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Rings in which nilpotents belong to Jacobson radical

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Abstract Let R be a ring with identity and $J(R)$ denote the Jacobson radical of R. A ring R is called *J-reduced* if $a^n = 0$ ($n \in \mathbb{N}$) implies $a \in J(R)$ for any $a \in R$. We give, in this article, many characterizations of such rings. We construct some families of J-reduced rings. Furthermore, the transposes of invertible matrices over such rings are studied.

Keywords Reduced ring · J-reduced ring · Matrix extension

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1 Introduction

Throughout this paper, all rings are associative with identity unless otherwise stated. A ring is *reduced* if it has no nonzero nilpotent elements. In this paper, we introduce a class of rings which generalizes the so-called reduced rings. A ring R is called *J-reduced* if all the nilpotent elements of R belong to the Jacobson radical $J(R)$. Clearly, reduced rings are J-reduced, but the converse is not true in general. As is well known, in a commutative ring all nilpotent elements belong to the Jacobson radical. We investigate characterizations of J-reduced rings, and that many families of J-reduced rings are presented.

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Recall that a ring R is an *exchange ring* provided that for any $a \in R$ there exists an idempotent $e \in R$ such that $e \in aR$ and $1-e \in (1-a)R$. Furthermore, the transposes of invertible matrices over J-reduced exchange rings are investigated.

In what follows, $\mathbb{Z}$ and $\mathbb{Q}$ denote the ring of integers and the ring of rational numbers and for a positive integer n, $\mathbb{Z}_n$ is the ring of integers modulo n. We write $M_n(R)$ for the ring of all $n \times n$ matrices and $T_n(R)$ for the ring of all $n \times n$ upper triangular matrices over R. Also we write $R[x]$, $R[[x]]$, $N(R)$, $U(R)$, $P(R)$ and $J(R)$ for the polynomial ring, the power series ring over a ring R, the set of all nilpotent elements, the set of all invertible elements, the prime radical and the Jacobson radical of R, respectively.

2 *J*-reduced rings

We start with a definition and examples of a class of rings larger than the class of reduced rings. The goal of this section is to introduce and characterize J-reduced rings and investigate relations between this class of rings and other ones.

Definition 2.1 *A ring R is called J-reduced if $a^n = 0$ for $a \in R$ and for some positive integer n, then $a \in J(R)$.*

Every reduced ring is J-reduced and every 2-primal ring (ring with the prime radical consists of all nilpotent elements) is J-reduced.

We begin with a simple result.

Lemma 2.2 *If $R/J(R)$ is a reduced ring, then R is J-reduced.*

Proof. Assume that $a^n = 0$ for some $a \in R$ and $n \geq 2$. Then $\overline{a^n} = \overline{0} \in R/J(R)$. Since $R/J(R)$ is reduced, $\overline{a} = \overline{0}$ and so $a \in J(R)$, as asserted. □

The next example shows that the converse implication of Lemma 2.2 is not true in general, i.e., if R is J-reduced, then $R/J(R)$ need not be a reduced ring. Also we have that there are nontrivial rings which are not J-reduced.

Example 2.1 (1) Let R denote the localization of $\mathbb{Z}$ at $3\mathbb{Z}$. Consider the quaternions Q over the ring R, that is, a free R-module with basis $1, i, j, k$. Then Q is a noncommutative domain, and so it is J-reduced. On the other hand, $J(Q) = 3Q$, and $Q/J(Q)$ is isomorphic to 2×2 full matrix ring over $\mathbb{Z}_3$ via an isomorphism f defined by $f\big((a_0/b_0)1 + (a_1/b_1)i + (a_2/b_2)j + (a_3/b_3)k + 3Q\big) = \begin{bmatrix} a_0b_0^{-1}+a_1b_1^{-1}-a_2b_2^{-1} & a_1b_1^{-1}+a_2b_2^{-1}-a_3b_3^{-1} \\ a_1b_1^{-1}+a_2b_2^{-1}+a_3b_3^{-1} & a_0b_0^{-1}-a_1b_1^{-1}+a_2b_2^{-1} \end{bmatrix}$ for any $(a_0/b_0)1 + (a_1/b_1)i + (a_2/b_2)j + (a_3/b_3)k + 3Q \in Q/3Q$, where the entries of the matrix are read modulo the ideal (3) of $\mathbb{Z}$. Hence $Q/J(Q)$ has a nonzero nilpotent element. Therefore $Q/J(Q)$ is not reduced.

(2) Let R be a ring with an identity and A denote the matrix in $M_n(R)$ whose $(1, n)$-entry is 1 and all other entries are 0. Clearly $A^2 = 0$. Since $J(M_n(R)) = M_n(J(R))$, A can not belong to $J(M_n(R))$. Therefore $M_n(R)$ is not J-reduced.

The next example shows that there are J-reduced rings but not reduced.

