An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat. (N.S.)
Tomul LXII, 2016, f. 2, vol. 2

Coverings and actions of structured Lie groupoids

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Received: 4.I.2014 / Revised: 27.VIII.2014 / Accepted: 20.XII.2014

Abstract In this work we deal with coverings and actions of Lie group-groupoids and Lie ring-groupoids being two sorts of the structured Lie groupoids. After we give definition of a structured Lie groupoid, we prove some characterizations of structured Lie groupoids. Then, we present the concept of covering of a structured Lie groupoid. As first main result of this work, we show that the category $\mathcal{LS}Cov(M)$ of the smooth coverings of Lie group M is equivalent to the category $\mathcal{SLG}Cov(\pi_1 M)$ of the coverings of structured Lie groupoid $\pi_1 M$, where it is assumed that $\pi_1 M$ is a Lie group-groupoid, specially. Furthermore, we define an action of a structured Lie groupoid on a connected Lie structure. Finally, we show that the category $\mathcal{SLG}Cov(G)$ of the coverings of a structured Lie groupoid G and the category $\mathcal{SLG}Op(G)$ of the actions of G on Lie structures (group or ring) are equivalent.

Keywords Lie groupoid · Covering · Action

Mathematics Subject Classification (2010) 22A22 · 57M10

1 Introduction

The theory of covering space has an important role in the algebraic topology. This theory plays an important role in the category theory when is studied relations between the fundamental groupoids of covering spaces and their base spaces. Many mathematician studied the theory of covering spaces using that of covering groupoids (see [1,8]).

The first papers in this field were written by Brown and Higgins [1,2,8]. Brown defined the fundamental groupoid $\pi_1 X$ associated to a topological space X and obtained the covering morphism $\pi_1 p : \pi_1 \widetilde{X} \to \pi_1 X$ of groupoids for a given covering map $p : \widetilde{X} \to X$ of topological spaces. Thus he proved the equivalence of the category $TCov(X)$ of coverings of X and the category $GdCov(\pi_1 X)$ of coverings of fundamental groupoid $\pi_1 X$, where X has universal covering space (see [1]).

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Another key result in theory of covering groupoids was given by Gabriel and Zisman [7]. They proved that if G is a groupoid, there exists an equivalence between the category $GdOp(G)$ of actions of G on sets and the category $GdCov(G)$ of covering groupoids of G (see [7]).

A concept considered in this paper is group-groupoid notion which is a group object in the category of groupoids. It was defined by Brown and Spencer [4]. They proved that if X is a topological group, then the fundamental groupoid $\pi_1 X$ becomes a group-groupoid.

In [13], it was given notion of a ring-groupoid which is a ring object in the category of groupoids. Also it was proved by Mucuk that if R is a topological ring whose underlying space has a universal covering space, then the category $TRCov(R)$ of topological ring coverings of R is equivalent to the category $RGdCov(\pi_1 R)$ of ring-groupoid coverings of $\pi_1 R$.

After defining the topological groupoid by Ehresmann all of the above results were also given for topological groupoids (see [2,10,14]). In [10], it was proved that the topological group structure of a topological group-groupoid how was lifted to a topological universal covering groupoid.

In this work we deal with smoothness of all these results. It is well-known that for a connected smooth manifold M there exist a universal covering manifold $\widetilde{M}$ and a smooth covering map $p : \widetilde{M} \to M$. Also, from the manifold theory there exist a simply connected universal covering Lie group $\widetilde{G}$ of a connected Lie group G and a smooth covering map $p : \widetilde{G} \to G$ which is a Lie group homomorphism at the same time (see [5]). It is well known that these situations are hold for Lie rings, too.

After we define smooth versions of notions of group-groupoid and ring-groupoid, we combine the concepts of Lie group-groupoid and Lie ring-groupoid under the one name, called structured Lie groupoid. Then we prove smooth versions of the above works.

More precisely, firstly, we show that the fundamental Lie groupoid $\pi_1 M$ associated to a connected Lie group M is a structured Lie groupoid, specially a Lie group-groupoid. Thus we constitute the category of smooth coverings of a connected Lie structure M and the category of smooth coverings of structured Lie groupoid $\pi_1 M$. Then we prove the equivalence of these categories, where M is a Lie group and $\pi_1 M$ is a Lie group-groupoid. Secondly, we define action of a structured Lie groupoid on a connected Lie structure (group or ring) M. Furthermore, we show that the category $\mathcal{SLG}Cov(G)$ of coverings of a structured Lie groupoid G and the category $\mathcal{SLG}Op(G)$ of actions of the structured Lie groupoid G on Lie structures are equivalent.

Throughout the paper, all our manifolds we consider are assumed to be smooth and second countable.

2 Structured Lie groupoids

In this section, we will give the basic definitions and concepts related to structured Lie groupoids; Lie group-groupoids and Lie ring-groupoids. In fact, these groupoids are internal groupoids in the category $\mathcal{LG}$ of Lie groupoids. Namely, these are Lie groupoids having a second algebraic structure such as Lie group, Lie ring. For this reason we recall, in advance, the basic facts about groupoids and Lie groupoids.

A groupoid is a category in which every arrow is invertible. More precisely, a groupoid consists of two sets G and G_0 called the set of arrows (or morphisms) and the set of objects of groupoid respectively, together with two maps $\alpha, \beta : G \to G_0$ called source and target maps respectively, a map $\epsilon : G_0 \to G, x \mapsto \epsilon(x) = 1_x$ called the object map, an inverse map $i : G \to G, a \mapsto a^{-1}$ and a composition $G_2 = G \ {}_\alpha\!\times_\beta G \to G, (b, a) \mapsto b \circ a$ defined on the pullback $G \ {}_\alpha\!\times_\beta G = \{(b, a) \mid \alpha(b) = \beta(a)\}$.

These maps should satisfy the following conditions:

1. $\alpha(b \circ a) = \alpha(a)$ and $\beta(b \circ a) = \beta(b)$, for all $(b, a) \in G_2$,
2. $c \circ (b \circ a) = (c \circ b) \circ a$ such that $\alpha(b) = \beta(a)$ and $\alpha(c) = \beta(b)$, for all $a, b, c \in G$,
3. $\alpha(1_x) = \beta(1_x) = x$, for all $x \in G_0$,
4. $a \circ 1_{\alpha(a)} = a$ and $1_{\beta(a)} \circ a = a$, for all $a \in G$,
5. $\alpha(a^{-1}) = \beta(a)$ and $\beta(a^{-1}) = \alpha(a)$, $a^{-1} \circ a = 1_{\alpha(a)}$ and $a \circ a^{-1} = 1_{\beta(a)}$ (see [3]).

Let G be a groupoid. For all $x, y \in G_0$, we denote $G(x, y)$ the set of all arrows $a \in G$ such that $\alpha(a) = x$ and $\beta(a) = y$. For $x \in G_0$, we write $St_G x$ for the set of all arrows started at x, and $CoSt_G x$ for the set of all arrows ended at x. The object or vertex group at x is $G\{x\} = \{a \in G \mid \alpha(a) = \beta(a) = x\}$.

Let G and H be two groupoids. A groupoid morphism from H to G is a pair (f, f_0) of maps $f : H \to G$ and $f_0 : H_0 \to G_0$ such that $\alpha_G \circ f = f_0 \circ \alpha_H$, $\beta_G \circ f = f_0 \circ \beta_H$ and $f(b \circ a) = f(b) \circ f(a)$ for all $(b, a) \in H_2$ (see [1,3]).

Throughout the work we shall assume that the set of objects G_0 and α-fibers $\alpha^{-1}(x) = St_G x$, $x \in G_0$ are Hausdorff.

Definition 2.1 *A groupoid G over G_0 is called Lie groupoid if G and G_0 are manifolds, α and β are surjective submersions and the composition map is smooth (see [3]).*

It follows that, ϵ is an immersion, the inverse map is a diffeomorphism, the sets $St_G x$, $CoSt_G x$ and $G(x, y)$ are closed submanifolds of G for all $x, y \in G_0$ and all vertex groups are Lie groups. Also since α and β are submersions, G_2 is a closed submanifold of $G \times G$ (see [11]).

