. We also assume that mean curvature is not constant. From (2.9), it is easy to see that grad H is an eigenvector of the shape operator A with the corresponding principal curvature $-2\varepsilon H$. The grad H can be spacelike or timelike. Without losing generality, we choose e_1 in the direction of grad H and therefore shape operator A of hypersurfaces will take the following form with respect to a suitable frame $\{e_1, e_2, e_3, e_4\}$

$$A_H = \begin{pmatrix} -2\varepsilon H & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{pmatrix}. \tag{3.1}$$

Since M_r^4 has three distinct principal curvatures, we can assume that $\lambda_2 = \lambda_3 = \lambda \neq \lambda_4$.

The grad H can be expressed as

$$\text{grad}\, H = \sum_{i=1}^{4} e_i(H) e_i. \tag{3.2}$$

As we have taken e_1 parallel to grad H, consequently

$$e_1(H) \neq 0, e_2(H) = 0, e_3(H) = 0, e_4(H) = 0. \tag{3.3}$$

We express

$$\nabla_{e_i} e_j = \sum_{k=1}^{4} \epsilon_k \omega_{ij}^k e_k, \qquad i, j = 1, 2, 3, 4. \tag{3.4}$$

From Codazzi equation (2.6), (3.1), (3.4) and the compatibility conditions $\nabla_{e_k} g(e_i, e_i) = 0$ and $\nabla_{e_k} g(e_i, e_j) = 0$, we obtain

$$\omega_{ki}^i = 0, \qquad \omega_{ki}^j + \omega_{kj}^i = 0, \tag{3.5}$$

for $i \neq j$, and $i, j, k = 1, 2, 3, 4$,

$$\epsilon_j e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j, \tag{3.6}$$

$$(\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j, \tag{3.7}$$

for distinct $i, j, k = 1, 2, 3, 4$.

Since $\lambda_1 = -2\varepsilon H$, from (3.3), we get

$$e_1(\lambda_1) \neq 0, e_2(\lambda_1) = 0, e_3(\lambda_1) = 0, e_4(\lambda_1) = 0. \tag{3.8}$$

Also, it is easy to show that

$$[e_i, e_j](\lambda_1) = 0, \qquad i, j = 2, 3, 4,$$

which gives

$$\omega_{ij}^1 = \omega_{ji}^1, \tag{3.9}$$

for $i \neq j$ and $i, j = 2, 3, 4$.

Now, we show that $\lambda_j \neq \lambda_1, j = 2, 3, 4$. In fact, if $\lambda_j = \lambda_1$ for $j \neq 1$, from (3.6), we find

$$\epsilon_j e_1(\lambda_j) = (\lambda_1 - \lambda_j)\omega_{j1}^j = 0, \tag{3.10}$$

which contradicts the first expression of (3.8).

Since $\lambda_2 = \lambda_3 = \lambda \neq \lambda_4$, from (2.4), we obtain that

$$\lambda_4 = 6\varepsilon H - 2\lambda, \qquad \lambda \neq -2\varepsilon H, 2\varepsilon H, 4\varepsilon H. \tag{3.11}$$

Putting $i, j = 2, 3$ and $i \neq j$ in (3.6), we get

$$e_j(\lambda) = 0. \tag{3.12}$$

Putting $i \neq 1, j = 1$ in (3.6) and using (3.8) and (3.5), we find

$$\omega_{1i}^1 = 0, \qquad i = 1, 2, 3, 4. \tag{3.13}$$

Putting $i = 2, 3, j = 4$ in (3.6) and using (3.12), we obtain

$$\omega_{4i}^4 = 0, \qquad i = 2, 3. \tag{3.14}$$

Putting $i=1, j=2,3,4$, in (3.6), we have

$$\omega_{21}^2=-\frac{\epsilon_2 e_1(\lambda)}{2\varepsilon H+\lambda},\qquad \omega_{31}^3=-\frac{\epsilon_3 e_1(\lambda)}{2\varepsilon H+\lambda},\qquad \omega_{41}^4=-\frac{\epsilon_4 e_1(3\varepsilon H-\lambda)}{4\varepsilon H-\lambda}. \tag{3.15}$$

Putting $i=4, j=2,3$, in (3.6), we find

$$\omega_{24}^2=\frac{\epsilon_2 e_4(\lambda)}{6\varepsilon H-3\lambda},\qquad \omega_{34}^3=\frac{\epsilon_3 e_4(\lambda)}{6\varepsilon H-3\lambda}. \tag{3.16}$$