Example 2.2 Let $R = \mathbb{Z}_4$. Since $J(R) = 2R$, $R/J(R) \cong \mathbb{Z}_2$ and so $R/J(R)$ is reduced. By Lemma 2.2, R is J-reduced but it is not reduced.

Let $J^{\#}(R)$ denote the subset $\{x \in R \mid \exists\, n \in \mathbb{N}$ such that $x^n \in J(R)\}$ of R. It is obvious that $J(R) \subseteq J^{\#}(R)$, but the following example shows that the reverse inclusion does not hold. Let R denote the ring $M_2(\mathbb{Z}_2)$. Then

$$J^{\#}(R) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}, \text{ while } J(R) = 0.$$

Proposition 2.3 *Let R be a ring. If $J(R) = J^{\#}(R)$, then R is J-reduced.*

Proof. Assume that $J(R) = J^{\#}(R)$ and $a^n = 0$ for some $n \geq 2$. Then $a^n \in J(R)$ and so $a \in J^{\#}(R) = J(R)$, as desired. □

By Proposition 2.3, we can say that J-reduced rings are abundant, for example, every commutative ring and every local ring are J-reduced. A ring R is *J-clean* if for any element $a \in R$, there exists an idempotent $e \in R$ such that $a - e \in J$.

Corollary 2.4 *Every J-clean ring is J-reduced.*

Proof. Assume that R is J-clean and let $a^n = 0$ for some $n \geq 2$. Then there exists an idempotent $e \in R$ such that $a - e = j \in J$. Since $a^n = 0$, $1 - a = 1 - e - j \in U(R)$ and so $1 - e \in U(R)$. Hence $e = 0$. That is, $a = j \in J(R)$, as asserted. □

Example 2.1 reveals that $R/J(R)$ may be not abelian even if R is J-reduced.

Proposition 2.5 *Let R be a ring in which idempotents lift modulo $J(R)$. If R is J-reduced, then $R/J(R)$ is abelian.*

Proof. Assume that R is J-reduced and let $\overline{e}^2 = \overline{e} \in R$. Since idempotents lift modulo $J(R)$, we may assume that $e^2 = e \in R$. If $r \in R$, then $(er - ere)^2 = 0$ and $(re - ere)^2 = 0$ and so $er - ere \in J(R)$ and $re - ere \in J(R)$ by assumption. Thus $er - re \in J(R)$ and so $\overline{er} = \overline{re}$, as required. □

Recall that a ring R is called to be *2-primal* if $P(R) = N(R)$.

Proposition 2.6 *Every 2-primal ring is J-reduced. The converse holds for left Artinian rings.*

Proof. If R is 2-primal, then $N(R) = P(R) = J(R)$, as desired. Conversely, assume that R is a left Artinian ring. By [4], we have $P(R) = J(R)$. If R is J-reduced, then $N(R) \subseteq J(R)$, and so it is 2-primal. □

A ring R is *semicommutative* in the case that $aRb = 0$ implies either a or b zero. In [7, Theorem 1.5] it is proved that every semicommutative ring is 2-primal. Immediately, we deduce that every semicommutative ring is J-reduced. But the converse is not true. Because the ring $T_2(\mathbb{Z})$ is J-reduced but not semicommutative.

Let $S(R)$ be the nonempty set of all ideals of a ring R generated by central idempotents. By Zorn's Lemma, $S(R)$ contains maximal elements. If P is a maximal element of the set $S(R)$, as usual, we say that R/P is a *Pierce stalk* of R, and that P is called a *Pierce ideal* of R.

Proposition 2.7 *Consider the following conditions for a ring R.*

(1) *R is J-reduced.*
(2) *R/P is J-reduced for every proper ideal P of R generated by central idempotents of R.*
(3) *All Pierce stalks of R are J-reduced.*

Then $(1) \Rightarrow (2) \Rightarrow (3)$.

Proof. $(1) \Rightarrow (2)$ Let $a \in R$ with $a^n \in P$ for some $n \geq 2$. Suppose that the ideal P is generated by central idempotents e_i, $i \in I$. Then there is a finite subset J of the set I such that $a^n \in \sum_{i\in J} e_iR$. Since J is a finite set, there is a central idempotent e of R such that $\sum_{i\in J} e_iR = eR \subseteq P$ by [8, Remark 3.8(3)]. Then $\big((1-e)a\big)^n = (1-e)a^n \in (1-e)eR = 0$. Since R is J-reduced, $(1-e)a \in J(R)$. Hence $\overline{a} - \overline{ea} = \overline{a} \in \big(J(R) + P\big)/P \subseteq J(R/P)$. Therefore R/P is J-reduced, as desired.

