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More accurate converses of the Jensen and the Lah-Ribarič operator inequality

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Abstract In this paper we derive several converses of the Jensen and the Lah-Ribarič operator inequality regarding convexity in the classical real sense. In such a way, we obtain refinements of some recent results known from the literature.

Keywords Jensen operator inequality · Lah-Ribarič operator inequality · Convexity · Operator convexity · Converse · Refinement

Mathematics Subject Classification (2010) 47A63 · 47A64

1 Introduction

One of the most significant inequalities in contemporary mathematics is the well-known Jensen inequality (see, e.g., [8]). Although classical, this inequality also plays an important role in functional analysis. The starting point in this paper is a rather general operator form of the Jensen inequality, established in paper [3]. In order to state that result, we first introduce the appropriate setting.

Let T be a locally compact Hausdorff space and let $\mathcal{A}$ be a C^*-algebra. A field $(x_t)_{t\in T}$ of elements in $\mathcal{A}$ is said to be continuous if the function $t \to x_t$ is norm continuous on T. In addition, if T is equipped with a Radon measure μ and the function $t \to \|x_t\|$ is integrable, then, the so-called Bochner integral $\int_T x_t d\mu(t)$ can be formed. More

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precisely, the Bochner integral is the unique element in $\mathcal{A}$ such that the relation

$$\varphi\left(\int_T x_t d\mu(t)\right) = \int_T \varphi\left(x_t\right) d\mu(t)$$

holds for every linear functional φ in the norm dual $\mathcal{A}^*$ (see [2]).

Assume furthermore that there is a field $(\phi_t)_{t\in T}$ of positive linear mappings $\phi_t : \mathcal{A} \to \mathcal{B}$ from $\mathcal{A}$ to another C^*-algebra $\mathcal{B}$. Such field is said to be continuous if the function $t \to \phi_t(x)$ is continuous for every $x \in \mathcal{A}$. In addition, if the C^*-algebras are unital and the field $t \to \phi_t(\mathbf{1})$ is integrable with integral $\mathbf{1}$, we say that $(\phi_t)_{t\in T}$ is unital.

Now, if $f : I \to \mathbb{R}$ is an operator convex function, where I is a real interval of any type, and $(\phi_t)_{t\in T}$ is a unital field, then the Jensen operator inequality (see Hansen et.al. [3]) asserts that

$$f\left(\int_T \phi_t(x_t) d\mu(t)\right) \leq \int_T \phi_t(f(x_t)) d\mu(t), \tag{1.1}$$

holds for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathcal{A}$ with spectra contained in I. Moreover, if $f : I \to \mathbb{R}$ is an operator concave function, then the sign of inequality in (1.1) is reversed.

Note that the above inequality refers to operator convex function. Recall that a continuous function $f : I \to \mathbb{R}$ is operator convex if the inequality $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ holds for each $\lambda \in [0,1]$ and every pair of self-adjoint operators x and y (acting) on an infinite dimensional Hilbert space $\mathcal{H}$ with spectra in I. The ordering is defined by setting $x \leq y$ if $y - x$ is positive semi-definite.

In the same paper, Hansen *et.al.* obtained the following inequality that holds for a convex function $f : [m, M] \to \mathbb{R}$ in a classical real sense (see [3], the proof of Theorem 2):

$$\int_T \phi_t(f(x_t)) d\mu(t) \leq \alpha_f \int_T \phi_t(x_t) d\mu(t) + \beta_f \mathbf{1}, \tag{1.2}$$

where $\alpha_f = \frac{f(M)-f(m)}{M-m}$ and $\beta_f = \frac{Mf(m)-mf(M)}{M-m}$. Clearly, the inequality (1.2) can be rewritten in the following form:

$$\begin{aligned}\int_T \phi_t(f(x_t)) d\mu(t) \leq{}& \frac{M\mathbf{1} - \int_T \phi_t(x_t) d\mu(t)}{M-m} f(m) \\ &+ \frac{\int_T \phi_t(x_t) d\mu(t) - m\mathbf{1}}{M-m} f(M). \end{aligned}\tag{1.3}$$