The left-translation (right translation) for $a \in G(x, y)$ is the map $L_a : CoSt_G x \to CoSt_G y$, $b \mapsto a \circ b$ ($R_a : St_G y \to St_G x$, $b \mapsto b \circ a$) which is a diffeomorphism (see [3]).

Example 2.1 Let M be a manifold. The product manifold $M \times M$ is a Lie groupoid over M in the following way: α is the second projection and β is the first projection; $1_x = (x, x)$ for all $x \in M$ and $(x, y) \circ (y, z) = (x, z)$ (see [12]).

Definition 2.2 *A morphism between Lie groupoids H and G is a groupoid morphism (f, f_0) such that f and f_0 are smooth (see [11]).*

Thus, we can construct the category $\mathcal{LG}$ of Lie groupoids. In this category, the objects are Lie groupoids and the morphisms are morphisms of Lie groupoids.

While we deal with structured Lie groupoids, we need the concepts of Lie groups and Lie rings which exist in the structures of these Lie groupoids. Since the structure of a Lie ring contains the structure of Lie group, we only recall the concept of Lie ring.

Definition 2.3 *A Lie ring G is a ring G whose the underlying set has a structure of manifold such that the group operation $m : G \times G \to G$, $(x, y) \mapsto x + y$, the ring operation $n : G \times G \to G$, $(x, y) \mapsto xy$ and the inverse map $\overline{u} : G \to G$, $x \mapsto -x$ are smooth.*

A morphism of Lie rings from H to G is a ring morphism $p : H \to G$ of the underlying rings of H and G such that p is a smooth map. Hence, there exists a category $LRing(G)$ of Lie rings G. Objects of this category are Lie rings G, and its morphisms are Lie ring morphisms.

Similarly, we have the category $LGrp(G)$ of Lie groups where G is a Lie group.

Specially in this paper we deal with connected Lie groups and Lie rings. By a connected Lie group or ring, we means that manifold structure is connected. From now on, unless we specify otherwise, all Lie groups and Lie rings will be assumed to be connected.

It is obvious that these categories are subcategories of the category of smooth manifolds. In other words, the categories of Lie groups and Lie rings are internal groups and internal rings in the category of smooth manifolds, respectively. Thus, we will use unique notation $\mathcal{L}$ to show above the categories. Namely, $\mathcal{L}$ will denote the category of Lie groups if G is a Lie group, and will denote the category of Lie rings if G is a Lie ring.

We can now give definitions of structured Lie groupoids. Firstly, we will present definition of a Lie group-groupoid which is an internal groupoid in the category $\mathcal{LG}$ of Lie groupoids.

Definition 2.4 *A Lie group-groupoid is a Lie groupoid endowed with a structure of Lie group such that the addition $m : G \times G \to G$, $(a, b) \mapsto a + b$, the unit map $e : * \to G$, $* \mapsto e(*) = 1_e$ and inverse map $\overline{u} : G \to G$, $a \mapsto -a$, which are the structure maps of Lie group, are Lie groupoid morphisms. Also there exists an interchange law $(b \circ a) + (d \circ c) = (b + d) \circ (a + c)$.*

Secondly, the precise definition of a Lie ring-groupoid which is a ring object in the category $\mathcal{LG}$ of Lie groupoids is following way

Definition 2.5 *A Lie ring-groupoid G is a Lie groupoid endowed with a structure of Lie ring such that following ring structure maps are Lie groupoid morphisms:*

i) $m : G \times G \to G$, $(a, b) \mapsto a + b$, *group operation;*
ii) $n : G \times G \to G$, $(a, b) \mapsto ab$, *ring operation;*
iii) $\overline{u} : G \to G$, $a \mapsto -a$, *inverse at group;*
iv) $e : * \to G$.

Also there exist following interchange laws in a Lie ring-groupoid G:

1. $(c \circ a) + (d \circ b) = (c + d) \circ (a + b)$,
2. $(c \circ a)(d \circ b) = (cd) \circ (ab)$.

It is obvious that a Lie ring-groupoid is an internal groupoid in $\mathcal{LG}$.

We can unify the definitions above into one.

Definition 2.6 *A structured Lie groupoid G is a Lie groupoid endowed with another algebraic structure whose the structure maps are Lie groupoid morphisms such that there exist interchange law between the composition of groupoid and the operations of other algebraic structure.*

From above the definition, it is understood that if the second algebraic structure has an operation, then there is an interchange law, and if it has two operation, then there are two interchange laws.

It is obvious that the structured Lie groupoids are internal groupoids in $\mathcal{LG}$. Thus, we can consider the categories of structured Lie groupoids as subcategories of $\mathcal{LG}$. In general, we will denote the category of structured Lie groupoids by $\mathcal{SLG}$.

Let G and H be two Lie group(or ring)-groupoids. A morphism $f : G \to H$ of Lie group(or ring)-groupoids is a morphism of underlying Lie groupoids preserving Lie group (or ring) structure.

The basic example to structured Lie groupoids is obtained by the product.

Example 2.2 Let G be a Lie structure (group or ring). Then we constitute a structured Lie groupoid $G \times G$ with object manifold G.

i) Firstly, let us consider G as a Lie group. Then, it is obtained a Lie group-groupoid.

A morphism from an object x to another object y is a pair of (y, x). The source map is defined by $\alpha(y, x) = x$, the target map is defined by $\beta(y, x) = y$, the object map is defined by $x \mapsto (x, x)$ for any $x \in G$, the inverse of (y, x) is defined by (x, y) and the composition is defined by $(z, y) \circ (y, x) = (z, x)$ for (y, x), $(z, y) \in G \times G$. Since G is a Lie group, $G \times G$ is also a Lie group with the operation $(x, y) + (z, t) = (x + z, y + t)$ defined by the operation of G. The unit element of this group is (e, e) where e is the unit element of G, and the inverse of (y, x)in the group is $(-y, -x)$. Also $G \times G$ is the Lie groupoid called the banal groupoid. Now let us show that the group structure maps of $G \times G$ are groupoid morphisms.

For $m : (G \times G) \times (G \times G) \to G \times G$, $m(((z, y), (z', y')) \circ ((y, x), (y', x'))) = m((z, y) \circ (y, x), (z', y') \circ (y', x')) = m((z, x), (z', x')) = (z + z', x + x')$ and $m((z, y), (z', y')) \circ m((y, x), (y', x')) = ((z, y) + (z', y')) \circ ((y, x) + (y', x')) = (z + z', y + y') \circ (y + y', x + x') = (z + z', x + x')$.

Similarly, it can be shown that the unit map and the inverse map of the group are also groupoid morphisms. Furthermore, since the group structure maps of $G \times G$ are defined by the operations of Lie group G, they are also smooth. Consequently, $G \times G$ is a Lie group-groupoid.

ii) Secondly, if G is a Lie ring, then $G \times G$ is a Lie ring-groupoid. In addition to (i), if we define ring operation of G by $(x, y)(z, t) = (xz, yt)$, it is clear that distribution property is satisfied as follows:

$$\begin{aligned}(x, y)((z, t) + (z', t')) &= (x, y)(z + z', t + t') = (x(z + z'), y(t + t'))\\ &= (xz + xz', yt + yt') = (xz, yt) + (xz', yt')\\ &= ((x, y)(z, t)) + ((x, y)(z', t')).\end{aligned}$$

Thus, $G \times G$ is also a Lie ring with structure of product manifold. Let us now show that ring structure maps of $G \times G$ are groupoid morphisms. By (i), since $G \times G$ is a Lie group-groupoid, it is only enough to show that the ring operation $n_1 = n \times n$ is a groupoid morphism. For $n_1 : (G \times G) \times (G \times G) \to G \times G$, we have $n_1(((z, y), (z', y')) \circ ((y, x), (y', x'))) = n_1((z, y) \circ (y, x), (z', y') \circ (y', x')) = n_1((z, x), (z', x')) = (zz', xx')$ and

$$\begin{aligned}n_1((z, y), (z', y')) \circ n_1((y, x), (y', x')) &= ((z, y)(z', y')) \circ ((y, x)(y', x'))\\ &= (zz', yy') \circ (yy', xx') = (zz', xx').\end{aligned}$$

Therefore $G\times G$ is a ring-groupoid. It is obvious that the ring structure maps of $G\times G$ are smooth, because they are defined by those of Lie ring G. Hence $G\times G$ is a Lie ring-groupoid.