$(2) \Rightarrow (3)$ For every Pierce stalk R/P, the proper ideal P is generated by central idempotents. □

Theorem 2.8 *Let I be an ideal of a J-reduced ring S, and let R be a subring of S containing I. If R/I is J-reduced, then so is R.*

Proof. Given $a^n = 0$ in R, then $a \in J(S)$ as S is J-reduced. For any $r \in R$, we can find some $s \in S$ such that $s(1-ar) = 1$. Furthermore, $\overline{a} \in J\big(R/I\big)$. Hence, we can find some $t \in R$ such that $1-(1-ar)t \in I$. This implies that $s - s(1-ar)t \in I$, and so $s - t \in I$. We infer that $s \in R$. Therefore $1 - ar \in R$ is left invertible. Likewise, $1 - ar \in R$ is right invertible. Hence, $1 - ar \in U(R)$, and then $a \in J(R)$. Therefore R is J-reduced. □

Corollary 2.9 *Let I be an ideal of a J-reduced ring S, and let R be a J-reduced subring of S. Then $I + R$ is J-reduced.*

Proof. Obviously, $I \subseteq I + R \subseteq S$. It is easy to check that $(I + R)/I$ is J-reduced. Therefore $I + R$ is J-reduced, by Theorem 2.8. □

Corollary 2.10 *Every finite subdirect product of J-reduced rings is J-reduced.*

Proof. Suppose that R/I, R/J be J-reduced and $I \cap J = 0$. Let $\varphi : R \to R/I \oplus R/J, r \mapsto \big(r+I, r+J\big)$. Then $R \cong im(\varphi)$. By hypothesis, $R/I \oplus R/J$ and $im(\varphi)/\varphi(I) \cong R/I$ are J-reduced. In addition, $\varphi(I) \subseteq im(\varphi) \subseteq R/I \oplus R/J$. Accordingly, we complete the proof by Theorem 2.8. □

Theorem 2.11 *Let R be a ring. Then the following are equivalent:*

(1) *R is J-reduced.*
(2) *The ring $\{(x, y) \in R \times R \mid x - y \in J(R)\}$ is J-reduced.*

Proof. $(1) \Rightarrow (2)$ $S = \{(x, y) \in R \times R \mid x - y \in J(R)\}$ forms a subring of $R \times R$ with $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $(x_1, y_1)(x_2, y_2) = (x_1x_2, y_1y_2)$. Note that $x_1x_2 - y_1y_2 = x_1x_2 - x_1y_2 + x_1y_2 - y_1y_2 = x_1(x_2 - y_2) + (x_1 - y_1)y_2 \in J(R)$. Choose $P_1 = J(R) \times 0$ and $P_2 = 0 \times J(R)$. Then $P_1 \cap P_2 = 0$, $S/P_1 \cong S/P_2 \cong R$. Therefore S is the subdirect product of R and R, hence the result by Corollary 2.10.

$(2) \Rightarrow (1)$ Assume that $a^n = 0 \in R$. Then $(a, a)^n = (0, 0) \in S$. By hypothesis, $(a, a) \in J(S)$. For any $r \in R$, we see that $(1, 1) - (a, a)(r, r) \in U(S)$, and then $1 - ar \in U(R)$. This implies that $a \in J(R)$. Therefore R is J-reduced. □

Example 2.1 reveals that the homomorphic image of a J-reduced ring need not be J-reduced. Furthermore, we have

Lemma 2.12 *Let I be a nil ideal of a ring R. Then the following are equivalent:*

(1) *R is J-reduced.*
(2) *R/I is J-reduced.*

Proof. (1) $\Rightarrow$ (2) Write $\overline{a}^n = \overline{0}$, then $a^n \in I$. As I is nil, there exists some $m \in \mathbb{N}$ such that $a^{mn} = 0$. Hence, $a \in J(R)$, and so $\overline{a} \in J(R/I)$, as desired.

(2) $\Rightarrow$ (1) Assume that $a^n = 0$ for some $n \in \mathbb{N}$ and $a \in R$. Then $\overline{a}^n = \overline{0}$ and so $\overline{a} \in J(R/I)$. For any $r \in R$, $\overline{1} - \overline{ar} \in U(R/I)$, we have some $t \in R$ such that $1-(1-ar)t \in I$. Since I is nil, we get $(1-ar)t \in U(R)$, and so $1-ar \in U(R)$. This implies that $a \in J(R)$, as desired. □

But one can easily checks that a ring R is J-reduced if and only if $R/P(R)$ is J-reduced.

Lemma 2.13 *Let I and J be ideals of a ring R. If R/I and R/J are J-reduced, then so is $R/(IJ)$.*

Proof. Let $\varphi : R/(I \cap J) \to R/I$ given by $r + I \cap J \mapsto r + I$ and $\psi : R/(I \cap J) \to R/J$ given by $r + I \cap J \mapsto r + J$. Then φ and ψ are epimorphisms with $ker(\varphi) \cap ker(\psi) = 0$. Thus, $R/(I \cap J)$ is the subdirect product of R/I nd R/J. In view of Corollary 2.10, $R/(I \cap J)$ are J-reduced.

