An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat. (N.S.)
Tomul LXII, 2016, f. 2, vol. 2

Korovkin type theorems in weighted L_p-spaces via statistical A-summability

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Received: 8.V.2014 / Accepted: 4.VI.2014

Abstract In this paper, we study Korovkin type approximation theorems on weighted spaces $L_{p,\omega}(\mathbb{R})$ and $L_{p,\Omega}(\mathbb{R}^n)$, with help of statistical A-summability which is stronger than A -statistical convergence. Also, we construct examples such that our new approximation result works but its statistical case does not work.

Keywords Statistical A-summability · Positive linear operator · Korovkin type approximation theorem

Mathematics Subject Classification (2010) 41A25 · 41A36 · 47B38

1 Introduction

Approximation theory has important applications in theory of polynomial approximation, in various areas of functional analysis, in numerical solutions of differential and integral equations, etc [5]. The well-known Korovkin theorem [5,20] on approximation of continuous functions on a compact interval, is mainly based on the existence of the limit $\lim_n L_n(f;x) = f(x)$. Many researchers have extended this theorem for various operators on different spaces [1,2,8,12,14,15]. One of the most important paper in these extensions is given by Gadjiev [13].

Recently, some Korovkin type approximation theorems have been studied via statistical convergence [3,4,7,10,17–19]. Also, Dirik, Aral and Demirci [6] have studied a Korovkin type approximation theorem on weighted spaces $L_{p,\omega}(\mathbb{R})$ and $L_{p,\Omega}(\mathbb{R}^n)$

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using the concept of $I-$convergence. Those results which obtained are stronger than the classical Korovkin theorem.

The purpose of the paper is to give Korovkin type approximation theorems on weighted spaces $L_{p,\omega}(\mathbb{R})$ and $L_{p,\Omega}(\mathbb{R}^n)$ using the concept of statistical A-summability which is stronger than A-statistical convergence.

We now recall some basic definitions and notations used in the paper.

Let $\mathbb{R}$ denote the set of real numbers. The function ω is called a weight function if it is a positive continuous function on the whole real axis and, for a fixed $p \in [1, \infty)$, satisfying the condition

$$\int_{\mathbb{R}} t^{2p} \omega(t)\, dt < \infty. \tag{1.1}$$

We denote by $L_{p,\omega}(\mathbb{R})$ $(1 \leq p < \infty)$ the linear space of measurable, $p-$absolutely integrable functions on $\mathbb{R}$ with respect to weight function ω, i.e.

$$L_{p,\omega}(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R};\ \|f\|_{p,\omega} = \left(\int_{\mathbb{R}} |f(t)|^p \omega(t)\, dt \right)^{\frac{1}{p}} < \infty \right\}. \tag{1.2}$$

The analogous of (1.1) and (1.2) in multidimensional space are given as follows. Let Ω be a positive continuous function in $\mathbb{R}^n$, satisfying the condition

$$\int_{\mathbb{R}^n} |t|^{2p} \Omega(t)\, dt < \infty, \tag{1.3}$$

and

$$L_{p,\Omega}(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{R};\ \|f\|_{p,\Omega} = \left(\int_{\mathbb{R}^n} |f(t)|^p \Omega(t)\, dt \right)^{\frac{1}{p}} < \infty \right\}. \tag{1.4}$$

Let $A = \{a_{kj}\}, k, j{=}1, 2, ...$ be an infinite summability matrix. For a given sequence $x{:=}\{x_j\}$, the $A - transform$ of x, denoted by $Ax := \{(Ax)_k\}$ is given by $(Ax)_k = \sum_{j=1}^{\infty} a_{kj} x_j$, provided the series converges for each $k \in \mathbb{N}$. We say that A is regular (see [16]) if $\lim Ax = L$ whenever $\lim x = L$.

