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Natarajan method of summability for double sequences and series

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Abstract Throughout this paper, entries of infinite matrices, sequences, series, double sequences and double series are real or complex numbers. Natarajan introduced the (M, λ_n) method of summability and studied some of its properties. In the present paper, we define $(M, \lambda_{m,n})$ method of summability for double sequences and double series and extend some properties of the (M, λ_n) method to $(M, \lambda_{m,n})$ method.

Keywords Regularity of 4-dimensional infinite matrices · Silverman-Toeplitz theorem · Consistency · Inclusion and equivalence

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1 Introduction and preliminaries

Throughout the present paper, entries of infinite matrices, sequences, series, double sequences and double series are real or complex numbers. We introduce a new definition of limit of a double sequence and a double series and record a few results on convergent double sequences and Silverman-Toeplitz theorem for double sequences and double series [6]. In this context, it is worthwhile to refer to the papers of Kojima [2] and Robison [7].

Definition 1.1 *Let* $\{x_{m,n}\}$ *be a double sequence. We say that*

$$\lim_{m+n\to\infty} x_{m,n} = x,$$

if for every $\epsilon > 0$*, the set* $\{(m, n) \in \mathbb{N}^2 : |x_{m,n} - x| \geq \epsilon\}$ *is finite,* $\mathbb{N}$ *being the set of positive integers. In such a case,* x *is unique and* x *is called the limit of* $\{x_{m,n}\}$*.*

Definition 1.2 *Let* $\{x_{m,n}\}$ *be a double sequence. We say that* $s = \sum_{m,n=0}^{\infty,\infty} x_{m,n}$ *if* $s = \lim_{m+n\to\infty} s_{m,n}$*, where* $s_{m,n} = \sum_{k,\ell=0}^{m,n} x_{k,\ell}, m, n = 0, 1, 2, \ldots$

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Remark 1.1 If $\lim_{m+n\to\infty} x_{m,n} = x$, then the double sequence $\{x_{m,n}\}$ is automatically bounded.

It is now easy to prove the following result.

Theorem 1.3 $\lim_{m+n\to\infty} x_{m,n} = x$ *if and only if*

(i) $\lim_{m\to\infty} x_{m,n} = x$, $n = 0, 1, 2, \ldots$;
(ii) $\lim_{n\to\infty} x_{m,n} = x$, $m = 0, 1, 2, \ldots$; *and*
(iii) *for any* $\epsilon > 0$, *there exists an* $N \in \mathbb{N}$ *such that* $|x_{m,n} - x| < \epsilon$, $m, n \geq N$, *which we write as* $\lim_{m,n\to\infty} x_{m,n} = x$ *(Note that this is Pringsheim's definition of limit of a double sequence.)*

Theorem 1.4 *If the double series* $\sum_{m,n=0}^{\infty,\infty} x_{m,n}$ *converges, then*

$$\lim_{m+n\to\infty} x_{m,n} = 0.$$

However, the converse is not true.

Definition 1.5 *The double series* $\sum_{m,n=0}^{\infty,\infty} x_{m,n}$ *is said to converge absolutely if* $\sum_{m,n=0}^{\infty,\infty} |x_{m,n}|$ *converges.*

Note that if $\sum_{m,n=0}^{\infty,\infty} x_{m,n}$ *converges absolutely, then* $\sum_{m,n=0}^{\infty,\infty} x_{m,n}$ *converges. Converse is not true.*

Definition 1.6 *Given the 4-dimensional infinite matrix* $A = (a_{m,n,k,\ell})$ *and a double sequence* $x = \{x_{k,\ell}\}$, *by the A-transform of* $x = \{x_{k,\ell}\}$, *we mean the double sequence* $A(x) = \{(Ax)_{m,n}\}$, *where*

$$(Ax)_{m,n} = \sum_{k,\ell=0}^{\infty,\infty} a_{m,n,k,\ell} x_{k,\ell}, \quad m, n = 0, 1, 2, \ldots,$$

assuming that the double series on the right converge. If $\lim_{m+n\to\infty}(Ax)_{m,n} = s$, *we say that the double sequence* $x = \{x_{k,\ell}\}$ *is A-summable or summable A to s, written as* $x_{k,\ell} \to s(A)$. *If* $\lim_{m+n\to\infty}(Ax)_{m,n} = s$, *whenever* $\lim_{k+\ell\to\infty} x_{k,\ell} = s$, *we say that the 4-dimensional infinite matrix* $A = (a_{m,n,k,\ell})$ *is regular.*

Natarajan [6] proved the following important theorem.