It is easy to verify that $IJ \subseteq I \cap J$, and that $R/(I \cap J) \cong R/(IJ)/(I \cap J)/(IJ)$. In addition, $\big((I \cap J)/(IJ)\big)^2 = 0$. It follows from Lemma 2.12 that $R/(IJ)$ is J-reduced. □

Theorem 2.14 *Let I be an ideal of a ring R. Then the following are equivalent:*

(1) *R/I is J-reduced.*
(2) *R/I^n is J-reduced for all $n \in \mathbb{N}$.*
(3) *R/I^n is J-reduced for some $n \in \mathbb{N}$.*

Proof. (1) $\Rightarrow$ (2) is clear by Lemma 2.13 and induction.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Let $a \in R$ be such that $\overline{a}^m = 0$ in R/I. Then $a^m \in I$; hence, $a^{mn} \in I^n$. That is, $\overline{a}^{mn} = 0$ in R/I^n. By hypothesis, $\overline{a} \in J(R/I^n)$. For any $r \in R$, $\overline{1} - \overline{ar} \in U(R/I^n)$. Thus, we can find some $b \in R$ such that $1-(1-ar)b \in I^n \subseteq I$. This implies that $\overline{1-ar} \in U(R/I)$. Accordingly, $\overline{a} \in J(R/I)$. Therefore R/I is J-reduced. □

3 Certain extensions

The goal of this section is to consider some extensions of J-reduced rings, and then construct more of such rings.

Lemma 3.1 *A ring R is J-reduced if and only if so is eRe for all idempotent $e \in R$.*

Proof. If $(eae)^n = 0$ in eRe, then $(eae)^n = 0$ in R. Since R is J-reduced, $eae \in J(R)$, and so $eae \in eJ(R)e$. That is, $eae \in J(eRe)$. Therefore eRe is J-reduced. The converse is trivial. □

Proposition 3.2 *Let R be a ring. Then the following are equivalent:*

(1) *R is J-reduced.*
(2) $S = \left\{ \begin{bmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{bmatrix} \mid a, a_{ij} \in R \ (i < j) \right\}$ *is J-reduced.*

Proof. One direction is obvious from Lemma 3.1. Conversely, choose

$$I = \left\{ \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \mid a_{ij} \in R \ (i < j) \right\}.$$

Then $I^n = 0$ and $S/I \cong R$, and so we complete the proof by Lemma 2.12. □

Corollary 3.3 *Let R be a ring. Then the following are equivalent:*

(1) *R is J-reduced.*
(2) *$R[x]/(x^n)$ is J-reduced for all $n \geq 2$.*

Proof. Since

$$R[x]/(x^n) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ 0 & 0 & a_1 & \ddots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 & a_2 \\ 0 & 0 & 0 & \cdots & 0 & a_1 \end{bmatrix} \mid a_i \in R \right\},$$

we get the result by Proposition 3.2. □

Let S and T be any rings, M an S-T-bimodule and R the formal triangular matrix ring $\begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$. It is well-known that $J(R) = \begin{bmatrix} J(S) & M \\ 0 & J(T) \end{bmatrix}$.

Proposition 3.4 *Let $R = \begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$. Then R is J-reduced if and only if S and T are J-reduced.*

Proof. The necessity is obvious by Lemma 3.1. Assume that S and T are J-reduced and $A = \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \in R$ with $A^n = \begin{bmatrix} a^n & * \\ 0 & b^n \end{bmatrix} = 0$ for some $n \geq 2$. Then $a^n = 0 = b^n$ and so $a \in J(S)$ and $b \in J(T)$ by assumption. Thus $A \in J(R)$ and so R is J-reduced. □

Theorem 3.5 *Let R be a ring. Then the following are equivalent:*
(1) *R is J-reduced.*
(2) *$T_n(R)$ is J-reduced for all $n \in \mathbb{N}$.*

Proof. (1) $\Rightarrow$ (2) Let $[a_{ij}] \in T_n(R)$ with $[a_{ij}]^m = 0$ for some $m \geq 2$. Then $a_{ii}^m = 0$ and so by (1), $a_{ii} \in J(R)$ for any $i = 1, 2, \ldots, n$. Thus $[a_{ij}] \in J(T_n(R))$ and so $T_n(R)$ is J-reduced for all $n \in \mathbb{N}$.

(2) $\Rightarrow$ (1) is obvious from Lemma 3.1. □

A ring R is called *semipotent* if each left ideal (resp., right ideal) not contained in $J(R)$ contains a non-zero idempotent.