It should be noticed here that the operator inequality (1.3) follows after applying the functional calculus to the well-known inequality

$$f(t) \leq \frac{M-t}{M-m} f(m) + \frac{t-m}{M-m} f(M), \tag{1.4}$$

that holds for every convex function f on the interval $[m, M]$. Observe that $l(t) = \frac{M-t}{M-m} f(m) + \frac{t-m}{M-m} f(M)$ is the linear function limiting convex function $f(t)$ on interval

$[m, M]$ from above. On the other hand, inequality (1.4) is a particular case of the so-called Lah-Ribarič inequality which asserts that if $f : [m, M] \to \mathbb{R}$ is a convex function, $\sum_{i=1}^{n} p_i = 1$, $p_i > 0$, $i = 1, 2, \ldots, n$, then $\sum_{i=1}^{n} p_i f(t_i) \leq \frac{M-\bar{t}}{M-m} f(m) + \frac{\bar{t}-m}{M-m} f(M)$, where $t_i \in [m, M]$, $i = 1, 2, \ldots, n$, and $\bar{t} = \sum_{i=1}^{n} p_i t_i$. For more details about the Lah-Ribarič inequality, the reader is referred to [5], [6] and [8]. Therefore, in this paper the operator inequality (1.3) will be referred to as the Lah-Ribarič operator inequality.

While the Jensen operator inequality (1.1) holds for operator convex functions, it is interesting that the corresponding converse relations are related with the convexity in the classical real sense. More precisely, with regard to the above operator setting, Jakšić et al. [4], proved that the series of inequalities

$$\begin{aligned}
&\int_T \phi_t(f(x_t))d\mu(t) - f\left(\int_T \phi_t(x_t)d\mu(t)\right) \\
&\leq \frac{f'_-(M) - f'_+(m)}{M-m}\left(M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}\right) \qquad (1.5)\\
&\leq \frac{1}{4}(M-m)(f'_-(M) - f'_+(m))\mathbf{1}
\end{aligned}$$

holds for a continuous convex function $f : I \to \mathbb{R}$ such that the interval $[m, M]$ belongs to the interior of the interval I and provided that $(x_t)_{t\in T}$ is bounded continuous field of self-adjoint elements in $\mathcal{A}$ with spectra in $[m, M]$. Obviously, the series of inequalities in (1.5) represents the converse of the Jensen inequality (1.1).

In the same paper the authors also obtained the following converse of the Lah-Ribarič operator inequality

$$\begin{aligned}
0 &\leq \frac{M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)}{M-m} f(m) \\
&\quad + \frac{\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}}{M-m} f(M) - \int_T \phi_t(f(x_t))d\mu(t) \\
&\leq \frac{f'_-(M) - f'_+(m)}{M-m} \int_T \phi_t\left([M\mathbf{1} - x_t][x_t - m\mathbf{1}]\right) d\mu(t) \qquad (1.6)\\
&\leq \frac{f'_-(M) - f'_+(m)}{M-m}\left(M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}\right) \\
&\leq \frac{1}{4}(M-m)(f'_-(M) - f'_+(m))\mathbf{1},
\end{aligned}$$

that holds under the same assumptions as (1.5).

The main objective of this paper is to derive another class of converses for the Jensen inequality (1.1) and the Lah-Ribarič inequality (1.3), that are also valid for convex functions in the classical real sense. In such a way we shall obtain a more accurate converses than the converses presented in this Introduction.

The paper is organized in the following way: after this introduction, in Sections 2 and 3 we derive our main results, that is, we obtain the whole series of converses that correspond to the Jensen and the Lah-Ribarič operator inequality, some of which improve stated converses (1.5) and (1.6).

The techniques that will be used in the proofs are mainly based on the classical real and functional calculus, especially on the well-known monotonicity principle for self-adjoint elements of a C^*-algebra $\mathcal{A}$: If $x \in \mathcal{A}$ with a spectra $Sp(x)$, then

$$f(t) \geq g(t),\ t \in \mathrm{Sp}(x) \Longrightarrow f(x) \geq g(x), \tag{1.7}$$

where f and g are real valued continuous functions (for more details, see [7]).

2 Converses of the Jensen operator inequality

The main objective of this section is to derive two different converses of the Jensen operator inequality in a general setting. As we shall see below, one of them refines the series of relations (1.5), presented in the introduction. Such improved relations are also accompanied with a convexity in the classical real sense.