Consequently, if G is a Lie structure, then $G\times G$ is a structured Lie groupoid over G.

This example defines a functor from the category $\mathcal{L}$ of Lie structures (resp., group or ring) to the category $\mathcal{SLG}$ of structured Lie groupoids. Let us give it by the following proposition

Proposition 2.7 *There exists a functor $\Omega : \mathcal{L} \to \mathcal{SLG}$ from the category $\mathcal{L}$ of Lie structures to the category $\mathcal{SLG}$ of structured Lie groupoids, where $\mathcal{L}$ and $\mathcal{SLG}$ have finite products.*

Proof. i) Let Lie structure G be a Lie group. Then, from Example 2.2, $G\times G$ is a Lie group-groupoid. If $f : G \to H$ is a morphism of Lie groups, then $\Omega(f) : G\times G \to H\times H$ is a morphism of Lie group-groupoids. Indeed, $\Omega(f)$ is defined by $(y, x) \mapsto (f(y), f(x))$ and $\Omega(f)$ preserves the group structure. That is,

$$\begin{aligned}\Omega(f)((y,x)+(y',x')) &= \Omega(f)(y+y',x+x') = (f(y+y'),f(x+x')) \\ &= (f(y)+f(y'),f(x)+f(x')) \\ &= (f(y),f(x))+(f(y'),f(x')) \\ &= \Omega(f)(y,x)+\Omega(f)(y',x').\end{aligned}$$

By the same idea, $\Omega(f)((z,y)\circ(y,x)) = \Omega(f)(z,x) = (f(z),f(x))$ and $\Omega(f)(z,y)\circ \Omega(f)(y,x) = (f(z),f(y))\circ(f(y),f(x)) = (f(z),f(x))$. Hence we obtain that $\Omega(f)((z, y)\circ(y,x)) = \Omega(f)(z,\ y)\circ\Omega(f)(y,x)$. Thus $\Omega(f)$ preserves the groupoid structure. $\Omega(f) = (f,f)$ is smooth, since f is smooth. Consequently $\Omega(f)$ is a morphism of Lie group-groupoids.

ii) Let Lie structure G be a Lie ring. By Example 2.2, $G\times G$ is a Lie ring-groupoid. If $f : G \to H$ is a morphism of Lie rings, then $\Omega(f) : G\times G \to H\times H$ is a morphism of Lie ring-groupoids. In addition to (i), it is only enough to show that $\Omega(f)$ preserves ring structure

$$\begin{aligned}\Omega(f)((y,x)(y',x')) &= \Omega(f)(yy',xx') = (f(yy'),f(xx')) \\ &= (f(y)f(y'),f(x)f(x')) = (f(y),f(x))(f(y'),f(x')) \\ &= \Omega(f)(y,x)\ \Omega(f)(y',x').\end{aligned}$$

Thus $\Omega(f)$ is a morphism of Lie ring-groupoids. □

2.1 Some results on structured Lie groupoids

In this section, we will prove some results related to structured Lie groupoids.

Theorem 2.8 *The transitive component $C_e(G)$ of unit element e in a structured Lie groupoid G has structure of the same structured Lie groupoid.*

Proof. Firstly, we will prove the situation of Lie group-groupoid. We know that $C_e(G)$ is the subgroupoid of G. For all object selected in $C_e(G)_0$, let us consider the arrow $T_x \in G(x, e)$. Let $a, b \in C_e(G)$ be arrows, where $a \in G(x, y)$ and $b \in G(x', y')$. Thus, we obtain $a-b \in G(x-x', y-y')$, where $T_x - T_{x'} \in G(x-x', e)$ and $T_y - T_{y'} \in G(y-y', e)$. Hence it follows that $x-x', y-y' \in C_e(G)_0$ and $a-b \in C_e(G)$. In other words, $C_e(G)$ is a subgroup. There are structures of submanifold on $C_e(G)_0$ and $C_e(G)$. Also, the addition in $C_e(G)$ is the addition of G, so it is smooth. Thus, $C_e(G)$ is a Lie group. In addition to this, let us show that $C_e(G)$ is normal. For any $a \in G(x, y)$, let $a \in C_e(G)$ and $g \in G(w, z)$. Thus, for any $g \in G(w, z)$, $T_x \in G(x, e)$ and $-g \in G(-w, -z)$ we have $g + T_x - g \in G(w + x - w, z + e - z)$, and hence $g + T_x - g \in G(w + x - w, e)$. It follows $g + a - g \in G(w + x - w, z + y - z)$. That is, we obtain $g + a - g \in C_e(G)$. We deduce that $C_e(G)$ is a normal Lie subgroup. Furthermore, the structure maps of the subgroupoid $C_e(G)$ are restrictions of structure maps in G, so they are smooth too. Consequently $C_e(G)$ is a Lie group-groupoid.

Secondly, if G is a Lie ring-groupoid, then $C_e(G)$ is also a Lie ring-groupoid. It is obvious that if G is a Lie ring-groupoid, $C_e(G)$ is a Lie group-groupoid. Thus, it suffices to prove that $C_e(G)$ has a structure of Lie ring. For all object selected in $C_e(G)_0$, let us consider the arrow $T_x \in G(x, e)$. Let $a, b \in C_e(G)$ be arrows, where $a \in G(x, y)$ and $b \in G(x', y')$. Then, one can obtain $ab \in G(xx', yy')$, where $T_x T_{x'} \in G(xx', e)$ and $T_y T_{y'} \in G(yy', e)$. Hence, it follows that $xx', yy' \in C_e(G)_0$ and $ab \in C_e(G)$. Similarly, it can be shown that distribution property is verified. Clearly, it is a Lie subring, because the addition and the multiplication operations in C_eG are restrictions of addition and multiplication operations in Lie ring G. Hence, they are also smooth. Consequently, C_eG is a Lie ring-groupoid. □

Here, we will give two propositions related to structured Lie groupoids.

Proposition 2.9 *All characteristic groups in a structured Lie groupoid G are linearly diffeomorphic to each other.*

Proof. It is enough to show that for any $x \in G_0$ the object group $G\{x\}$ is diffeomorphic to the vertex group $G\{e\}$. From the definition of left translation, we can write $L_{1_x} : G\{e\} \to G\{x\}$, $a \mapsto 1_x + a$. On the other hand $L_{1_x}(b \circ a) = 1_x + (b \circ a) = (1_x \circ 1_x) + (b \circ a) = (1_x + b) \circ (1_x + a)$, by the interchange law. Therefore, L_{1_x} is a homomorphism of groups. Since the operations $+$ and $\circ$ are the operations of Lie group and Lie groupoid, respectively, they are smooth. That is, L_{1_x} is smooth. Furthermore, there exists inverse of L_{1_x}, and it is also smooth. □

Proposition 2.10 *Let G be a structured Lie groupoid and let e be unit of G_0 (which is group or ring). Then G_e is a Lie structure (group or ring), where G_e denotes the set of arrows started at e.*

Proof. Let us firstly show that G_e is a Lie group with operations induced from Lie group-groupoid G. For any $a, b \in G_e$, we have $a \in G(e, x)$ and $b \in G(e, y)$. Hence, it follows that $a + b \in G(e + e, x + y) = G(e, x + y)$, that is, $a + b \in G_e$. It is obvious that unit of the addition operation is $1_e \in G(e, e)$ and inverse of an arrow $a \in G_e$ is $-a \in G(e, -x)$. So, G_e is a Lie group.