Corollary 3.6 *If R is an abelian semipotent ring, then $T_n(R)$ is J-reduced.*

Proof. Let $a^n = 0$ for some $n \geq 2$ and let $a \notin J(R)$. Then there is a non-zero idempotent $e \in aR$. Write $e = ar$ with $re = r$, so ra is an idempotent. Since R is abelian, ra is central and so $e = e^n = (ar)^n = r^{n-1}a^n r = 0$, a contradiction. Hence $a \in J(R)$. Therefore R is J-reduced, and then the result follows, by Theorem 3.5. □

Since every exchange ring is semipotent by [6, Proposition 1.9], the ring of all triangular matrices over abelian exchange rings are J-reduced.

Let R be a ring and V an R-R-bimodule which is a general ring (possibly with no unity) in which $(vw)r = v(wr)$, $(vr)w = v(rw)$ and $(rv)w = r(vw)$ hold for all $v, w \in V$ and $r \in R$. Then *ideal-extension* (it is also called *Dorroh extension*) $I(R; V)$ of R by V is defined to be the additive abelian group $I(R; V) = R \oplus V$ with multiplication $(r, v)(s, w) = (rs, rw + vs + vw)$.

Proposition 3.7 *Suppose that for any $v \in V$ there exists $w \in V$ such that $v+w+vw = 0$. Then the following are equivalent for a ring R:*

(1) *R is J-reduced.*
(2) *An ideal-extension $S = I(R; V)$ is J-reduced.*

Proof. (1) $\Rightarrow$ (2) Let $s = (r, v) \in S$ with $s^n = (r^n, *) = (0, 0)$ for some $n \geq 2$. Then $r^n = 0$ and so $r \in J(R)$ by (1). Note that $(0, V) \subseteq J(S)$ by hypothesis. Since $s = (r, v) = (r, 0) + (0, v)$, it suffices to show that $(r, 0) \in J(S)$. For any $(x, y) \in S$, $(1, 0) - (r, 0)(x, y) = (1 - rx, -ry) \in U(S)$ because $(1 - rx, -ry) = (1 - rx, 0)(1, (1 - rx)^{-1}(-ry))$ and $(1, (1 - rx)^{-1}(-ry)) = (1, 0) + (0, (1 - rx)^{-1}(-ry)) \in U(S)$ by $(0, V) \subseteq J(S)$. Thus $s = (r, v) \in J(S)$.

(2) $\Rightarrow$ (1) Suppose that S is J-reduced and let $a \in R$ with $a^n = 0$ for some $n \geq 2$. Then $(a, 0)^n = (a^n, 0) = (0, 0)$ and so $(a, 0) \in J(S)$ by (2). Therefore $a \in J(R)$, as desired. □

If R is a ring and $\sigma : R \to R$ is a ring homomorphism, let $R[[x, \sigma]]$ denote the ring of skew formal power series over R; that is all formal power series in x with coefficients from R with multiplication defined by $xr = \sigma(r)x$ for all $r \in R$. In particular, $R[[x]] = R[[x, 1_R]]$ is the ring of formal power series over R. Note that $J(R[[x, \sigma]]) = J(R) + < x >$. Since $R[[x, \sigma]] \cong I(R; < x >)$ where $< x >$ is the ideal generated by x, Proposition 3.7 gives the next result.

Corollary 3.8 *Let R be a ring and $\sigma : R \to R$ a ring homomorphism. Then the following are equivalent:*

(1) *R is J-reduced.*
(2) *$R[[x,\sigma]]$ is J-reduced.*

We say that B is a *subring* of a ring A if $\emptyset \neq B \subseteq A$ and for any $x, y \in B$, $x-y, xy \in B$ and $1_A \in B$. Let A be a ring and B a subring of A and $R[A,B]$ denote the set $\{(a_1, a_2, \cdots, a_n, b, b, \cdots) : a_i \in A, b \in B,\ 1 \leq i \leq n\}$. Then $R[A,B]$ is a ring under the componentwise addition and multiplication. Also $J(R[A,B]) = R[J(A), J(A) \cap J(B)]$.

Proposition 3.9 *Let A be a ring and a subring B of A. The following are equivalent:*

(1) *A and B are J-reduced.*
(2) *$R[A,B]$ is J-reduced.*

Proof. (1) $\Rightarrow$ (2) Let $(a_1, \cdots, a_n, b, b, \cdots) \in R[A,B]$ with $(a_1, \cdots, a_n, b,\ b, \cdots)^m = 0$ for some $m \geq 2$. Then $(a_1^m, \cdots, a_n^m, b^m, b^m, \cdots) = 0$. This implies that $a_i^m = 0 = b^m$ for $i = 1, \ldots, n$. By assumption, $a_i, b \in J(A)$ for $i = 1, \ldots, n$ and $b \in J(B)$. Therefore $(a_1, \cdots, a_n, b, b, \cdots) \in J(R[A,B])$.