Putting $i=1, j\neq k$, and $j,k=2,3$ in (3.7), we obtain

$$\omega_{k1}^j=0,\qquad j\neq k,\ \text{and } j,k=2,3. \tag{3.17}$$

Putting $i=4, j\neq k$, and $j,k=2,3$ in (3.7), we have

$$\omega_{k4}^j=0,\qquad j\neq k,\ \text{and } j,k=2,3. \tag{3.18}$$

Putting $i=4, j=1$, and $k=2,3$ in (3.7), and using (3.9) we get

$$\omega_{k4}^1=\omega_{4k}^1=0,\qquad k=2,3. \tag{3.19}$$

Putting $i=1, j=4$ and $k=2,3$ in (3.7), and using (3.9), we find

$$\omega_{1k}^4=\omega_{k1}^4=0,\qquad k=2,3. \tag{3.20}$$

Now, we have the following:

Lemma 3.1 *Let $M_r^4, r=0,1,2,3,4$, be an 4-dimensional biharmonic non-degenerate hypersurface with non-constant mean curvature in semi-Euclidean space E_s^5, $s=0$, 1, 2, 3, 4, 5, having the shape operator given by* (3.1) *with respect to suitable orthonormal frame $\{e_1,e_2,e_3,e_4\}$. Then, we obtain*

$$\nabla_{e_1}e_1=0,\ \ \nabla_{e_i}e_1=-\alpha e_i, i=2,3,\ \ \nabla_{e_4}e_1=-\beta e_4, \tag{3.21}$$

$$\nabla_{e_i}e_i=\alpha e_1\epsilon_i\epsilon_1+\sum_{i\neq j,j=2}^{3}\omega_{ii}^j e_j\epsilon_j-\frac{e_4(\lambda)}{6\varepsilon H-3\lambda}e_4\epsilon_i\epsilon_4,\ \ i=2,3, \tag{3.22}$$

$$\nabla_{e_i}e_j=\omega_{ij}^i e_i\epsilon_i,\ \ i,j=2,3, and\ i\neq j, \tag{3.23}$$

$$\nabla_{e_1}e_4=0,\ \nabla_{e_4}e_4=\beta e_1\epsilon_4\epsilon_1,\ \nabla_{e_i}e_4=\frac{e_4(\lambda)}{6\varepsilon H-3\lambda}e_i,\ \ i=2,3, \tag{3.24}$$

where ω_{ij}^k satisfies (3.5) *for $i,j,k=1,2,3,4$ and $\alpha=\frac{e_1(\lambda)}{2\varepsilon H+\lambda}, \beta=\frac{e_1(3\varepsilon H-\lambda)}{4\varepsilon H-\lambda}$.*

Using Lemma 3.1, Gauss equation and comparing the coefficients with respect to the orthonormal frame $\{e_1,e_2,e_3,e_4\}$, we find the following:

- $X=e_1, Y=e_2, Z=e_1$,

$$-e_1(\alpha)+\alpha^2=2\varepsilon\epsilon_1 H\lambda. \tag{3.25}$$

- $X=e_1, Y=e_2, Z=e_4$,

$$e_1\Big(\frac{e_4(\lambda)}{6\varepsilon H-3\lambda}\Big)-\alpha\frac{e_4(\lambda)}{6\varepsilon H-3\lambda}=0. \tag{3.26}$$

• $X = e_1, Y = e_4, Z = e_1,$

$$-e_1(\beta) + \beta^2 = 2\varepsilon\epsilon_1 H(6\varepsilon H - 2\lambda). \tag{3.27}$$

• $X = e_3, Y = e_4, Z = e_1,$

$$e_4(\alpha) + \frac{e_4(\lambda)}{6\varepsilon H - 3\lambda}(\alpha - \beta) = 0. \tag{3.28}$$

• $X = e_4, Y = e_2, Z = e_4,$

$$e_4(\frac{e_4(\lambda)}{6\varepsilon H - 3\lambda}) + \alpha\beta\epsilon_1\epsilon_4 + (\frac{e_4(\lambda)}{6\varepsilon H - 3\lambda})^2 = -\lambda\epsilon_4(6\varepsilon H - 2\lambda). \tag{3.29}$$

Using (2.8), (2.10), (3.1) and Lemma 3.1, we find

$$-\epsilon_1 e_1 e_1(H) + (2\alpha + \beta)\epsilon_1 e_1(H) + \varepsilon H(4H^2 + 2\lambda^2 + (6\varepsilon H - 2\lambda)^2) = 0. \tag{3.30}$$

From (3.3) and Lemma 3.1, we obtain

$$e_i e_1(H) = 0, \qquad i = 2, 3, 4. \tag{3.31}$$

Now, we find $e_4(\beta)$ as follows:

Differentiating $\alpha = \frac{e_1(\lambda)}{2\varepsilon H + \lambda}, \beta = \frac{e_1(3\varepsilon H - \lambda)}{4\varepsilon H - \lambda}$ along e_4, we get equations

$$(2\varepsilon H + \lambda)e_4(\alpha) + \alpha e_4(\lambda) = e_4 e_1(\lambda),$$
$$(4\varepsilon H - \lambda)e_4(\beta) = -e_4 e_1(\lambda) + \beta e_4(\lambda),$$

respectively and eliminating $e_4 e_1(\lambda)$ from above equations and using (3.28), we have

$$e_4(\beta) = \frac{4e_4(\lambda)(\alpha - \beta)(\lambda - \varepsilon H)}{(4\varepsilon H - \lambda)(6\varepsilon H - 3\lambda)}.$$

Differentiating (3.30) along e_4, and using (3.31), (3.28) and $e_4(\beta)$, we get

$$e_4(\lambda)[\frac{-2(\alpha - \beta)\epsilon_1 e_1(H)}{4\varepsilon H - \lambda} + 12\varepsilon H(\lambda - 2\varepsilon H)] = 0. \tag{3.32}$$

From (3.32), it follows that if $e_4(\lambda) \neq 0$, then

$$\frac{(\alpha - \beta)\epsilon_1 e_1(H)}{4\varepsilon H - \lambda} - 6\varepsilon H(\lambda - 2\varepsilon H) = 0. \tag{3.33}$$

Now, differentiating (3.33) along e_4, we have

$$(\alpha - \beta)\lambda - \frac{12\varepsilon H(\lambda - 3\varepsilon H)(\lambda^2 - 6\varepsilon H\lambda + 8H^2)}{\epsilon_1 e_1(H)} = 0. \tag{3.34}$$

Eliminating $e_1(H)$ from (3.33) and (3.34), we obtain

$$\lambda - 2\varepsilon H = 0,$$

which is not possible since $\lambda \neq 2\varepsilon H$, consequently, $e_4(\lambda) = 0$. Therefore, (3.29) reduces to

$$\alpha\beta = \epsilon_1 \lambda(2\lambda - 6\varepsilon H). \tag{3.35}$$

Now, eliminating $e_1e_1(H)$ and $e_1e_1(\lambda)$, using (3.35), (3.30), (3.27) and (3.25), we obtain

$$(\alpha + 2\beta)\epsilon_1 e_1(H) = 12H(7H^2 - 7\varepsilon H\lambda + 2\lambda^2). \tag{3.36}$$

Differentiating (3.36) along e_1 and using (3.35), (3.30), (3.27), (3.25) and (3.36), we get

$$\begin{aligned}
&[(210 + 20\varepsilon)H^3 + (12 + 3\varepsilon)H\lambda^2 - (12 + 132\varepsilon)H^2\lambda]\alpha \\
&\quad + [(84 + 40\varepsilon)H^3 + (24 + 6\varepsilon)H\lambda^2 - (24 + 84\varepsilon)H^2\lambda]\beta \\
&\quad = e_1(H)(138H^2 - 96\varepsilon H\lambda + 15\lambda^2).
\end{aligned} \tag{3.37}$$

Also, we have

$$3\varepsilon e_1(H) = \alpha(2\varepsilon H + \lambda) + \beta(4\varepsilon H - \lambda). \tag{3.38}$$

Combining (3.37) and (3.38), we obtain

$$\begin{aligned}
&[(118 + 20\varepsilon)H^3 + (34 + 3\varepsilon)H\lambda^2 - (12 + 114\varepsilon)H^2\lambda - 5\varepsilon\lambda^3]\alpha \\
&\quad + [(-100 + 40\varepsilon)H^3 + (-28 + 6\varepsilon)H\lambda^2 \\
&\quad + (-24 + 90\varepsilon)\lambda + 5\varepsilon\lambda^3]\beta = 0.
\end{aligned} \tag{3.39}$$

For simplicity, we denote by

$$\begin{aligned}
p_1 &= (118 + 20\varepsilon)H^3 + (34 + 3\varepsilon)H\lambda^2 - (12 + 114\varepsilon)H^2\lambda - 5\varepsilon\lambda^3, \\
p_2 &= (-100 + 40\varepsilon)H^3 + (-28 + 6\varepsilon)H\lambda^2 + (-24 + 90\varepsilon)\lambda + 5\varepsilon\lambda^3.
\end{aligned}$$

Therefore, (3.39) can be rewritten as

$$\alpha p_1 + \beta p_2 = 0. \tag{3.40}$$

On the other hand, combining (3.38) with (3.36) and using (3.35), we find

$$\epsilon_1[\alpha^2(2\epsilon H + \lambda) + 2\beta^2(4\epsilon H - \lambda)] = q, \tag{3.41}$$

where q is given by

$$q = 252\epsilon H^3 - 204H^2\lambda + 62\lambda^2 H\epsilon - 2\lambda^3.$$

Using (3.40) and (3.35), we get

$$\alpha^2 = \frac{\epsilon_1 p_2}{p_1}(6\varepsilon H\lambda - 2\lambda^2), \qquad \beta^2 = \frac{\epsilon_1 p_1}{p_2}(6\varepsilon H\lambda - 2\lambda^2).$$