Assume that A is a nonnegative regular summability matrix. The A-density of a subset $K \subset \mathbb{N}$, denoted by $\delta_A(K)$, is given by

$$\delta_A(K) = \lim_k \sum_{j \in K} a_{kj},$$

provided the limit exists. Then the sequence $x = \{x_j\}$ is called A-statistically convergent to L provided that, for every $\varepsilon > 0$,

$$\delta_A(\{j \in \mathbb{N} : |x_j - L| \geq \varepsilon\}) = 0. \tag{1.5}$$

In this case we write $st_A - \lim x = L$.

Note that if we take $A = (C, 1)$, which is the Cesáro matrix, then $(C, 1)-$statistical convergence coincides with the notion of statistical convergence, which was introduced in [11]. Finally, if we replace the matrix A by the identity matrix, then $A-$statistical convergence reduces to the usual convergence.

Definition 1.1 ([9]) *Let $A = \{a_{kj}\}$ be a non-negative regular summability matrix and $x = \{x_j\}$ be a sequence. We say that x is statistically $A-$summable to L if for every $\varepsilon > 0$, $\delta(\{k \in \mathbb{N} : |(Ax)_k - L| \geq \varepsilon\}) = 0$, i.e., $\lim_N \frac{1}{N} |\{k \leq N : |(Ax)_k - L| \geq \varepsilon\}| = 0$. Thus $x = \{x_j\}$ is statistically $A-$summable to L if and only if Ax is statistically convergent to L. In this case we write $(A)_{st} - \lim_j x_j = L$ or $st - \lim_k (Ax)_k = L$.*

We note that if we take $A = (C, 1)$ then statistical $A-$ summability is reduced to the statistical $(C, 1)-$ summability.

Let $A = \{a_{kj}\}$ be a non-negative regular summability matrix and $\{L_j\}$ be a sequence of positive linear operators from $L_{p,\omega}$ into $L_{p,\omega}$. By $A_k(f; x)$ we denote

$$A_k(f; x) = \sum_{j=1}^{\infty} a_{kj} L_j (f(t); x). \tag{1.6}$$

2 Main results

Now we first recall the classical and statistical cases of Korovkin type results introduced in [15,6], respectively.

Theorem 2.1 ([15]) *Let $\{L_j\}_{j\in\mathbb{N}}$ be the sequence of positive linear operators $L_j : L_{p,\omega}(\mathbb{R}) \to L_{p,\omega}(\mathbb{R})$ and let the sequence $\{\|L_j\|\}$ be uniformly bounded. If $\lim_j \|L_j(t^i, x) - x^i\|_{p,\ \omega} = 0,\ i = 0, 1, 2,$ then for any function $f \in L_{p,\omega}(\mathbb{R})$, we have $\lim_j \|L_j(f) - f\|_{p,\ \omega} = 0$.*

Theorem 2.2 ([6]) *Let $A = \{a_{kj}\}$ be a non-negative regular summability matrix. Let $\{L_j\}_{j\in\mathbb{N}}$ be the sequence of positive linear operators $L_j : L_{p,\omega}(\mathbb{R}) \to L_{p,\omega}(\mathbb{R})$ and let the sequence $\{\|L_j\|\}$ be uniformly bounded. If $st_A - \lim_j \|L_j(t^i,\ x) - x^i\|_{p,\ \omega} = 0,\ i = 0, 1, 2,$ then for any function $f \in L_{p,\omega}(\mathbb{R})$, we have $st_A - \lim_j \|L_j(f) - f\|_{p,\ \omega} = 0$.*

Theorem 2.3 *Let $A = \{a_{kj}\}$ be a non-negative regular summability matrix and $\{L_j\}$ be a sequence of positive linear operators from $L_{p,\omega}$ into $L_{p,\omega}$. Assume that*

$$\sup_k \|A_k\|_{L_{p,\omega}\to L_{p,\omega}} < \infty. \tag{2.1}$$

If

$$st - \lim_k \left\|A_k\left(t^i; x\right) - x^i\right\|_{p,\omega} = 0,\ i = 0, 1, 2, \tag{2.2}$$

then for any function $f \in L_{p,\omega}(\mathbb{R})$, we have $st - \lim_k \|A_k f - f\|_{p,\omega} = 0$.