(2) $\Rightarrow$ (1) Let $a \in A$ with $a^t = 0$ for some $t \geq 2$. Then $(a, 0, 0, \cdots)^t = (a^t, 0, 0, \cdots) = 0$. By (2), we have $(a, 0, 0, \cdots) \in J(R[A,B])$, and so $a \in J(A)$. Therefore A is J-reduced. Let $b \in B$ with $b^k = 0$ for some $k \geq 2$. Then $(0, b, b, \cdots)^k = (0, 0, 0, \cdots) \in J(R[A,B])$. Since $R[A,B]$ is J-reduced, $(0, b, b, \cdots) \in J(R[A,B])$. Hence $b \in J(B)$, as desired. $\square$

Let R be a ring, and let $s \in R$ be central. Following Krylov and Tuganbaev [3], we use $K_s(R)$ to denote the set $\{[a_{ij}] \in M_2(R) \mid \text{each } a_{ij} \in R\}$ with the following operations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix},$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} aa'+sbc' & ab'+bd' \\ ca+dc' & scb'+dd' \end{bmatrix}.$$

Theorem 3.10 *Let R be a ring and $s \in R$ be a central nilpotent element. The following are equivalent:*

(1) *R is J-reduced.*
(2) *$K_s(R)$ is J-reduced.*

Proof. (1) $\Rightarrow$ (2) Suppose $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^n = 0$ in $K_s(R)$. Then we can find some $r, t \in R$ such that $\begin{bmatrix} a^n + sr & * \\ * & d^n + st \end{bmatrix} = 0$; hence, $a^n + sr = d^n + st = 0$. As $s \in R$ is a central nilpotent element, we write $s^m = 0$ for some $m \in \mathbb{N}$. Then $a^{mn} = d^{mn} = 0$. Since R is J-reduced, we get $a, d \in J(R)$. Therefore $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in J\big(K_s(R)\big)$, by [3]. Accordingly, $K_s(R)$ is J-reduced.

(2) $\Rightarrow$ (1) Choose $e = diag(1,0) \in K_s(R)$. Then $R \cong eK_s(R)e$, and we therefore obtain the result by Lemma 3.1. $\square$

Corollary 3.11 *Let R be a ring. The following are equivalent:*

(1) *R is J-reduced.*
(2) *$K_0(R)$ is J-reduced.*

Proof. It is an immediate consequence of Theorem 3.10. □

For instance, $K_0(\mathbb{Z}_4) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{Z}_4 \right\}$ is J-reduced, where the additive and multiplicative operations are defined as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix},$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} aa' & ab'+bd' \\ ca+dc' & dd' \end{bmatrix}.$$

But we note that $M_2(\mathbb{Z}_4)$ is not J-reduced.

Corollary 3.12 *Let R be a ring. The following are equivalent:*

(1) *R is J-reduced.*
(2) *$K_{x^m}\big(R[x]/(x^n)\big)$ is J-reduced for all $1 \leq m \leq n$.*

Proof. (1) $\Rightarrow$ (2) In view of Corollary 3.3, $R[x]/(x^n)$ is J-reduced. Choose $s = x^m$. Applying Theorem 3.10 to $R[x]/(x^n)$, $K_{x^m}\big(R[x]/(x^n)\big)$ is J-reduced.

(2) $\Rightarrow$ (1) Clearly, $R[x]/(x^n) \cong eK_{x^m}\big(R[x]/(x^n)\big)e$ for some idempotent $e \in K_{x^m}\big(R[x]/(x^n)\big)$. In light of Lemma 3.1, $R[x]/(x^n)$ is J-reduced. Therefore we complete the proof, by Corollary 3.3. □

4 Exchange properties

The class of exchange rings is very large. It includes all regular rings, all π-regular rings, all strongly π-regular rings, all semiperfect rings, all left or right continuous rings, all clean rings, all unit C^*-algebras of real rank zero and all right semi-Artinian rings. The aim of this section is to consider J-reduceness for such rings.

Theorem 4.1 *Let R be an exchange ring. Then the following are equivalent:*
(1) *R is J-reduced.*
(2) *$R/J(R)$ is reduced.*

Proof. (2) $\Rightarrow$ (1) is obvious from Lemma 2.2.

(1) $\Rightarrow$ (2) Suppose that $R/J(R)$ is not reduced. Then there exists some $a \in R/J(R)$ such that $a^2 = 0$, but $a \neq 0$. In view of [5, Theorem 2.1], the principal ideal (a) contains a system $\{e_{11}, e_{12}, e_{21}, e_{22}\}$ of 2×2 matrix units. As $e_{11}^2 = e_{11} \in R/J(R)$, we can find an idempotent $f \in R$ such that $e_{11} = \overline{f}$. For any $r \in R$, $\big(fr(1-f)\big)^2 = 0$, and so $fr(1-f) \in J(R)$. This implies that $\overline{fr} = \overline{frf}$. Likewise, $\overline{rf} = \overline{frf}$. Hence $e_{11}\overline{r} = \overline{fr} = \overline{frf} = \overline{rf} = \overline{r}e_{11}$. Choose $\overline{r} = e_{12}$. Then $e_{11}e_{12} = e_{12} = e_{12}e_{11} = 0$, a contradiction. Therefore $R/J(R)$ is reduced. □