Eliminating α^2 and β^2 from (3.41), we obtain

$$\begin{aligned} &H^9(-2772000\varepsilon + 685440) + H^8\lambda(-1792224\varepsilon + 6724368) \\ &\quad + H^7\lambda^2(-5824768\varepsilon + 1808032) \\ &\quad + H^6\lambda^3(-971208\varepsilon + 1923408) + H^5\lambda^4(160376\varepsilon + 330664) \\ &\quad + H^4\lambda^5(-81224\varepsilon - 363028) \\ &\quad + H^3\lambda^6(141352\varepsilon + 17280) + H^2\lambda^7(-2838\varepsilon - 26452) \\ &\quad + H\lambda^8(180\varepsilon + 270) = 0, \end{aligned} \tag{3.42}$$

which is a homogeneous equation of degree 9 in terms of λ and H. Here, we point out that $\lambda \neq 0$. In fact, if $\lambda = 0$ then (3.42) gives $H = 0$, which is contradiction to our assumption. We put $Y = \frac{H}{\lambda}$.

(a) For spacelike normal vector ξ: In this case $\varepsilon = 1$, then (3.42) reduces to an algebraic equation of degree 8 in Y

$$\begin{aligned} &Y^8(-2086560) + Y^7(4932144) + Y^6(-4016736) \\ &\quad + Y^5(952200) + Y^4(491040) \\ &\quad + Y^3(-444252) + Y^2(158632) + Y(-29290) + 450 = 0, \end{aligned} \tag{3.43}$$

and without having to solve (3.43) explicitly, even in the case of the existence of a real solution, H will be proportional to λ with a numerical factor μ, where μ be the root of the equation (3.43). Hence, we can assume that $H = \mu\lambda$ and substituting it in (3.25), (3.27), (3.30), we obtain

$$-e_1e_1(\lambda) + \frac{e_1^2(\lambda)}{\lambda}[\frac{1}{2\mu+1} + 1] = 2\mu(2\mu+1)\lambda^3\epsilon_1, \tag{3.44}$$

$$-e_1e_1(\lambda) + \frac{e_1^2(\lambda)}{\lambda}[\frac{3\mu-1}{4\mu-1} + 1] = 4\mu(4\mu-1)\lambda^3\epsilon_1, \tag{3.45}$$

$$-e_1e_1(\lambda) + \frac{e_1^2(\lambda)}{\lambda}[\frac{2}{2\mu+1} + \frac{3\mu-1}{4\mu-1}] = -\lambda^3\frac{(4\mu^2+2+(6\mu-2)^2)}{\epsilon_1}. \tag{3.46}$$

From (3.44)∼(3.46), we find

$$e_1^2(\lambda) = 2(8\mu^2 + 2\mu - 1)\lambda^4\epsilon_1, \tag{3.47}$$

$$e_1^2(\lambda) = \frac{(2\mu+1)(56\mu^2 - 28\mu + 6)\lambda^4}{(2\mu-1)\epsilon_1}. \tag{3.48}$$

Using (3.47) and (3.48), we get

$$10\mu^2 - 4\mu + 1 = 0. \tag{3.49}$$

On the other hand, differentiating (3.47) along e_1, we obtain

$$e_1e_1(\lambda) = 4(8\mu^2 + 2\mu - 1)\lambda^3\epsilon_1, \tag{3.50}$$

From (3.44), (3.47), and (3.50), we find

$$10\mu - 1 = 0, \tag{3.51}$$

which gives $\mu = \frac{1}{10}$. However, this contradicts the equation (3.49). Hence H must be constant.

(b) For timelike normal vector ξ: In this case $\varepsilon = -1$, then (3.42) reduces to an algebraic equation of degree 8 in Y

$$\begin{aligned} &Y^8(3457440) + Y^7(8516592) + Y^6(7632800) \\ &\quad + Y^5(2894616) + Y^4(170288) \\ &\quad + Y^3(-281804) + Y^2(-124072) + Y(-23614) + 90 = 0, \end{aligned} \tag{3.52}$$

we proceed analogously to case (a), where equation (3.51) will be replace by

$$10\mu + 1 = 0, \tag{3.53}$$

which gives $\mu = -\frac{1}{10}$.

Now, we have main result as follows:

Theorem 3.2 *There exist no proper biharmonic non-degenerate hypersurfaces M_r^4, $r = 0, 1, 2, 3, 4$, with three distinct principal curvatures in the semi-Euclidean space E_s^5, $s = 0, 1, 2, 3, 4, 5$ of the diagonal shape operator.*

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