Proof. We give the proof of theorem as similarly as the proof of theorem in [15]. Let $\chi_1^B(t)$ be the characteristic function of the interval $[-B, B]$ and $\chi_2^B(t) = 1 - \chi_1^B(t)$ for any $B \geq 0$. We can choose a such large B such that for every $\varepsilon > 0$, $\left\|f\chi_2^B\right\|_{p,\omega} < \varepsilon$. Using the assumption of the convergence of the series in (1.6) for each k, f and the linearity of the operators L_j, we get

$$\begin{aligned}\|A_k f - f\|_{p,\omega} &= \left\|A_k\left(\chi_1^B + \chi_2^B\right) f - \left(\chi_1^B + \chi_2^B\right) f\right\|_{p,\omega} \\ &\leq \left\|A_k\left(\chi_1^B f\right) - \chi_1^B f\right\|_{p,\omega} + \left\|A_k\left(\chi_2^B f\right) - \chi_2^B f\right\|_{p,\omega} \\ &= I_k' + I_k''.\end{aligned} \tag{2.3}$$

From the condition (2.1), there exists a constant $K > 0$ such that

$$\sup_k \|A_k\|_{p,\omega} \leq K. \tag{2.4}$$

Hence, from (2.1), we have $I_k'' \leq \|A_k \chi_2^B f\|_{p,\omega} + \|\chi_2^B f\|_{p,\omega} \leq (K+1)\|\chi_2^B f\|_{p,\omega} < (K+1)\varepsilon$. For every function $f \in L_{p,\omega}(\mathbb{R})$ the inequality $\|\chi_1^B f\|_p \leq \omega_{\min}^{-1/p}\|f\|_{p,\omega}$ implies $L_{p,\omega}(\mathbb{R}) \subset L_p(-B,B)$. Since the space of continuous functions on $[-B,B]$ is dense in $L_p(-B,B)$, given $f \in L_{p,\omega}(\mathbb{R})$, for each $\varepsilon' > 0$, there exists a continuous function φ on $[-B,B]$ satisfying the condition $\varphi(x) = 0$ for $|x| > B$ such that

$$\left\|(f-\varphi)\chi_1^B\right\|_p < \frac{\varepsilon'}{(K+1)\,\omega_{\max}^{1/p}}. \tag{2.5}$$

Using the inequalities (2.4) and (2.5), we get

$$\begin{aligned} I_k' &= \left\|A_k\left(\chi_1^B f\right) - \chi_1^B f\right\|_{p,\omega} \\ &\leq \left\|A_k(f-\varphi)\chi_1^B\right\|_{p,\omega} + \left\|A_k\left(\varphi\chi_1^B\right) - \varphi\chi_1^B\right\|_{p,\omega} + \left\|(f-\varphi)\chi_1^B\right\|_{p,\omega} \\ &\leq \left\|A_k\left(\varphi\chi_1^B\right) - \varphi\chi_1^B\right\|_{p,\omega} + \varepsilon'. \end{aligned} \tag{2.6}$$

On the other hand, since $\chi_2^{B_1}\chi_1^B\varphi = 0$ for some $B_1 > B$, we get the equality

$$\begin{aligned} \left\|A_k\left(\varphi\chi_1^B\right) - \varphi\chi_1^B\right\|_{p,\omega} &= \left\|\left(\chi_1^{B_1} + \chi_2^{B_1}\right)A_k\left(\varphi\chi_1^B\right) - \left(\chi_1^{B_1} + \chi_2^{B_1}\right)\varphi\chi_1^B\right\|_{p,\omega} \\ &\leq \left\|\left[A_k\left(\varphi\chi_1^B\right) - \varphi\chi_1^B\right]\chi_1^{B_1}\right\|_{p,\omega} + \left\|\chi_2^{B_1}A_k\left(\varphi\chi_1^B\right)\right\|_{p,\omega}. \end{aligned}$$