In order to present our basic results, we define

$$\varDelta_f(t;m,M) = \frac{1}{M-m}\left[\frac{f(M)-f(t)}{M-t} - \frac{f(t)-f(m)}{t-m}\right], \tag{2.1}$$

where $m < M$ and $f : I \to \mathbb{R}$ is a continuous convex function such that the interval $[m, M]$ belongs to the interior of interval I. Observe that expression (2.1) is actually the second order divided difference of the function f at points m, t, and M, for every $t \in (m, M)$.

Remark 2.1 Observe that the function f is defined on the interval I whose interior contains interval $[m, M]$. This condition ensures finiteness of one-sided derivatives at points m and M. Then,

$$\lim_{t\to m^+} \varDelta_f(t;m,M) = \frac{1}{M-m}\left[\frac{f(M)-f(m)}{M-m} - f'_+(m)\right],$$
$$\lim_{t\to M^-} \varDelta_f(t;m,M) = \frac{1}{M-m}\left[f'_-(M) - \frac{f(M)-f(m)}{M-m}\right],$$

so $\varDelta_f(\cdot;m,M)$ may be regarded as a continuous function (in parameter t) on the interval $[m, M]$. Therefore, if x is a self-adjoint element in C^*-algebra with spectra contained in $[m, M]$, then the expression $\varDelta_f(x;m,M)$ is also meaningful. Clearly, this assertion holds due to functional calculus.

Now we give two series of converses for the Jensen operator inequality. One of them refines series (1.5), established in paper [4]. The classical real version of the following theorem was proved by Dragomir in recent paper [1]. In fact, such scalar series of inequalities will be exploited in establishing the corresponding operator forms.

Theorem 2.1 *Let $f : I \to \mathbb{R}$ be a continuous convex function, and let $m, M \in \mathbb{R}$, $m < M$, be such that interval $[m, M]$ belongs to the interior of interval I. Further, suppose $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras, and $(\phi_t)_{t\in T}$ is a unital field of positive linear* *mappings* $\phi_t : \mathcal{A} \to \mathcal{B}$ *defined on a locally compact Hausdorff space* T *with a bounded Radon measure* μ. *Then the series of inequalities*

$$\begin{aligned} &\int_T \phi_t(f(x_t))d\mu(t) - f\left(\int_T \phi_t(x_t)d\mu(t)\right) \\ &\le \sup_{m<t<M} \Delta_f(t;m,M)\left(M\mathbf{1}- \int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t)-m\mathbf{1}\right) \\ &\le \frac{f'_-(M)-f'_+(m)}{M-m}\left(M\mathbf{1}- \int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t)-m\mathbf{1}\right) \\ &\le \frac{1}{4}(M-m)(f'_-(M)-f'_+(m))\mathbf{1} \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &\int_T \phi_t(f(x_t))d\mu(t) - f\left(\int_T \phi_t(x_t)d\mu(t)\right) \\ &\quad\le \frac{1}{4}(M-m)^2\Delta_f\left(\int_T \phi_t(x_t)d\mu(t);m,M\right) \end{aligned} \tag{2.3}$$

$$\le \frac{1}{4}(M-m)(f'_-(M)-f'_+(m))\mathbf{1}$$

hold for every bounded continuous field $(x_t)_{t\in T}$ *of self-adjoint elements in* $\mathcal{A}$ *with spectra contained in* $[m,M]$. *If* f *is concave on* I, *then the signs of inequalities in* (2.2) *and* (2.3) *are reversed.*

Proof. Taking into account the operator version of the Lah-Ribarič inequality (1.3), it follows that

$$\begin{aligned} &\int_T \phi_t(f(x_t))d\mu(t) - f\left(\int_T \phi_t(x_t)d\mu(t)\right) \\ &\quad\le \frac{M\mathbf{1}-\int_T \phi_t(x_t)d\mu(t)}{M-m}f(m) \\ &\quad+\frac{\int_T \phi_t(x_t)d\mu(t)-m\mathbf{1}}{M-m}f(M) - f\left(\int_T \phi_t(x_t)d\mu(t)\right). \end{aligned} \tag{2.4}$$