Secondly, we know that every Lie ring-groupoid contains a Lie group-groupoid in itself. To prove that G_e is a Lie ring in the Lie ring-groupoid G, in addition to above mentioned facts, it is enough to show that distribution property is verified. Because it is clear that if any $a, b \in G_e$, then we have $ab \in G(ee, xy) = G(e, xy)$, namely; $ab \in G_e$. Let us now prove that distribution property $a(b+c) = ab + ac$ is verified. We have $a(b+c) \in G(e(e+e), x(y+z)) = G(e, x(y+z)) = G(e, xy+xz)$. Since $ab \in G(ee, xy) = G(e, xy)$ and $ac \in G(ee, xz) = G(e, xz)$, we have also $ab + ac \in G(e+e, xy+xz) = G(e, xy+xz)$. Hence we obtain $a(b+c) = ab + ac$. Thus, G_e is a ring. There is smooth structure induced from smooth structure of Lie ring-groupoid G on G_e. The ring structure maps of G_e are smooth, because they are restrictions of those of Lie ring G. Therefore, G_e is a Lie ring. □

Proposition 2.11 *Let M be a connected Lie structure (group or ring) such that its underlying manifold is connected. Then $\pi_1 M$ is a structured Lie groupoid.*

Proof. From [2], it is known that $\pi_1 M$ is a group-groupoid. Let us denote the atlas making the manifold M smooth and consisting of the liftable charts by $\mathcal{A}$. Since M is connected manifold, the fundamental groupoid $\pi_1 M$ is a Lie groupoid (see [6]). So there exists an atlas $\mathcal{A}$ lifted from $\mathcal{A}$ over $\pi_1 M$. Further this atlas makes it smooth manifold. Now we can show that $\pi_1 M$ is a Lie group-groupoid. For this, it is enough to show that the group operation $\pi_1 m : \pi_1 M \times \pi_1 M \to \pi_1 M$ is smooth. Since $m : M \times M \to M$ is the addition of Lie group, it is smooth. In fact, it is easily seen by the following diagram:

$$\begin{array}{ccc} M \times M & \xrightarrow{m} & M \\ {\scriptstyle \varphi\times\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathbb{R}^{2n} & \xrightarrow{pr_1} & \mathbb{R}^n \end{array}$$

At this diagram, φ is a coordinate chart with domain U selected from the liftable atlas of M. So φ is a diffeomorphism onto the open subset $\varphi(U) \subset \mathbb{R}^n$. From here, $\varphi \times \varphi$ is also smooth. Furthermore, pr_1 is smooth, because it is a projection onto the first factor. Thus, $m = \varphi^{-1} \circ pr_1 \circ (\varphi \times \varphi)$. That is, m is smooth map. Since $\pi_1 M$ is a Lie groupoid over M, and it is the covering manifold of the product manifold $M \times M$, then we can give the following diagram:

$$\begin{array}{ccc} \pi_1(M) \times \pi_1(M) & \xrightarrow{\pi_1 m} & \pi_1 M \\ {\scriptstyle \widetilde{\varphi}\times\widetilde{\varphi}}\downarrow & & \downarrow{\scriptstyle \widetilde{\varphi}} \\ \mathbb{R}^{4n} & \xrightarrow{pr_{1,2}} & \mathbb{R}^{2n} \end{array}$$

which is lifting of the above diagram. $\widetilde{\varphi}$ is the coordinate chart lifted from the coordinate chart φ. So $\widetilde{\varphi}$ is smooth. This brings the map $\widetilde{\varphi} \times \widetilde{\varphi}$ is smooth too. $pr_{1,2}$ is also smooth, because it is a projection. Hence $\pi_1 m$ is smooth. Therefore, $\pi_1 M$ is a Lie group-groupoid.

Secondly, since a Lie ring M is a topological ring at the same time, from [13], it is known that $\pi_1 M$ is a ring-groupoid. We also showed that $\pi_1 M$ is a Lie group-groupoid.

For this reason, it is enough to prove that ring operation $\pi_1 n : \pi_1 M \times \pi_1 M \to \pi_1 M$ is smooth. This is also easily proved as similar to above.

Consequently, if M is a connected Lie structure, $\pi_1 M$ is a structured Lie-groupoid. $\square$

From the following proposition, structural Lie groupoid $\pi_1 G$ is functorial.

Proposition 2.12 *Let $f : H \to G$ be a morphism of connected Lie structures. A morphism $\pi_1 f : \pi_1 H \to \pi_1 G$ induced from f is a morphism of structured Lie groupoids.*

Proof. In [4], it was showed that $\pi_1 f$ is a morphism of group-groupoids. Also, Mucuk was proved that $\pi_1 f$ is the morphism of ring-groupoids in [13]. For this reason, it is enough to show that $\pi_1 f$ is smooth. Since $f : H \to G$ is smooth, we have $f(U) \subset V$ for the charts (U, φ) and (V, ψ) on H and G, respectively. And hence $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is a smooth map.

$$\begin{array}{ccc} H & \xrightarrow{\ f\ } & G \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ \mathbb{R}^n & \xrightarrow{\ I\ } & \mathbb{R}^n \end{array}$$

There exist the fundamental Lie groupoids $\pi_1 H$ and $\pi_1 G$ corresponding to H and G, respectively. So we can define groupoid morphism $\pi_1 f$ as $\widetilde{\psi}^{-1} \circ Id \circ \widetilde{\varphi}$ by coordinate charts $(\widetilde{U}, \widetilde{\varphi})$ and $(\widetilde{V}, \widetilde{\psi})$ which are liftings of the coordinate charts (U, φ) and(V, ψ), respectively. $\widetilde{\psi}$ and $\widetilde{\varphi}$ are smooth, because they are the chart maps of Lie groupoids $\pi_1 G$ and $\pi_1 H$, respectively. Since the Id is the unit map, it is smooth. Thus $\pi_1 f$ is smooth. $\square$

A structured Lie groupoid G is called transitive, 1-transitive or simply transitive, if the underlying Lie groupoid of G is transitive, 1 -transitive or simply transitive, respectively.

3 Coverings and actions of structured Lie groupoids

Primarily, we recall the coverings of Lie structures (groups or rings).

Let $p : \widetilde{G} \to G$ be a morphism of Lie structures. If p is a covering map of underlying manifolds of $\widetilde{G}$ and G, then p is called a covering morphism of Lie structures.

Further, let $\widetilde{G}$ and G be connected Lie structures. By connected Lie structure, we means that the underlying manifold is connected. Then, we obtain a category $\mathcal{LS}Cov(G)$ of smooth coverings of connected Lie structures G. In this category, objects are covering morphisms $p : \widetilde{G} \to G$ of Lie structures, and a morphism from $p : \widetilde{G} \to G$ to $q : \hat{G} \to G$ is a morphism $r : \widetilde{G} \to \hat{G}$ of Lie structures such that it is satisfied condition $p = q \circ r$.

We are going to use certain pullbacks of smooth maps in the proofs of theorems which will be given in the sequel.

Since the category of smooth manifolds has all the pullbacks of smooth submersions, the category $\mathcal{LS}Cov(G)$, the category of manifolds with an additional structure, is to possess all the required pullbacks.

It will be useful to present the definition of covering morphism of Lie groupoids before the definition of covering morphism of structured Lie groupoids.

Definition 3.1 *Let $p : \widetilde{G} \to G$ be a morphism of Lie groupoids. For each $\widetilde{x} \in \widetilde{G}_0$, if the restriction $\widetilde{G}_{\widetilde{x}} \to G_{p(\widetilde{x})}$ of p is a diffeomorphism, p is called the covering morphism of Lie groupoids. Then $\widetilde{G}$ is called the covering of Lie groupoid G (see [9]).*

Let us give an equivalent criterion to the covering of Lie groupoids.

Let $p : H \to G$ be a covering morphism of Lie groupoids. Take the pullback

$$G\ {}_{\alpha}\times_{p_0} H_0 = \{(a, x) \in G \times H_0 \mid \alpha(a) = p_0(x)\}.$$

Since α is a submersion, $G\ {}_{\alpha}\times_{p_0} H_0$ is a manifold. Then the map $s_p : G\ {}_{\alpha}\times_{p_0} H_0 \to H$ is the lifting function assigning the unique element $h \in H_x$ to the pair (a, x) such that $p(h) = a$. It is clear that s_p is inverse of the map $(p, \alpha) : H \to G\ {}_{\alpha}\times_{p_0} H_0$.