Now, by denoting $M_\varphi = \max_{t\in\mathbb{R}} |\varphi(t)|\,\chi_1^B(t)$, we get

$$\begin{aligned} \left\|\chi_2^{B_1}A_k\left(\varphi\chi_1^B\right)\right\|_{p,\omega} &= \left(\int_{|t|>B_1}\left|A_k\left(\varphi\chi_1^B;t\right)\right|^p\omega(t)\,dt\right)^{\frac{1}{p}} \\ &\leq M_\varphi\left(\int_{|t|>B_1}|A_k(1;t)-1|^p\omega(t)\,dt\right)^{\frac{1}{p}} + M_\varphi\left(\int_{\mathbb{R}}\chi_2^{B_1}\omega(t)\,dt\right)^{\frac{1}{p}}. \end{aligned}$$

Since $\omega \in L_1(\mathbb{R})$, we can choose a number B_1 such that

$$\left(\int_{\mathbb{R}}\chi_2^{B_1}\omega(t)\,dt\right)^{\frac{1}{p}} < \frac{\varepsilon'}{M_\varphi}.$$

Using this inequality we have $\|\chi_2^{B_1}A_k(\varphi\chi_1^B)\|_{p,\omega} \leq M_\varphi\|A_k(1;x)-1\|_{p,\omega} + \varepsilon'$. As a corollary, we get the following inequality for I_k'

$$I_k' \leq 2\varepsilon' + M_\varphi\|A_k(1;x)-1\|_{p,\omega} + \left\|\left[A_k\left(\varphi\chi_1^B\right) - \varphi\chi_1^B\right]\chi_1^{B_1}\right\|_{p,\omega}.$$

Since $\varphi\chi_1^B$ is a continuous function on $[-B,B]$, for given any $\varepsilon'>0$ there exist a $\delta>0$ such that $\left|\varphi(t)\chi_1^B(t)-\varphi(x)\chi_1^B(x)\right|<\varepsilon'+2M_\varphi\frac{(t-x)^2}{\delta^2}$. So we have

$$\begin{aligned}\|[A_k(\varphi\chi_1^B)-\varphi\chi_1^B]\chi_1^{B_1}\|_{p,\omega} &\le \|[A_k(|\varphi(t)\chi_1^B(t)-\varphi(x)\chi_1^B(x)|;x)]\chi_1^{B_1}(x)\|_{p,\omega}\\ &+\|\varphi(x)\chi_1^B(x)(A_k(1;x)-1)\|_{p,\omega}\\ &\le\left(\varepsilon'+\frac{2M_\varphi}{\delta^2}B^2+M_\varphi\right)\|A_k(1;x)-1\|_{p,\omega} \qquad (2.7)\\ &+\frac{4M_\varphi}{\delta^2}B\|A_k(t;x)-x\|_{p,\omega}+\frac{2M_\varphi}{\delta^2}\|A_k(t^2;x)-x^2\|_{p,\omega}.\end{aligned}$$

Using (2.7), we can write

$$\begin{aligned}I_k'\le 2\varepsilon'&+\left(\varepsilon'+\frac{2M_\varphi}{\delta^2}B^2+2M_\varphi\right)\|A_k(1;x)-1\|_{p,\omega}\\ &+\frac{4M_\varphi}{\delta^2}B\|A_k(t;x)-x\|_{p,\omega}+\frac{2M_\varphi}{\delta^2}\left\|A_k\left(t^2;x\right)-x^2\right\|_{p,\omega}.\end{aligned}$$