On the other hand, the scalar inequality

$$\begin{aligned} &\frac{M-t}{M-m}f(m)+\frac{t-m}{M-m}f(M)-f(t) \\ &\quad= \frac{(M-t)(t-m)}{M-m}\left[\frac{f(M)-f(t)}{M-t}-\frac{f(t)-f(m)}{t-m}\right] \\ &\quad= (M-t)(t-m)\Delta_f(t;m,M)\le(M-t)(t-m)\sup_{m<t<M}\Delta_f(t;m,M) \end{aligned} \tag{2.5}$$

holds for all $t \in [m, M]$. In addition, since

$$\begin{aligned}
&\sup_{m<t<M} \Delta_f(t; m, M) \\
&= \frac{1}{M-m} \sup_{m<t<M} \left[\frac{f(M)-f(t)}{M-t} - \frac{f(t)-f(m)}{t-m} \right] \\
&\leq \frac{1}{M-m} \left[\sup_{m<t<M} \frac{f(M)-f(t)}{M-t} + \sup_{m<t<M} \left(-\frac{f(t)-f(m)}{t-m} \right) \right] \\
&= \frac{1}{M-m} \left[\sup_{m<t<M} \frac{f(M)-f(t)}{M-t} - \inf_{m<t<M} \frac{f(t)-f(m)}{t-m} \right] \\
&= \frac{f'_-(M)-f'_+(m)}{M-m},
\end{aligned} \tag{2.6}$$

we have the following series of inequalities:

$$\begin{aligned}
&\frac{M-t}{M-m} f(m) + \frac{t-m}{M-m} f(M) - f(t) \\
&\leq (M-t)(t-m) \sup_{m<t<M} \Delta_f(t; m, M) \\
&\leq \frac{f'_-(M)-f'_+(m)}{M-m} (M-t)(t-m) \\
&\leq \frac{1}{4}(M-m)(f'_-(M)-f'_+(m)).
\end{aligned} \tag{2.7}$$

Clearly, the last inequality sign in (2.7) holds due to the arithmetic-geometric mean inequality, that is, $(M-t)(t-m) \leq \frac{1}{4}(M-m)^2$.

Now, since $m\mathbf{1} \leq x_t \leq M\mathbf{1}$ for every $t \in T$, it follows that $m\phi_t(\mathbf{1}) \leq \phi_t(x_t) \leq M\phi_t(\mathbf{1})$, that is, $m\mathbf{1} \leq \int_T \phi_t(x_t)d\mu(t) \leq M\mathbf{1}$. Hence, applying the functional calculus to the above series of inequalities, that is, putting $\int_T \phi_t(x_t)d\mu(t)$ instead of t, we have

$$\begin{aligned}
&\frac{M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)}{M-m} f(m) \\
&+ \frac{\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}}{M-m} f(M) - f\left(\int_T \phi_t(x_t)d\mu(t) \right) \\
&\leq \sup_{m<t<M} \Delta_f(t; m, M) \left(M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t) \right) \left(\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1} \right) \\
&\leq \frac{f'_-(M)-f'_+(m)}{M-m} \left(M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t) \right) \left(\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1} \right) \\
&\leq \frac{1}{4}(M-m)(f'_-(M)-f'_+(m))\mathbf{1}.
\end{aligned} \tag{2.8}$$

Finally, comparing (2.4) and (2.8), we obtain (2.2), as claimed.

To prove (2.3), we start with the scalar series of inequalities

$$\begin{aligned}
&\frac{M-t}{M-m}f(m)+\frac{t-m}{M-m}f(M)-f(t)\\
&\quad\le \frac{1}{4}(M-m)^2\Delta_f(t;m,M) \qquad (2.9)\\
&\quad\le \frac{1}{4}(M-m)(f'_-(M)-f'_+(m)),\ t\in[m,M],
\end{aligned}$$

which obviously follows from (2.5), (2.6), and the arithmetic-geometric mean inequality. Finally, setting $\int_T \phi_t(x_t)d\mu(t)$ in (2.9) and utilizing (2.4), we obtain (2.3) and the proof is completed. □