Thus, the morphism $p : H \to G$ is covering morphism of Lie groupoids iff the morphism (p, α) is a diffeomorphism.

Definition 3.2 *A morphism $p : \widetilde{G} \to G$ of structured Lie groupoids is called a covering morphism of structured Lie groupoids if it is a covering morphism of underlying Lie groupoids. In other words, if the morphism $s_p : G\ {}_{\alpha}\times_{p_0} \widetilde{G}_0 \to \widetilde{G}$ defined on underlying Lie groupoids of $\widetilde{G}$ and G is a diffeomorphism, then p is called covering morphism of structured Lie groupoids. The inverse of s_p is $(p, \alpha) : \widetilde{G} \to G\ {}_{\alpha}\times_{p_0} \widetilde{G}_0$.*

It is obvious that coverings of structured Lie groupoids are internal groupoids in the category of coverings of Lie groupoids.

Example 3.1 Let $p : G \to G$ be unit morphism of structured Lie groupoids. Then, it is clear that projection $s_p : G\ {}_{\alpha}\times_{p_0} G_0 \to G$ is one-to-one, surjective and smooth. Further, p and α are smooth, because p is the morphism of structured Lie groupoids and α is the source map of the structured Lie groupoid G. Hence the inverse $(p, \alpha) : G \to G\ {}_{\alpha}\times_{p_0} G_0$ of s_p is smooth. Therefore, s_p is a diffeomorphism. Consequently, p is a covering morphism of structured Lie groupoids.

Proposition 3.3 *Let $p : \widetilde{G} \to G$ be a covering morphism of connected Lie structures. Then the morphism $\pi_1 p : \pi_1\widetilde{G} \to \pi_1 G$ is a covering morphism of structured Lie groupoids.*

Proof. We suppose that $p : \widetilde{G} \to G$ be covering morphism of the connected Lie structures. It is obvious that p is smooth and a Lie structure (group or ring) homomorphism. Since $\widetilde{G}$ and G are connected smooth manifolds, from Proposition 2.11, $\pi_1\widetilde{G}$ and $\pi_1 G$ are structured Lie groupoids. It is known that $\pi_1 p : \pi_1\widetilde{G} \to \pi_1 G$ is covering morphism of groupoids (see [1]). Also by Proposition 2.12, $\pi_1 p$ is the morphism of structured Lie groupoids. More precisely, we can show that $\pi_1 p$ is preserved the group or ring structures as follows:

The group or ring operation on objects is preserved, because p is the morphism of Lie structures (resp., group or ring homomorphism). Let us now show that the group and ring operations on morphisms are preserved $(\pi_1 p)[a+b] = [p(a+b)] = [p(a)+p(b)] = [p(a)] + [p(b)] = (\pi_1 p)([a]) + (\pi_1 p)([b])$ and

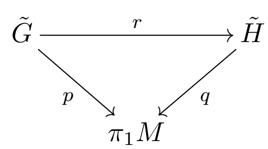
$$(\pi_1 p)[ab] = [p(ab)] = [p(a)p(b)] = [p(a)][p(b)] = (\pi_1 p)([a])(\pi_1 p)([b]).$$

Finally, let us prove that $\pi_1 p$ is covering morphism of structured Lie groupoids. Since α is the source map of the structured Lie groupoid, it is obvious that it is smooth too. So the map $(\pi_1 p, \alpha) : \pi_1 \widetilde{G} \to \pi_1 G \ {}_{\alpha}\times_{p} \widetilde{G}$ is smooth. Further, it is bijection, because $\pi_1 p$ is the covering morphism of structured groupoids. Hence there exists an inverse $s_{\pi_1 p} : \pi_1 G \ {}_{\alpha}\times_{p} \widetilde{G} \to \pi_1 \widetilde{G}$ of $(\pi_1 p, \alpha)$. $s_{\pi_1 p}$ is the function assigning to the unique homotopy class $[h]_{\widetilde{x}}$ started at $\widetilde{x}$ of smooth paths h each pair $([a], \widetilde{x})$ such that $\pi_1 p([h]) = [a]$. By the homotopy lifting property and the unique lifting property, it is obvious that $s_{\pi_1 p}$ is well-defined. Also, we can write $s_{\pi_1 p}$ as the composition of the smooth maps in the following diagram:

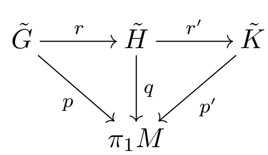
$$\begin{array}{ccccccc} \pi_1 G \ {}_{\alpha}\times_{p} \tilde{G} & \xrightarrow{I\times\epsilon} & \pi_1 G \times \pi_1 \widetilde{G} & \xrightarrow{I\times L_{\widetilde{a}}} & \pi_1 G \times \pi_1 \widetilde{G} & \xrightarrow{pr_2} & \pi_1 \widetilde{G} \\ ([a], \widetilde{x}) & \longmapsto & ([a], [1_{\widetilde{x}}]) & \longmapsto & ([a], [\widetilde{a}]) & \longmapsto & [\widetilde{a}] \end{array}$$

Hence $s_{\pi_1 p}$ is smooth. Thus, $(\pi_1 p, \alpha)$ is a diffeomorphism. Consequently, $\pi_1 p$ is the covering morphism of structured Lie groupoids. $\square$

After this proposition, we obtain a category as follows: If M is a connected Lie structure, by Proposition 2.11 the fundamental groupoid $\pi_1 M$ is a structured Lie groupoid. Then, there exists a category $\mathcal{SLG}Cov(\pi_1 M)$. In this category, objects are covering morphisms $p : \widetilde{G} \to \pi_1 M$ of structured Lie groupoids, and a morphism from an object $p : \widetilde{G} \to \pi_1 M$ to an object $p : \widetilde{H} \to \pi_1 M$ is a morphism $r : \widetilde{G} \to \widetilde{H}$ of structured Lie groupoids such that $p = q \circ r$, where $\widetilde{M} = \widetilde{G}_0$ is connected manifold.



In this category, the source map is $\alpha(r) = p$, the target map is $\beta(r) = q$ and object map is $1_{(p)} : \widetilde{G} \to \widetilde{G}$. Also, if $r : \widetilde{G} \to \widetilde{H}$ and $r' : \widetilde{H} \to \widetilde{K}$ are two morphisms, then the composition is defined by the following commutative diagram:

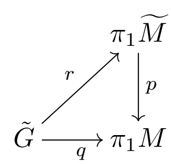


More generally, let G be a structured Lie groupoid. Then we obtain a category $\mathcal{SLG}Cov(G)$ whose objects are covering morphisms $p : H \to G$ of structured Lie groupoids. In this category, a morphism from an object $p : H \to G$ to an object $q : K \to G$ is a morphism $r : H \to K$ of structured Lie groupoids satisfying the condition $p = q \circ r$.

Let us now state a proposition from [9], which is necessary for the proof of the following theorem

Proposition 3.4 *Let M be a connected manifold and let $q : \widetilde{G} \to \pi_1 M$ be covering morphism of groupoids. Let $\widetilde{M} = \widetilde{G}_0$ and $p = q_0 : \widetilde{M} \to M$. Let $\mathcal{A}$ denotes an atlas consisting of the liftable charts. Then the smooth structure over $\widetilde{M}$ is the unique structure such that the followings are hold:*

1. *$p : \widetilde{M} \to M$ is a covering map.*
2. *There exists an isomorphism $r : \tilde{G} \to \pi_1 \tilde{M}$ which is the identical on objects such that the following diagram is commutative:*



Let us now give first main result of this paper.