Then we obtain the following equality for (2.3) as $\|A_kf-f\|_{p,\omega}\le 2\varepsilon'+(K+1)\varepsilon+C\{\|A_k(1;x)-1\|_{p,\omega}+\|A_k(t;x)-x\|_{p,\omega}+\|A_k(t^2;x)-x^2\|_{p,\omega}\}$, where $C:=\max\{\varepsilon'+\frac{2M_\varphi}{\delta^2}B^2+2M_\varphi,\frac{4M_\varphi}{\delta^2}B,\frac{2M_\varphi}{\delta^2}\}$. Let $r>0$ be a number such that $(2\varepsilon'+(K+1)\varepsilon)<r$. Then, let

$$\begin{aligned}D&:=\left\{k\le N:\sum_{i=0}^{2}\left\|A_k\left(t^i;x\right)-x^i\right\|_{p,\omega}\ge\frac{r-(2\varepsilon'+(K+1)\varepsilon)}{C}\right\},\\ D_1&:=\left\{k\le N:\|A_k(1;x)-1\|_{p,\omega}\ge\frac{r-(2\varepsilon'+(K+1)\varepsilon)}{3C}\right\},\\ D_2&:=\left\{k\le N:\|A_k(t;x)-x\|_{p,\omega}\ge\frac{r-(2\varepsilon'+(K+1)\varepsilon)}{3C}\right\},\\ D_3&:=\left\{k\le N:\left\|A_k\left(t^2;x\right)-x^2\right\|_{p,\omega}\ge\frac{r-(2\varepsilon'+(K+1)\varepsilon)}{3C}\right\}.\end{aligned}$$

It is easy to see that $D\subset D_1\cup D_2\cup D_3$ and we have

$$\begin{aligned}&\left|\left\{k\le N:\|A_kf-f\|_{p,\omega}\ge r\right\}\right|\\ &\le\left|\left\{k\le N:\|A_k(1;x)-1\|_{p,\omega}\ge\frac{r-(2\varepsilon'+(K+1)\varepsilon)}{3C}\right\}\right|\\ &+\left|\left\{k\le N:\|A_k(t;x)-x\|_{p,\omega}\ge\frac{r-(2\varepsilon'+(K+1)\varepsilon)}{3C}\right\}\right|\\ &+\left|\left\{k\le N:\left\|A_k\left(t^2;x\right)-x^2\right\|_{p,\omega}\ge\frac{r-(2\varepsilon'+(K+1)\varepsilon)}{3C}\right\}\right|,\end{aligned}$$

where $|A|$ denotes the cardinality of the set A. Then taking the limit $N \longrightarrow \infty$, using the hypothesis of theorem, we obtain

$$\lim_{N} \frac{1}{N} \left| \left\{ k \leq N : \|A_k f - f\|_{p,\omega} \geq r \right\} \right| = 0$$

which is the desired result. □

Now we give an example of a sequence of positive linear operators which satisfy the conditions of Theorem 2.3 in the weighted space $L_{p,\omega}(\mathbb{R})$.

Example 2.1 We choose $\omega(x) = e^{-x}$. Note that this selection of ω satisfies the condition (1.1). Also note that for $1 \leq p < \infty$, $L_{p,\omega}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : \, e^{-x} f(x) \in L_p(\mathbb{R})\}$. Also, $A = (C, 1)$ is the Cesáro matrix, i.e.,

$$c_{kj} = \begin{cases} \dfrac{1}{k}, & 1 \leq j \leq k, \\ 0, & \text{otherwise.} \end{cases}$$

and $\alpha = \{\alpha_j\}$ is defined by $\alpha_j = (-1)^j$, then we can easily see that $st - \lim_k ((C,1)\alpha)_k = 0$. However, the sequence (α_j) does not converge in usual and statistical sense. The Kantorovich variant of the Szasz-Mirakyan operators [21] by replacing $f(\frac{sb_j}{j})$ with an integral mean of $f(x)$ over the interval $[(s+1)b_j/j, \; sb_j/j]$ is as follows:

$$S_j(f; x) := \frac{j}{b_j} \sum_{s=0}^{\infty} P_{j,s}(x) \int_{sb_j/j}^{(s+1)b_j/j} f(t)\, dt, \quad j \in \mathbb{N}, \; x \in [0, b_j), \tag{2.8}$$

where $\{b_j\}$ is a sequence of positive real numbers satisfying the conditions $\lim_{j\to\infty} \frac{b_j}{j} = 0$ and $\lim_{j\to\infty} b_j = \infty$ and $P_{j,s}(x) := e^{-jx/b_j} \frac{(jx)^s}{s! b_j^s}$, $s = 0, 1, 2, ...$ It is known that $S_j(1; x) = 1$, $S_j(t; x) = x + \frac{b_j}{2j}$ and $S_j(t^2; x) = x^2 + \frac{2b_j}{j} x + \frac{b_j^2}{3j^2}$. Then using the operators S_j and the sequence $\alpha = (\alpha_j)$, we define the sequence of positive lineer operators $L_j(f; x) = (1 + \alpha_j) S_j(f; x)$ for $f \in L_{p,\omega}(\mathbb{R})$ and $j \in \mathbb{N}$. By some simple calculations, we obtain

$$\|C_k(1; x) - 1\|_{p,\omega} = \left| \frac{1}{k} \sum_{j=1}^{k} \alpha_j \right| \|1\|_{p,\omega},$$

$$\|C_k(t; x) - x\|_{p,\omega} \leq \frac{1}{k} \sum_{j=1}^{k} \frac{b_j}{j} \|1\|_{p,\omega} + \left| \frac{1}{k} \sum_{j=1}^{k} \alpha_j \right| \|x\|_{p,\omega},$$

$$\|C_k(t^2; x) - x^2\|_{p,\omega} \leq \frac{4}{k} \sum_{j=1}^{k} \frac{b_j}{j} \|x\|_{p,\omega} + \frac{2}{3k} \sum_{j=1}^{k} \frac{b_j^2}{j^2} \|1\|_{p,\omega} + \left| \frac{1}{k} \sum_{j=1}^{k} \alpha_j \right| \|x^2\|_{p,\omega},$$

where $C_k(f; x) = \sum_{j=1}^{\infty} c_{kj} L_j(f(t); x)$.
Also, $\sup_k \|C_k\|_{L_{p,\omega} \to L_{p,\omega}} = \sup_k \sup_{\|f\|_{p,\omega}=1} \|C_k(f; \, x)\|_{p,\omega} < \infty$. Hence (2.1), (2.2) conditions are provided. For any function $f \in L_{p,\omega}(\mathbb{R})$, we have $st - \lim_k \|C_k f - f\|_{p,\omega} = 0$.

Also, an analogue of Theorem 2.3 for the space of function of several variables can be obtained. Now we establish this theorem.

Theorem 2.4 *Let $A = \{a_{kj}\}$ be an infinite matrix with non-negative real entries and $\{L_j\}$ be a sequence of positive linear operators from $L_{p,\Omega}(\mathbb{R}^n)$ into $L_{p,\Omega}(\mathbb{R}^n)$. Assume that*

$$\sup_k \|A_k\|_{L_{p,\Omega}\to L_{p,\Omega}} < \infty. \tag{2.9}$$

If

$$\begin{aligned} &st-\lim_k \|A_k(1;x)-1\|_{p,\Omega}=0,\ i=0,1,2,\\ &st-\lim_k \left\|A_k\left(t^i;x\right)-x^i\right\|_{p,\Omega}=0,\ i=1,2,...,n,\\ &st-\lim_k \|A_k(|t|^2;x)-|x|^2\|_{p,\Omega}=0,\ i=0,1,2, \end{aligned} \tag{2.10}$$

then for any function $f \in L_{p,\Omega}(\mathbb{R}^n)$, we have $st-\lim_k \|A_kf-f\|_{p,\Omega}=0$.