Remark 2.2 Observe that the series of inequalities in (2.2) refines the series (1.5), since $\sup_{m<t<M}\Delta_f(t;m,M)\le \frac{f'_-(M)-f'_+(m)}{M-m}$. For example, if $f(t)=t^2$ and $m<M$, then

$$1=\sup_{m<t<M}\Delta_f(t;m,M)<\frac{f'_-(M)-f'_+(m)}{M-m}=2,$$

while for $f(t)=t^3$ we have

$$m+2M=\sup_{m<t<M}\Delta_f(t;m,M)<\frac{f'_-(M)-f'_+(m)}{M-m}=3(m+M),$$

provided that $0<m<M$. However, a convex function need not be differentiable. To see the corresponding example, let $m<0<M$ and let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(t)=\begin{cases} t^2, & t\ge 0\\ -t, & t<0\end{cases}.$$

Then,

$$\Delta_f(t;m,M)=\begin{cases} 1-\frac{m(m+1)}{(M-m)(t-m)}, & t\ge 0\\ \frac{M(M+1)}{(M-m)(M-t)}, & t<0\end{cases},$$

and consequently,

$$\sup_{m<t<M}\Delta_f(t;m,M)=\begin{cases} \frac{M^2-2Mm-m}{(M-m)^2}, & \text{if } m<-1\\ \frac{M+1}{M-m}, & \text{if } -1\le m<0\end{cases}.$$

On the other hand,

$$\frac{f'_-(M)-f'_+(m)}{M-m}=\frac{2M+1}{M-m},$$

which implies that $\sup_{m<t<M}\Delta_f(t;m,M)<\frac{f'_-(M)-f'_+(m)}{M-m}$, since $M>0$.

Remark 2.3 It should be noticed here that the first line in the series of inequalities (2.2) and (2.3), that is, the element $\int_T \phi_t(f(x_t))d\mu(t)-f\left(\int_T \phi_t(x_t)d\mu(t)\right)$ is not positive in general. This element is positive if f is in addition operator convex function, due to the Jensen operator inequality (1.1).

3 Converses of the Lah-Ribarič operator inequality

This section is devoted to converses of the Lah-Ribarič operator inequality. The following result provides several converse series of inequalities for the Lah-Ribarič operator inequality (1.3). As we shall see below, one of them improves the series (1.6). Although the Theorem 2.1 and the Theorem 3.1 refer to different inequalities, it appears that the corresponding series of converses are closely connected.

Theorem 3.1 *Suppose $f : I \to \mathbb{R}$ is a continuous convex function, and $m, M \in \mathbb{R}$, $m < M$, are such that interval $[m, M]$ belongs to the interior of interval I. Further, let $(\phi_t)_{t\in T}$ be a unital field of positive linear mappings $\phi_t : \mathcal{A} \to \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras, defined on a locally compact Hausdorff space T with a bounded Radon measure μ. Then the series of inequalities*

$$
\begin{aligned}
0 &\le \frac{M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)}{M-m} f(m) \\
&\quad + \frac{\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}}{M-m} f(M) - \int_T \phi_t(f(x_t))d\mu(t) \\
&\le \sup_{m<t<M} \Delta_f(t;m,M) \int_T \phi_t\left([M\mathbf{1}-x_t][x_t-m\mathbf{1}]\right)d\mu(t) \\
&\le \frac{f'_-(M)-f'_+(m)}{M-m} \int_T \phi_t\left([M\mathbf{1}-x_t][x_t-m\mathbf{1}]\right)d\mu(t) \\
&\le \frac{f'_-(M)-f'_+(m)}{M-m} \left(M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}\right) \\
&\le \frac{1}{4}(M-m)(f'_-(M)-f'_+(m))\mathbf{1},
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
0 &\le \frac{M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)}{M-m} f(m) \\
&\quad + \frac{\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}}{M-m} f(M) - \int_T \phi_t(f(x_t))d\mu(t) \\
&\le \sup_{m<t<M} \Delta_f(t;m,M) \int_T \phi_t\left([M\mathbf{1}-x_t][x_t-m\mathbf{1}]\right)d\mu(t) \\
&\le \sup_{m<t<M} \Delta_f(t;m,M) \left(M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}\right)
\end{aligned}
\tag{3.2}
$$