Corollary 4.2 *Let R be an exchange ring. Then the following are equivalent:*
(1) *R is J-reduced.*
(2) *R is right (left) quasi-duo.*
(3) *R is quasi-normal.*

Proof. In light of Theorem 4.1, R is J-reduced if and only if $R/J(R)$ is reduced. Therefore the result follows from [9, Lemma 3.5]. □

Let R be a ring and $f(x) = \sum_{i=0}^{n} a_i x^i,\ g(x) = \sum_{j=0}^{m} b_j x^j \in R[x]$. Then R is called *J-Armendariz* if $f(x)g(x) = 0$, then $a_i b_j \in J(R)$ for all $0 \le i \le n$, $0 \le j \le m$.

Corollary 4.3 *Every J-reduced exchange ring is J-Armendariz.*

Proof. Let R be a J-reduced exchange ring. Then $R/J(R)$ is reduced by Theorem 4.1. Suppose that $f(x)g(x) = 0$ and without loss of generality we can assume that $n = m$. Then we have

$$\begin{aligned} a_0b_0 &= 0 && (1)\\ a_0b_1 + a_1b_0 &= 0 && (2)\\ a_0b_2 + a_1b_1 + a_2b_0 &= 0 && (3)\\ &\dots && \dots\\ a_nb_0 + \ldots + a_0b_n &= 0 && (n) \end{aligned}$$

Since $a_0b_0 = 0$ and $R/J(R)$ is reduced, $(b_0a_0)^2 = 0$ and so $b_0a_0 \in J(R)$. Hence left multiplying (2) by a_1b_0 yields $(a_1b_0)^2 + a_1b_0a_0b_1 = 0$ hence $(a_1b_0)^2 \in J(R)$ and so $a_1b_0 \in J(R)$. Since $a_1b_0 \in J(R)$ and $R/J(R)$ is reduced, $(b_0a_1)^2 = b_0a_1b_0a_1 \in J(R)$ and so $b_0a_1 \in J(R)$. Hence left multiplying (3) by a_2b_0 yields $(a_2b_0)^2 + a_2b_0a_1b_1 + a_2b_0a_0b_2 = 0$ hence $(a_2b_0)^2 \in J(R)$ and so $a_2b_0 \in J(R)$. Similarly we get $a_ib_0 \in J(R)$ for all $1 \le i \le n$. Thus

$$\begin{aligned} a_0b_1 &\in J(R) && (1')\\ a_1b_1 + a_0b_2 &\in J(R) && (2')\\ &\dots && \dots\\ a_{n-1}b_1 + \ldots + a_0b_n &\in J(R) && ((n-1)') \end{aligned}$$

Again using the fact that $a_0b_1 \in J(R)$ implies $b_1a_0 \in J(R)$ we conclude from $(2')$ that $a_1b_1 \in J(R)$ and then similarly that $a_ib_1 \in J(R)$ for all $1 \le i \le n$. Repetition yields $a_ib_j \in J(R)$ for all $0 \le i, j \le n$, as required. □

Corollary 4.3 is very useful to supply examples for J-Armendariz rings.

As is well known, the transpose of an invertible matrix over a noncommutative ring may be not invertible. In [1, Theorem 2.3], Gupta et al. proved that every transpose of an invertible matrix over a ring R is invertible if and only if $R/J(R)$ is commutative. In the next, we will characterize the J-reduced exchange rings over which the transpose of every invertible square matrix is invertible in term of commutators of invertible elements, and that give more several explicit results.

Lemma 4.4 *Let R be a semiprimitive exchange ring with all idempotents central. Then the following are equivalent:*

(1) *R is commutative.*
(2) *For any $u, v \in U(R)$, $[u, v] \in Z(R)$.*

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1) Clearly, R is a clean ring. Let $a, b \in R$. Then there exist idempotents $e, f \in R$ and units $u, v \in U(R)$ such that $a = e + u$ and $b = f + v$. Thus, $[a, b] = (e+u)(f+v) - (f+v)(e+u) = ef + ev + uf + uv - fe - fu - ve - vu = [u, v]$. By hypothesis, $[a, b] \in Z(R)$. According to [1, Theorem 2.2], R is commutative. $\square$

If R has stable range one, we note that $J(R) = \{x \in R \mid x - u \in U(R)$ for any $u \in U(R)\}$.

Theorem 4.5 *If R is a J-reduced exchange ring, then the following are equivalent:*

(1) *$R/J(R)$ is commutative.*
(2) *For all n, $GL_n(R)$ is closed under transposition.*
(3) *For all $a, b, c \in U(R)$, $c + [a, b] \in U(R)$.*
(4) *For all $a, b \in U(R)$, $[a, b] \in J(R)$.*

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3) For all $a, b, c \in U(R)$, $c - ba + ba = c \in U(R)$. We infer that $\begin{bmatrix} b & c-ba \\ -1 & a \end{bmatrix} \in GL_2(R)$; hence, $\begin{bmatrix} b & -1 \\ c-ba & a \end{bmatrix} \in GL_2(R)$. This implies that $c - ba + ab \in U(R)$, and so $c + [a, b] \in U(R)$.