Proof. Let χ_1^B be the characteristic function of the ball $|x| \le B$ and $\chi_2^B(t) = 1-\chi_1^B(t)$. Then, it is possible to choose a sufficient large B such that

$$\left\|f\chi_2^B(t)\right\|_{p,\Omega} < \varepsilon. \tag{2.11}$$

By the condition (2.9) there exists a positive constant K' such that $\sup_k \|A_k\|_{p,\Omega} \le K'$ and so, for given $\varepsilon' > 0$ there exists a continuous function θ on $|x| \le B$ satisfying the condition $\theta(x) = 0$, for $|x| > B$ and such that

$$\left\|(f-\theta)\chi_1^B\right\|_{p,\Omega} < \frac{\varepsilon'}{(K'+1)\left(\max_{|t|\le B}\Omega(t)\right)^{1/p}}.$$

Since the series (1.6) is convergent for each k, f and using the linearity of the operators L_j, which means the linearity of A_k, we obtain

$$\|A_kf-f\|_{p,\Omega} \le \left\|A_k\left(\chi_1^B\theta\right)-\chi_1^B\theta\right\|_{p,\Omega} + \left(K'+1\right)\varepsilon+\varepsilon'. \tag{2.12}$$

Let $B_1>B$, so we also have $\|A_k(\chi_1^B\theta)-\chi_1^B\theta\|_{p,\Omega} \le \|[A_k(\chi_1^B\theta)-\chi_1^B\theta]\chi_1^{B_1}\|_{p,\Omega} + M_\theta\|A_k(1)-1\|_{p,\Omega} + M_\theta\|\chi_2^{B_1}\|_{p,\Omega}$, where $M_\theta := \max_{t\in\mathbb{R}^n}|\theta(t)|\chi_1^B(t)$. Furthermore, we can choose B_1 such that $\|\chi_2^{B_1}\|_{p,\Omega} < \varepsilon'/M_\theta$, and for sufficiently large k, we estimate $\|A_k(1)-1\|_{p,\Omega} < \varepsilon'/M_\theta$. Substituting these estimates in (2.12), we obtain $\|A_kf-f\|_{p,\Omega} \le \|[A_k(\chi_1^B\theta)-\chi_1^B\theta]\chi_1^{B_1}\|_{p,\Omega} + (K'+1)\varepsilon + 3\varepsilon'$. Since $|\chi_1^B(t)\theta(t) - \chi_1^B(x)\theta(x)| < \varepsilon' + 2M_\theta\frac{|t-x|^2}{\delta^2}$ we can write

$$\begin{aligned} \|A_kf-f\|_{p,\Omega} \le{}& \left(K'+1\right)\varepsilon + 4\varepsilon'K'\|\Omega\|_1^{1/P} \\ &+C\left\{\left\|A_k\left(|t|^2;x\right)-|x|^2\right\|_{p,\Omega}\right. \\ &\left.+\sum_{i=1}^n\left\|A_k\left(t^i;x\right)-x^i\right\|_{p,\Omega}+\|A_k(1)-1\|_{p,\Omega}\right\}. \end{aligned}$$

where $C = 2M_\theta \frac{(1+B)^2}{\delta^2}$. Now for a given $r' > 0$ such that $(2\varepsilon' + (K'+1)\varepsilon) < r'$. Then,let us define the following sets

$$D := \left\{ k \leq N : \|A_k f - f\|_{p,\Omega} \geq r' \right\},$$

$$D_1 := \left\{ k \leq N : \|A_k(1;x) - 1\|_{p,\Omega} \geq \frac{r' - (2\varepsilon' + (K'+1)\varepsilon)}{3C} \right\},$$

$$D_2 := \left\{ k \leq N : \sum_{i=1}^{n} \left\|A_k(t^i;x) - x^i\right\|_{p,\Omega} \geq \frac{r' - (2\varepsilon' + (K'+1)\varepsilon)}{3C} \right\},$$

$$D_3 := \left\{ k \leq N : \left\|A_k\left(|t|^2;x\right) - |x|^2\right\|_{p,\Omega} \geq \frac{r' - (2\varepsilon' + (K'+1)\varepsilon)}{3C} \right\}.$$