Theorem 3.5 *Let M be a connected Lie group. Then the category $\mathcal{LSC}ov(M)$ of the smooth coverings of Lie group M is equivalent to the category $\mathcal{SLG}Cov(\pi_1 M)$ of the coverings of structured Lie groupoid $\pi_1 M$, where it is assumed that $\pi_1 M$ is a Lie group-groupoid, specially.*

Proof. Let us define a functor $\Gamma : \mathcal{LSC}ov(M) \to \mathcal{SLG}Cov(\pi_1 M)$ as follows. Let M, $\widetilde{M}$ be connected Lie groups. By Proposition 2.12, if $p : \widetilde{M} \to M$ is a covering morphism of Lie groups, then $\pi_1 p : \pi_1 \widetilde{M} \to \pi_1 M$ is a covering morphism of Lie group-groupoids. Hence $\Gamma(p) = \pi_1 p$ is a covering morphism of Lie group-groupoids. If $r : \widetilde{M} \to \widetilde{N}$ is a morphism of covering morphisms of Lie groups from $p : \widetilde{M} \to M$ to $q : \widetilde{N} \to M$, by the definition of category $\mathcal{LSC}ov(M)$, r is also covering morphism of Lie groups, and since $\widetilde{M}$, $\widetilde{N}$ are connected, $\pi_1 r$ is also covering morphism of Lie group-groupoids. Obviously $\Gamma(r)$ is a morphism of covering morphisms of Lie group-groupoids from $\pi_1 p$ to $\pi_1 q$. Let $r' : \widetilde{N} \to \widetilde{P}$ be another morphism of the covering morphisms of Lie groups, where $q' : \widetilde{P} \to M$. Since r and r' are covering morphisms of Lie groups, the composition $r' \circ r : \widetilde{M} \to \widetilde{P}$ is also covering morphism of Lie groups and is clearly morphism of covering morphisms of Lie groups from p to q'. Also, $\pi_1(r' \circ r) = \pi_1 r' \circ \pi_1 r$ is the covering morphism of Lie group-groupoids, because $\widetilde{M}$ and $\widetilde{P}$ are connected. Furthermore, it is a morphism of covering morphisms of Lie group-groupoids from $\pi_1 p$ to $\pi_1 q'$. Thus, we have $\Gamma(r' \circ r) = \Gamma r' \circ \Gamma r$, so that Γ is a functor.

Now let us set a functor $\Phi : \mathcal{SLG}Cov(\pi_1 M) \to \mathcal{LSC}ov(M)$. Suppose $\widetilde{G}_0 = \widetilde{M}$ and let $q : \widetilde{G} \to \pi_1 M$ be smooth covering morphism of Lie group-groupoids. Since M is connected, from Proposition 3.4, for $p = q_0 : \widetilde{M} \to M$ there exist a lifted manifold on $\widetilde{M}$ making p smooth covering map on the underlying manifolds of M and $\widetilde{M}$, and an isomorphism $r : \widetilde{G} \to \pi_1\widetilde{M}$. Furthermore, since $p = q_0$ and q are Lie group-groupoid morphisms, p is also morphism of Lie groups. Thus, $\Phi(q) = q_0 = p$ is a covering morphism of Lie groups. Let $f : \widetilde{G} \to \widetilde{H}$ be a morphism of smooth covering morphisms of Lie group-groupoids from $q : \widetilde{G} \to \pi_1 M$ to $q' : \widetilde{H} \to \pi_1 M$. From Proposition 2.11, for the liftable atlas $\mathcal{A}$ on M there exist lifted atlases $\widetilde{\mathcal{A}}_q$ and $\widetilde{\mathcal{A}}_{q'}$ on $\widetilde{M} = \widetilde{G}_0$ and $\widetilde{N} = \widetilde{H}_0$, respectively. These atlases consist of the lifted charts making the manifolds $\widetilde{M}$ and $\widetilde{N}$ are smooth. Let $\widetilde{x} \in \widetilde{M}$ and let $\widetilde{U}_{q'}$ be an element of $\widetilde{\mathcal{A}}_{q'}$ including $f(\widetilde{x})$. Then $U = q'(\widetilde{U}_{q'}) \in \mathcal{A}$ and U is lifted to a unique $\widetilde{U}_q \in \widetilde{\mathcal{A}}_q$, which contains $\widetilde{x}$. Also $f(\widetilde{U}_q) = \widetilde{U}_{q'}$. Hence $f : \widetilde{M} \to \widetilde{N}$ is smooth. That is, $\Phi(f)$ is a morphism of covering morphisms of Lie groups. Let $f' : \widetilde{H} \to \widetilde{H'}$ be another morphism from $q' : \widetilde{H} \to \pi_1 M$ to $q'' : \widetilde{H'} \to \pi_1 M$. Since f and f' are covering morphisms of Lie group-groupoids, the composition $f' \circ f$ is also covering morphism of Lie group-groupoids, and clearly it is a morphism of covering morphisms of Lie group-groupoids from q to q''. Furthermore, $\Phi(f' \circ f)$ is a morphism of covering morphisms of Lie groups as above. Thus, we have $\Phi(f' \circ f) = \Phi(f') \circ \Phi(f)$, so that Φ is a functor.

Now let us show natural equivalences $\Gamma\Phi \simeq 1_{\mathcal{SLG}Cov(\pi_1 M)}$ and $\Phi\Gamma \simeq 1_{\mathcal{LS}Cov(M)}$. Let $q : \widetilde{G} \to \pi_1 M$ and $q' : \widetilde{H} \to \pi_1 M$ be covering morphisms of Lie group-groupoids. Since M is connected, there exist the covering maps $p = q_0 : \widetilde{M} \to M$ and $p' = q'_0 : \widetilde{N} \to M$ of smooth manifolds and the isomorphisms $r : \widetilde{G} \to \pi_1\widetilde{M}$ and $r' : \widetilde{H} \to \pi_1\widetilde{N}$. Now let us show that the following diagram is commutative

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\quad r \quad} & \pi_1\tilde{M} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi_1 f_0} \\ \tilde{H} & \xrightarrow{\quad r' \quad} & \pi_1\tilde{N} \end{array}$$

Indeed, let $\widetilde{a}$ be an element of $\widetilde{G}$ started at $\widetilde{x}$ and let $a : I \to M$ be a representation of $q(\widetilde{a}) \in \pi_1 M$. Then a induces a morphism $\pi_1 a : \pi_1 I \to \pi_1 M$, $\pi_1 a(\imath) = q(\widetilde{a})$. Further, $\pi_1 a$ is lifted to unique morphism $a' : (\pi_1 I, 0) \to (\widetilde{G}, \widetilde{x})$. Then $r(\widetilde{a})$ is the equivalent class of the path $a'_0 : I \to \widetilde{M}$. Let $\widetilde{b} = f(\widetilde{a})$. We use the same method in order to obtain $b' : (\pi_1 I, 0) \to (\widetilde{H}, f(\widetilde{x}))$, where $b = f_0(a)$. Since b' is determined as unique by b, we obtain $r' f(\widetilde{a}) = (\pi_1 f_0) r(\widetilde{a})$. This means that we have $\Gamma\Phi \simeq 1_{\mathcal{SLG}Cov(\pi_1 M)}$. Finally, we must show $\Phi\Gamma \simeq 1_{\mathcal{LS}Cov(M)}$. But since $\widetilde{M} = (\pi_1\widetilde{M})_0$ and the structure manifold of $\widetilde{M}$ is the lifted manifold, it is obvious that $\Phi\Gamma = 1_{\mathcal{LS}Cov(M)}$. Thus, the proof is completed. □

Remark 3.1 We note that one can not extend the above equivalence to Lie ring-groupoid case. If one considers Lie ring instead of Lie group, then the above result is trivial. Indeed, in this case, connected Lie rings are simply connected. So, all of their coverings are trivial.

Now we will introduce a smooth action of a structured Lie groupoid on a connected Lie structure. Firstly, let us give the definition.