(3) $\Rightarrow$ (4) Let $a, b \in U(R)$. Then $c + [a, b] \in U(R)$ for all $c \in U(R)$. Clearly, $R/J(R)$ is an exchange ring. Let $\overline{e}^2 = \overline{e} \in R/J(R)$. Then we may assume that $e^2 = e \in R$. Let $x \in R$. Then $\big(ex(1-e)\big)^2 = 0$. As R is J-reduced, we deduce that $ex(1-e) \in J(R)$, and so $\overline{ex} = \overline{exe}$. Likewise, $\overline{xe} = \overline{exe}$. Hence, $\overline{ex} = \overline{xe}$. That is, $R/J(R)$ is abelian. Therefore it has stable range one. This yields that R has stable range one. In view of Lemma 4.4, $[a, b] \in J(R)$.

(4) $\Rightarrow$ (1) Let $S = R/J(R)$. Then S is an exchange ring with all idempotents central. Moreover, $J(S) = 0$, i.e., S is semiprimitive. For any $\overline{a}, \overline{b} \in U(S)$, we have $a, b \in U(R)$. By hypothesis, $1 + [a, b] \in U(R)$ as $[a, b] \in J(R)$. Set $w = 1 + [a, b]$. For any $r \in R$, in light of Lemma 4.4, it will suffice to show that $\overline{wr} = \overline{rw}$. Obviously, $R/J(R)$ is clean. Thus, we can find $e \in R$ such that $\overline{r} - \overline{e} \in U\big(R/J(R)\big)$ and $\overline{e} = \overline{e^2}$. Hence, $u := r - e \in U(R)$ and $e - e^2 \in J(R)$. One easily checks that $[w, r] = wr - rw = w(e+u) - (e+u)w = [w, u] + [w, e]$. Obviously, $[w, e] \in J(R)$. By hypothesis, $[w, u] \in J(R)$, and so $[w, r] \in J(R)$. Therefore $\overline{w} \in Z(S)$, and then $[\overline{a}, \overline{b}] \in Z(S)$. According to Lemma 4.4, S is commutative. $\square$

Let $R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$. Then R is a J-reduced exchange ring. For all $a, b \in U(R)$, it is easy to verify that $ab = ba$. That is, all units in R commutate. Thus, $[a, b] = 0 \in J(R)$. By Theorem 4.5, $R/J(R)$ is commutative. Choose $u = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in U(R)$. Then $u \in R$ is not central. In this case, R is not commutative.

Corollary 4.6 *Let R be an abelian exchange ring. Then the following are equivalent:*

(1) *$R/J(R)$ is commutative.*
(2) *For all n, $GL_n(R)$ is closed under transposition.*
(3) *For all $a, b, c \in U(R)$, $c + [a, b] \in U(R)$.*
(4) *For all $a, b \in U(R)$, $[a, b] \in J(R)$.*

Proof. Let $S = R/J(R)$. Then R is a J-reduced exchange ring by Corollary 3.6, and so the result follows. □

Let $a, b, c \in R$. Following Herstein, we write $[a, b, c] = abc - cba$, and call such elements *generalized commutators* in R (cf. [2]).

Corollary 4.7 *If R is a J-reduced exchange ring, then the following are equivalent:*

(1) *$R/J(R)$ is commutative.*
(2) *For all n, $GL_n(R)$ is closed under transposition.*
(3) *For all $a, b, c \in U(R)$, $1 + [a, b, c] \in U(R)$.*

Proof. (1) $\Leftrightarrow$ (2) is obvious from [1, Theorem 2.3].

(1) $\Rightarrow$ (3) For all $a, b, c \in U(R)$, $\overline{1 + [a, b, c]} = \overline{1 + abc - cba} = \overline{1} \in R/J(R)$. Therefore $1 + [a, b, c] \in U(R)$.

(3) $\Rightarrow$ (1) For all $a, b, c \in U(R)$, $c + [a, b] = c + ab - ba = c\big(1 + c^{-1}(ab - ba)\big) = c\big(1 + (c^{-1}a)c(c^{-1}b) - (c^{-1}b)c(c^{-1}a)\big) = c\big(1 + [c^{-1}a, c, c^{-1}b]\big) \in U(R)$. Thus, the result follows from Theorem 4.5. □

Let R be an abelian exchange ring. Then $R/J(R)$ is commutative if and only if for all n, $GL_n(R)$ is closed under transposition, if and only if for all $a, b, c \in U(R)$, $1 + [a, b, c] \in U(R)$.

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