Theorem 1.7 [Silverman-Toeplitz] *The 4-dimensional infinite matrix* $A = (a_{m,n,k,\ell})$ *is regular if and only if*

(i) $\sup_{m,n} \sum_{k,\ell=0}^{\infty,\infty} |a_{m,n,k,\ell}| < \infty$;
(ii) $\lim_{m+n\to\infty} a_{m,n,k,\ell} = 0$, $k, \ell = 0, 1, 2, \ldots$;
(iii) $\lim_{m+n\to\infty} \sum_{k,\ell=0}^{\infty,\infty} a_{m,n,k,\ell} = 1$;
(iv) $\lim_{m+n\to\infty} \sum_{k=0}^{\infty} |a_{m,n,k,\ell}| = 0$, $\ell = 0, 1, 2, \ldots$; *and*
(v) $\lim_{m+n\to\infty} \sum_{\ell=0}^{\infty} |a_{m,n,k,\ell}| = 0$, $k = 0, 1, 2, \ldots$

2 Natarajan's summability method (M, λ_n)

The concepts of regularity of a 2-dimensional infinite matrix $(a_{n,k})$, consistency, inclusion, equivalence are very familiar and they are taken for granted while developing the topic. One can refer to any standard text for these concepts (see, for instance, [1]).

NATARAJAN introduced his summability method (M, λ_n) for simple sequences in [4] and studied its properties in [3–5]. In this section, we present the definition of the Natarajan method (M, λ_n) and some results which are relevant to the present paper.

Definition 2.1 *Let $\{\lambda_n\}$ be a sequence such that $\sum_{n=0}^{\infty} |\lambda_n| < \infty$. The (M, λ_n) method is defined by the infinite matrix $(a_{n,k})$, where*

$$a_{n,k} = \begin{cases} \lambda_{n-k}, & k \leq n \\ 0, & k > n. \end{cases}$$

Remark 2.1 In this context, it is worthwhile to note that the (M, λ_n) method reduces to the Y-method, when $\lambda_0 = \lambda_1 = \frac{1}{2}$ and $\lambda_n = 0$, $n \geq 2$.

Theorem 2.2 *The (M, λ_n) method is regular if and only if $\sum_{n=0}^{\infty} \lambda_n = 1$.*

The following results are needed in the sequel.

Theorem 2.3 *Any two regular methods (M, λ_n), (M, μ_n) are consistent.*

Theorem 2.4 (inclusion theorem) *Given the methods $(M, \lambda_n), (M, \mu_n)$, $(M, \lambda_n) \subseteq (M, \mu_n)$ if and only if $\sum_{n=0}^{\infty} |k_n| < \infty$ and $\sum_{n=0}^{\infty} k_n = 1$, where*

$$\frac{\mu(x)}{\lambda(x)} = k(x) = \sum_{n=0}^{\infty} k_n x^n, \quad \lambda(x) = \sum_{n=0}^{\infty} \lambda_n x^n, \quad \mu(x) = \sum_{n=0}^{\infty} \mu_n x^n.$$

Consequently, we have

Theorem 2.5 (equivalence theorem) *The methods (M, λ_n), (M, μ_n) are equivalent if and only if*

$$\sum_{n=0}^{\infty} |k_n| < \infty, \quad \sum_{n=0}^{\infty} k_n = 1, \quad \sum_{n=0}^{\infty} |h_n| < \infty, \quad \sum_{n=0}^{\infty} h_n = 1,$$

where

$$\frac{\mu(x)}{\lambda(x)} = k(x) = \sum_{n=0}^{\infty} k_n x^n, \quad \frac{\lambda(x)}{\mu(x)} = h(x) = \sum_{n=0}^{\infty} h_n x^n,$$

$$\lambda(x) = \sum_{n=0}^{\infty} \lambda_n x^n, \quad \mu(x) = \sum_{n=0}^{\infty} \mu_n x^n.$$

3 Natarajan's summability method $(M, \lambda_{m,n})$ and main results of the paper

In this section, we introduce the $(M, \lambda_{m,n})$ methods and extend results of section 2 to these methods.