Definition 3.6 *Let G be a structured Lie groupoid and let M be a Lie structure (group or ring. An action of structured Lie groupoid G on Lie structure M via Lie structure homomorphism $w : M \to G_0$ consists of the action of the underlying Lie groupoid of G on the underlying smooth manifold of M via the submersion $w : M \to G_0$ satisfying the conditions $w({}^a x) = \beta(a)$, ${}^b({}^a x) = {}^{(b \circ a)}x$ and ${}^{1_{w(x)}}x = x$ such that the following interchange laws are hold:*

1. $({}^b y) + ({}^a x) = {}^{b+a}(y + x)$, for the group operation,
2. $({}^b y)({}^a x) = {}^{ba}(yx)$, for the ring operation.

Remark 3.2 We note that the condition (1) is valid both Lie group-groupoids and Lie ring-groupoids. But, the condition (2) is valid in case that structured Lie groupoid G is just a Lie ring-groupoid.

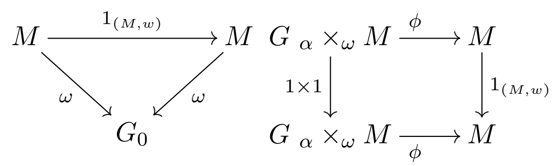
Example 3.2 If G is a structured Lie groupoid, then G acts on $M = G_0$ via the unit morphism $w = p_0 : M = G_0 \to G_0$. Indeed, since p is the unit morphism of structured Lie groupoids, p and p_0 are Lie structure homomorphisms. Hence w is a Lie structure homomorphism. The composition of the target map of structured Lie groupoid with the projection pr_1 gives action $\beta \circ pr_1 = \phi : G\ {}_\alpha\times_w G_0 \to G_0$ by $(a, x) = {}^a x = \beta(a)$ for any $a \in G$ and $x \in G_0$. Since pr_1 and β are smooth, the composition $\beta \circ pr_1 = \phi$ is also smooth. Now let us show that the conditions of the action are satisfied. We have $w({}^a x) = w(\beta(a)) = \beta(a)$, because w is the unit morphism. That is, the first condition is satisfied. The second condition is satisfied, namely ${}^b({}^a x) = {}^b(\beta(a)) = \beta(b) = \beta(b \circ a) = {}^{(b\circ a)}x$ for any $a, b \in G$ and $x \in G_0$. Finally, ${}^{1_{w(x)}}x = \beta(1_{w(x)}) = x$. Further, we must show that the interchange laws are verified. Let $a, b \in G$ and $x, y \in G_0$. If G is just a Lie group-groupoid, we have:

1. $({}^b y) + ({}^a x) = \beta(b) + \beta(a) = \beta(b + a) = {}^{b+a}(y + x)$.
   In addition to this condition, if G is a Lie ring-groupoid, it is seen that the interchange law between ring operation and action is verified as follows:
2. $({}^b y)({}^a x) = \beta(b)\beta(a) = \beta(ba) = {}^{ba}(yx)$.

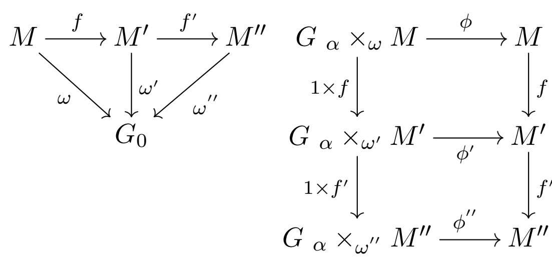
Consequently, G acts smoothly on the Lie structure $M = G_0$ via the unit morphism $w = p_0 : G_0 \to G_0$.

Let G be a structured Lie groupoid. The action of G on a Lie structure M via smooth structure homomorphism $w : M \to G_0$ is denoted by (M, w). A morphism from (M, w) to (M', w') is a morphism $f : M \to M'$ of Lie structures such that the conditions $w' \circ f = w$ and $f({}^a x) = {}^a f(x)$ are satisfied.

Thus, we obtain a category $\mathcal{SLGO}p(G)$ of the actions of the structured Lie groupoid G on Lie structures. In this category; for any morphism $f : M \to M'$, the source and target maps are defined by $\alpha(f) = (M, w)$ and $\beta(f) = (M', w')$, respectively. The object map is defined by $1_{(M,w)} : (M,w) \to (M,w)$ for any object (M,w). We can represent the object map by the following diagrams:



Finally, the composition is defined by the following commutative diagrams:



The following two examples show the relation between the concepts covering and action for any structured Lie groupoid.

Example 3.3 Let $p : \widetilde{G} \to G$ be covering morphism of structured Lie groupoids. Then there is an action of structured Lie groupoid G on Lie structure $M = \widetilde{G}_0$ via Lie structure homomorphism $w = p_0 : \widetilde{G}_0 \to G_0$. Indeed, since p is the covering morphism of structured Lie groupoids, p and $p_0 = w$ are Lie structure homomorphisms. Also, there exists diffeomorphism $s_p : G\ {}_{\alpha}\times_{p_0} \widetilde{G}_0 \to \widetilde{G}$. Since s_p and $\widetilde{\beta}$ are smooth, the composition of $\widetilde{\beta} : \widetilde{G} \to \widetilde{G}_0$ and s_p gives a smooth action $\phi = \widetilde{\beta} \circ s_p : G\ {}_{\alpha}\times_{p_0} \widetilde{G}_0 \to \widetilde{G}_0$, $(a, \widetilde{x}) \mapsto {}^{a}\widetilde{x} = \widetilde{\beta}(\widetilde{a})$. By [9], there exists a smooth action of the underlying Lie groupoid of G on the underlying manifold of $M = \widetilde{G}_0$ via smooth submersion $w = p_0 : \widetilde{G}_0 \to G_0$. So the conditions of the action are satisfied. Hence, we must only show the interchange laws:

1. $({}^{b}y) + ({}^{a}x) = \widetilde{\beta}(\widetilde{b}) + \widetilde{\beta}(\widetilde{a}) = \widetilde{\beta}(\widetilde{b} + \widetilde{a}) = {}^{b+a}(y + x)$.
   In addition to this condition, if structured Lie groupoid G is specially a Lie ring-groupoid, it is seen that the interchange law between ring operation and action is verified as follows:
2. $({}^{b}y)({}^{a}x) = \widetilde{\beta}(\widetilde{b})\widetilde{\beta}(\widetilde{a}) = \widetilde{\beta}(\widetilde{b}\widetilde{a}) = {}^{ba}(yx)$.

Therefore, structured Lie groupoid G acts smoothly on Lie structure $M = \widetilde{G}_0$ via Lie structure homomorphism $w = p_0 : \widetilde{G}_0 \to G_0$.

Example 3.4 Let G be a structured Lie groupoid acting on Lie structure M via Lie structure homomorphism $w : M \to G_0$. Then we have action Lie groupoid $G \ltimes M$ whose object manifold is Lie structure M. Further, action Lie groupoid $G \ltimes M$ is a structured Lie groupoid defined by group operation $(a,x) + (b,y) = (a+b, x+y)$ and ring operation $(a,x).(b,y) = (ab, xy)$, where the operations "+" and "." are defined by the group operation and ring operation of structured Lie groupoid G, respectively. By [9], the covering morphism $p : G \ltimes M \to G$ of underlying Lie groupoids of $G \ltimes M$ and G is defined. Now let us show that p is a covering morphism of structured Lie groupoids. For this, we must show that p preserves the group structure and ring structure. Since p is the projection map,

$$p((a,x)+(b,y)) = p(a+b, x+y) = a+b = p(a,x)+p(b,y).$$

Thus, p is a Lie group homomorphism. If we take the structured Lie groupoid G as a Lie ring-groupoid, one can easily show that p preserves the ring structure. Consequently, p is a covering morphism of structured Lie groupoids.

Let us now give second main result of this paper.