$$
\begin{aligned}
&\le \frac{f'_-(M)-f'_+(m)}{M-m} \left(M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}\right) \\
&\le \frac{1}{4}(M-m)(f'_-(M)-f'_+(m))\mathbf{1},
\end{aligned}
\tag{3.3}
$$

and

$$\begin{aligned}0 &\le \frac{M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)}{M-m} f(m) \\ &+ \frac{\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}}{M-m} f(M) - \int_T \phi_t(f(x_t))d\mu(t) \\ &\le \frac{1}{4}(M-m)^2 \int_T \phi_t\left(\varDelta_f(x_t; m, M)\right) d\mu(t) \\ &\le \frac{1}{4}(M-m)(f'_-(M) - f'_+(m))\mathbf{1}\end{aligned} \tag{3.4}$$

hold for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathcal{A}$ with spectra contained in $[m, M]$. Moreover, if f is concave on I, then the signs of inequalities in (3.1)*,* (3.3)*, and* (3.4) *are reversed.*

Proof. The first inequality in (3.1) holds by virtue of the Lah-Ribarič inequality (1.3). Further, starting from the scalar inequalities (2.5) and (2.6), it follows that the relation

$$\begin{aligned}&\frac{M\mathbf{1} - x_t}{M-m} f(m) + \frac{x_t - m\mathbf{1}}{M-m} f(M) - f(x_t) \\ &\quad\le \sup_{m<t<M} \varDelta_f(t; m, M)(M\mathbf{1} - x_t)(x_t - m\mathbf{1}) \\ &\quad\le \frac{f'_-(M) - f'_+(m)}{M-m}(M\mathbf{1} - x_t)(x_t - m\mathbf{1})\end{aligned}$$

holds for every $t \in T$. Now, applying the positive linear mappings ϕ_t to the above relation, we obtain

$$\begin{aligned}&\frac{M\phi_t(\mathbf{1}) - \phi_t(x_t)}{M-m} f(m) + \frac{\phi_t(x_t) - m\phi_t(\mathbf{1})}{M-m} f(M) - \phi_t(f(x_t)) \\ &\le \sup_{m<t<M} \varDelta_f(t; m, M)\phi_t([M\mathbf{1} - x_t][x_t - m\mathbf{1}]) \\ &\le \frac{f'_-(M) - f'_+(m)}{M-m}\phi_t([M\mathbf{1} - x_t][x_t - m\mathbf{1}]),\end{aligned}$$

while integrating yields

$$\begin{aligned}&\frac{M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)}{M-m} f(m) + \frac{\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}}{M-m} f(M) - \int_T \phi_t(f(x_t))d\mu(t) \\ &\le \sup_{m<t<M} \varDelta_f(t; m, M) \int_T \phi_t\left([M\mathbf{1} - x_t][x_t - m\mathbf{1}]\right) d\mu(t) \\ &\le \frac{f'_-(M) - f'_+(m)}{M-m} \int_T \phi_t\left([M\mathbf{1} - x_t][x_t - m\mathbf{1}]\right) d\mu(t).\end{aligned}$$

Therefore, the second and the third inequality sign in (3.1) hold.

Taking into account Theorem 2.1, that is, the series of inequalities in (2.2), it suffices to motivate the fourth inequality sign in (3.1). To prove our assertion, we note that the function $h(t) = (M-t)(t-m) = -t^2 + (M+m)t - Mm, t \in [m, M]$ is operator concave (see, e.g., [7]). Finally, applying the Jensen operator inequality (1.1) to the above function h, it follows that

$$\int_T \phi_t\left([M\mathbf{1}-x_t][x_t-m\mathbf{1}]\right)d\mu(t) \leq \left(M\mathbf{1}-\int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t)-m\mathbf{1}\right),$$

and the proof of (3.1) is completed.

Further, the series of inequalities in (3.3) is established in the same way as the proof of (3.1), except that we apply the above functional calculus to inequality (2.5) and utilize scalar inequality (2.6).

Finally, the series of inequalities in (3.4) follows by applying the functional calculus to the scalar series of inequalities in (2.9). □

Remark 3.1 Taking into account the Remark 2.2, the series of inequalities in (3.1) refines converse series (1.6).

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