Definition 3.1 *Let $\{\lambda_{m,n}\}$ be a double sequence such that $\sum_{m,n=0}^{\infty,\infty} |\lambda_{m,n}| < \infty$. The method $(M, \lambda_{m,n})$ is defined by the 4-dimensional infinite matrix $(a_{m,n,k,\ell})$, where*

$$a_{m.n,k,\ell} = \begin{cases} \lambda_{m-k,n-\ell}, & k \leq m, \ell \leq n \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3.2 *The methods $(M, \lambda_{m,n})$, $(M, \mu_{m,n})$ are said to be consistent if $s_{k,\ell} \to \sigma(M, \lambda_{m,n})$ and $s_{k,\ell} \to \sigma'(M, \mu_{m,n})$ imply that $\sigma = \sigma'$.*

Definition 3.3 *We say that $(M, \lambda_{m,n})$ is included in $(M, \mu_{m,n})$, written as $(M, \lambda_{m,n}) \subseteq (M, \mu_{m,n})$ if $s_{k,\ell} \to \sigma(M, \lambda_{m,n})$ implies that $s_{k,\ell} \to \sigma(M, \mu_{m,n})$. The two methods $(M, \lambda_{m,n})$, $(M, \mu_{m,n})$ are said to be equivalent if $(M, \lambda_{m,n}) \subseteq (M, \mu_{m,n})$ and vice versa.*

It is easy to prove the following result.

Theorem 3.4 *The method $(M, \lambda_{m,n})$ is regular if and only if*

$$\sum_{m,n=0}^{\infty,\infty} \lambda_{m,n} = 1.$$

In the sequel, let $(M, \lambda_{m,n})$, $(M, \mu_{m,n})$ be regular methods such that each row and each column of the infinite matrices $(\lambda_{m,n}), (\mu_{m,n})$ is a regular Natarajan method for simple sequences.

Theorem 3.5 *Any two such regular Natarajan methods are consistent.*

Proof. Let $(M, \lambda_{m,n})$, $(M, \mu_{m,n})$ be two regular methods such that each row and each column of the infinite matrices $(\lambda_{m,n})$, $(\mu_{m,n})$ is a regular Natarajan method for simple sequences. We now define a third method $(M, \gamma_{m,n})$ by the equation $\gamma_{m,n} = \sum_{i,j=0}^{m,n} \lambda_{i,j}\mu_{m-i,n-j}$, $m, n = 0, 1, 2, \ldots$ Now, for $s = \{s_{m,n}\}$, we get $(M, \gamma_{m,n})(s) = \sum_{i,j=0}^{m,n} u_{m,n,i,j}(M, \mu_{i,j})(s)$, where $u_{m,n,i,j} = \lambda_{m-i,n-j}Q_{i,j}$, $Q_{i,j} = \sum_{k,\ell=0}^{i,j} \mu_{k,\ell}$, $i, j = 0, 1, 2, \ldots$ Using Theorem 3.4, we can verify that $(M, \gamma_{m,n})$ is regular. Thus $s_{k,\ell} \to \sigma'(M, \mu_{m,n})$ implies that $s_{k,\ell} \to \sigma'(M, \gamma_{m,n})$. Similarly we can prove that $s_{k,\ell} \to \sigma(M, \lambda_{m,n})$ implies that $s_{k,\ell} \to \sigma(M, \gamma_{m,n})$. Consequently $\sigma = \sigma'$ and so the methods $(M, \lambda_{m,n})$ and $(M, \mu_{m,n})$ are consistent, completing the proof of the theorem. □

Let

$$\lambda(x, y) = \sum_{m,n=0}^{\infty,\infty} \lambda_{m,n}x^m y^n, \quad \mu(x, y) = \sum_{m,n=0}^{\infty,\infty} \mu_{m,n}x^m y^n.$$

It is clear that these series converge for $|x|$, $|y| < 1$. Let

$$k(x,y) = \sum_{m,n=0}^{\infty,\infty} k_{m,n} x^m y^n = \frac{\mu(x,y)}{\lambda(x,y)}, \quad h(x,y) = \sum_{m,n=0}^{\infty,\infty} h_{m,n} x^m y^n = \frac{\lambda(x,y)}{\mu(x,y)}.$$

We note that

$$\sum_{i,j=0}^{m,n} k_{i,j} \lambda_{m-i,n-j} = \mu_{m,n}, \quad \sum_{i,j=0}^{m,n} h_{i,j} \mu_{m-i,n-j} = \lambda_{m,n}.$$

We now have

Theorem 3.6 *If* $(M, \lambda_{m,n})$, $(M, \mu_{m,n})$ *are regular, then* $(M, \lambda_{m,n}) \subseteq (M, \mu_{m,n})$ *if and only if* $\sum_{m,n=0}^{\infty,\infty} |k_{m,n}| < \infty$ *and* $\sum_{m,n=0}^{\infty,\infty} k_{m,n} = 1$.