Theorem 3.7 *Let G be a structured Lie groupoid. Then the category $\mathcal{SLG}Cov(G)$ of the coverings of G and the category $\mathcal{SLG}Op(G)$ of the actions of G on Lie structures are equivalent.*

Proof. Let us define a functor $\Gamma : \mathcal{SLG}Op(G) \to \mathcal{SLG}Cov(G)$ as follows. Suppose $\phi : G\ {}_{\alpha}\times_{w} M \to M$, $(a,x) \mapsto \phi(a,x) = {}^{a}x$ be smooth action of structured Lie groupoid G on a Lie structure M via Lie structure homomorphism $w : M \to G_0$. Then, from Example 3.4, the structured Lie groupoid $G \ltimes M$ with object manifold M is defined. Since $p : G \ltimes M \to G$ is defined by $(a,x) \mapsto a$ on the morphisms and by w on the objects, it is a covering morphism of structured Lie groupoids. That is, $\Gamma(M,w)$ is the covering morphism of structured Lie groupoids. If (M,w) and (M',w') are smooth actions, then $\Gamma(M,w)$ and $\Gamma(M',w')$ are smooth covering morphisms of structured Lie groupoids. Let us denote these smooth covering morphisms by $p : G \ltimes M \to G$ and $q : G \ltimes M' \to G$, respectively. If $f : M \to M'$ is a morphism of the smooth actions, then $\Gamma(f) = r$ is also a morphism of smooth covering morphisms with $r_0 = f$ and $r = 1 \times f$. However, if $f : M \to M'$ and $g : M' \to N$ are morphisms of the smooth actions, then $\Gamma(g \circ f) = \Gamma(g) \circ \Gamma(f)$. If we denote by $\Gamma(M,w) = G \ltimes M$, $\Gamma(M',w') = G \ltimes M'$, $\Gamma(N,w'') = G \ltimes N$, $\Gamma(f) = r$ and $\Gamma(g) = r'$, then we have $g \circ f : M \to N$ and hence $\Gamma(g \circ f) = r' \circ r = \Gamma(g) \circ \Gamma(f)$. Thus, Γ is a functor.

Secondly, let us define a functor $\Phi : \mathcal{SLG}Cov(G) \to \mathcal{SLG}Op(G)$ as follows. Let $p : \widetilde{G} \to G$ be a covering morphism of structured Lie groupoids. Let us take $M = \widetilde{G}_0$ and $w = p_0 : \widetilde{G}_0 \to G_0$. One can obtain a smooth action $(M = \widetilde{G}_0, w = p_0)$ by Example 3.3. That is, $\Phi(p)$ is a smooth action of the structured Lie groupoid G on a Lie structure. If $p : \widetilde{G} \to G$ and $q : H \to G$ are covering morphisms of structured Lie groupoids, then $\Phi(p)$ and $\Phi(q)$ are actions of structured Lie groupoid G on Lie structures $\widetilde{G}_0$ and H_0 via Lie structure homomorphisms p_0 and q_0, respectively. Let $(\widetilde{G}_0, p_0)$ and (H_0, q_0) be these smooth actions, respectively. If p and q are covering morphisms of structured Lie groupoids, then $r : \widetilde{G} \to H$ is also a covering morphism of structured Lie groupoids. Thus, if r is the morphism of the covering morphisms of structured Lie groupoids, then $\Phi(r) = f$ is also morphism of the smooth actions with

$r_0 = f$. Indeed, the diagram

$$\begin{array}{ccc} \widetilde{G}_0 & \xrightarrow{r_0=f} & H_0 \\ & {\scriptstyle p_0}\searrow \quad \swarrow{\scriptstyle q_0} & \\ & G_0 & \end{array}$$

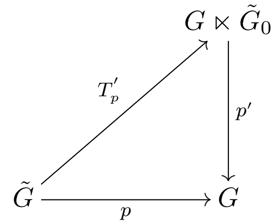
is commutative, because r is the morphism of covering morphisms of structured Lie groupoids. Furthermore, since r is the morphism of covering morphisms of structured Lie groupoids, we have $p = q \circ r$ and $p_0 = q_0 \circ r_0$. It is easily seen that the action is preserved by the following diagram

$$\begin{array}{ccc} G_\alpha \times_{p_0} \tilde{G}_0 & \xrightarrow{\phi} & \tilde{G}_0 \\ \Big\downarrow{\scriptstyle 1\times r_0} & & \Big\downarrow{\scriptstyle f=r_0} \\ G_\alpha \times_{q_0} H_0 & \xrightarrow{\phi'} & H_0 \end{array}$$

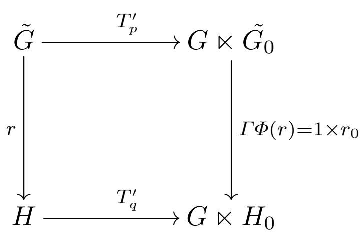
However, if a morphism from $p : \widetilde{G} \to G$ to $q : H \to G$ is $r : \widetilde{G} \to H$ and a morphism from $q : H \to G$ to $p' : H' \to G$ is $r' : H \to H'$, then $\Phi(r' \circ r) = \Phi(r') \circ \Phi(r)$. If we denote as $\Phi(p) = (\widetilde{G}_0, p_0)$, $\Phi(q) = (H_0, q_0)$, $\Phi(p') = (H'_0, p'_0)$, $\Phi(r) = f$ and $\Phi(r') = f'$, then $r' \circ r : \widetilde{G} \to H'$ is a covering morphism of structured Lie groupoids and hence $\Phi(r' \circ r) = f' \circ f = \Phi(r') \circ \Phi(r)$. Thus, Φ is a functor.

Let us now show that there exist natural equivalences $\Phi\Gamma \simeq 1_{\mathcal{SLGO}p(G)}$ and $\Gamma\Phi \simeq 1_{\mathcal{SLGC}ov(G)}$. Given a smooth action (M, w), there exist the structured Lie groupoid $G \ltimes M$ whose the object manifold is Lie structure $(G \ltimes M)_0 = M$, and the covering morphism $p : G \ltimes M \to G$ of structured Lie groupoids. Furthermore, $\Phi(\Gamma(M, w))$ gives a smooth action of structured Lie groupoid G on Lie structure $(G \ltimes M)_0 = M$ via smooth structure homomorphism $p_0 = w : (G \ltimes M)_0 = M \to G_0$. That is, $\Phi(\Gamma(M, w)) = (M, w)$. Thus, we have obtain $\Phi\Gamma = 1_{\mathcal{SLGO}p(G)}$.

Conversely, if $p : \widetilde{G} \to G$ is a covering morphism of structured Lie groupoids, then $\Phi(p)$ is a smooth action $\phi : G\ {}_\alpha{\times}_{p_0} \widetilde{G}_0 \to \widetilde{G}_0$ of the structured Lie groupoid G on Lie structure $\widetilde{G}_0$ via smooth structure homomorphism $p_0 : \widetilde{G}_0 \to G_0$. Furthermore, $\Gamma(\Phi(p))$ is also covering morphism of structured Lie groupoids, where the manifold of objects is $\widetilde{G}_0$ and the manifold of morphisms is $G \ltimes \widetilde{G}_0$. Now let us define natural transformation $T' : 1_{\mathcal{SLGC}ov(G)} \to \Gamma\Phi$. If $p : \widetilde{G} \to G$ is the covering morphism of structured Lie groupoids, then the map $T'_p : \widetilde{G} \to \Gamma\Phi(p) = G \ltimes \widetilde{G}_0$ is defined by identity on objects and by $\widetilde{a} \mapsto (a, \widetilde{x})$ on morphisms, where $\widetilde{a}$ is the lifting of a and $\widetilde{x}$ is the source of a. Since p and p' are smooth covering morphisms, T'_p is also a smooth covering morphism from [9]. For $\widetilde{a} \in \widetilde{G}$, $p(\widetilde{a}) = a$ and $p'(T'_p(a)) = p'(a, \widetilde{x}) = a$. That is, the following diagram is commutative



Thus, T'_p is a morphism of the covering morphisms of structured Lie groupoids. If $q : H \to G$ is another covering morphism of structured Lie groupoids, then the following diagram is commutative



Obviously, the inverse of T'_p is the morphism $(T'_p)^{-1} : G \ltimes \widetilde{G}_0 \to \widetilde{G}$ defined by identity on objects and by $(a, \widetilde{x}) \mapsto \widetilde{a}$ on morphisms. Thus, T' is a natural equivalence. So it follows $1_{\mathcal{SLG}Cov(G)} \simeq \Gamma\Phi$. $\square$

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