Proof. Let $s(x,y) = \sum_{m,n=0}^{\infty,\infty} s_{m,n} x^m y^n$. Then

$$\begin{aligned}\sum_{m,n=0}^{\infty,\infty} (M, \mu_{m,n})(s) x^m y^n &= \sum_{m,n=0}^{\infty,\infty} \left(\sum_{i,j=0}^{m,n} \mu_{m-i,n-j} s_{i,j} \right) x^m y^n \\ &= \mu(x,y) s(x,y).\end{aligned}$$

Similarly

$$\sum_{m,n=0}^{\infty,\infty} (M, \lambda_{m,n})(s) x^m y^n = \lambda(x,y) s(x,y).$$

Thus

$$\begin{aligned}\sum_{m,n=0}^{\infty,\infty} (M, \mu_{m,n})(s) x^m y^n &= k(x,y) \sum_{m,n=0}^{\infty,\infty} (M, \lambda_{m,n})(s) x^m y^n \\ &= \left(\sum_{m,n=0}^{\infty,\infty} k_{m,n} x^m y^n \right) \left(\sum_{m,n=0}^{\infty,\infty} (M, \lambda_{m,n})(s) x^m y^n \right),\end{aligned}$$

which implies that

$$(M, \mu_{m,n})(s) = \sum_{i,j=0}^{m,n} k_{m-i,n-j} (M, \lambda_{i,j})(s) = \sum_{i,j=0}^{m,n} c_{m,n,i,j} (M, \lambda_{i,j})(s),$$

where

$$c_{m,n,i,j} = \begin{cases} k_{m-i,n-j}, & i \le m, j \le n \\ 0, & \text{otherwise.} \end{cases}$$

If $(M, \lambda_{m,n}) \subseteq (M, \mu_{m,n})$, then $(c_{m,n,i,j})$ is regular. By Theorem 1.7 (iii),

$$\lim_{m+n\to\infty} \sum_{i,j=0}^{\infty,\infty} c_{m,n,i,j} = 1, \text{ i.e. } \lim_{m+n\to\infty} \sum_{i,j=0}^{m,n} k_{m-i,n-j} = 1, \text{ i.e.}$$

$$\lim_{m+n\to\infty} \sum_{i,j=0}^{m,n} k_{i,j} = 1, \text{ i.e. } \sum_{m,n=0}^{\infty,\infty} k_{m,n} = 1.$$

Again, by Theorem 1.7 (i), there is $H > 0$ such that

$$\sum_{i,j=0}^{\infty,\infty} |c_{m,n,i,j}| \leq H, \quad m, n = 0, 1, 2, \ldots,$$

$$\text{i.e. } \sum_{i,j=0}^{m,n} |k_{m-i,n-j}| \leq H, \quad m, n = 0, 1, 2, \ldots,$$

$$\text{i.e. } \sum_{i,j=0}^{m,n} |k_{i,j}| \leq H, \quad m, n = 0, 1, 2, \ldots,$$

from which it follows that $\sum_{m,n=0}^{\infty,\infty} |k_{m,n}| < \infty$.

Converesely, if $\sum_{m,n=0}^{\infty,\infty} |k_{m,n}| < \infty$ and $\sum_{m,n=0}^{\infty,\infty} k_{m,n} = 1$, it is easy to verify that $(c_{m,n,i,j})$ is regular and consequently $(M, \lambda_{m,n}) \subseteq (M, \mu_{m,n})$, completing the proof of the theorem. □

As a consequence of Theorem 3.6, we have,

Theorem 3.7 *The regular methods $(M, \lambda_{m,n})$, $(M, \mu_{m,n})$ are equivalent if and only if*

$$\sum_{m,n=0}^{\infty,\infty} |k_{m,n}| < \infty, \quad \sum_{m,n=0}^{\infty,\infty} k_{m,n} = 1$$

and

$$\sum_{m,n=0}^{\infty,\infty} |h_{m,n}| < \infty, \quad \sum_{m,n=0}^{\infty,\infty} h_{m,n} = 1.$$

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