It is easy to see that $D \subset D_1 \cup D_2 \cup D_3$ and we have

$$\begin{aligned} &\left|\left\{ k \leq N : \|A_k f - f\|_{p,\Omega} \geq r' \right\}\right| \\ &\leq \left|\left\{ k \leq N : \|A_k(1;x) - 1\|_{p,\Omega} \geq \frac{r' - (2\varepsilon' + (K'+1)\varepsilon)}{3C} \right\}\right| \\ &+ \left|\left\{ k \leq N : \sum_{i=1}^{n} \|A_k(t_i;x) - x_i\|_{p,\Omega} \geq \frac{r' - (2\varepsilon' + (K'+1)\varepsilon)}{3C} \right\}\right| \\ &+ \left|\left\{ k \leq N : \left\|A_k\left(|t|^2;x\right) - |x|^2\right\|_{p,\Omega} \geq \frac{r' - (2\varepsilon' + (K'+1)\varepsilon)}{3C} \right\}\right|, \end{aligned}$$

where $|A|$ denotes the cardinality of the set A.Then taking the limit as $N \longrightarrow \infty$, using the hypothesis of theorem, we obtain $\lim_N \frac{1}{N} |\{k \leq N : \|A_k f - f\|_{p,\Omega} \geq r'\}| = 0$ which is desired. □

Now we give an example such that:

Example 2.2 We choose $\Omega(x,y) = e^{-x-y}$. Note that this selection of Ω satisfies the condition (1.3). Also note that for $1 \leq p < \infty$, $L_{p,\Omega}(\mathbb{R}^2) = \{ f : \mathbb{R}^2 \to \mathbb{R} : \Omega(x,y)f(x) \in L_p(\mathbb{R}^2)\}$. Also, let $A = (C,1)$ and $\alpha = (\alpha_j)$ be as the Example 2.1. The Kantorovich variant of the double Szasz-Mirakyan operators by replacing $f(\frac{tb_j}{j}, \frac{sb_j}{j})$ with an integral mean of $f(x,y)$ over the interval $[(t+1)b_j/j,\ tb_j/j] \times [(s+1)b_j/j,\ sb_j/j]$ is as follows: for $j \in \mathbb{N}$, $x, y \in [0, b_j)$,

$$S_j(f;x,y) := \frac{j^2}{b_j^2} \sum_{t=0}^{\infty} \sum_{s=0}^{\infty} P_{j,t,s}(x,y) \int_{tb_j/j}^{(t+1)b_j/j} \int_{sb_j/j}^{(s+1)b_j/j} f(u,v)\,dudv, \tag{2.13}$$

where $\{b_j\}$ is a sequence of positive real numbers satisfying the conditions $\lim_{j\to\infty} \frac{b_j}{j} = 0$ and $\lim_{j\to\infty} b_j = \infty$ and

$$P_{j,t,s}(x,y) := e^{-\frac{j(x+y)}{b_j}} \frac{(jx)^t (jy)^s}{t!s!b_j^{t+s}},\ t,s = 0,1,2,....$$

It is known that

$$S_j\left(1;x,y\right)=1,$$
$$S_j\left(u;x,y\right)=x+\frac{b_j}{2j},$$
$$S_j\left(v;x,y\right)=y+\frac{b_j}{2j},$$
$$S_j\left(u^2+v^2;x,y\right)=x^2+y^2+\frac{2b_j}{j}\left(x+y\right)+\frac{2b_j^2}{3j^2}.$$

Then using the operators S_j and the sequence $\alpha=(\alpha_j)$, we define the sequence of positive linear operators $L_j(f;x,y)=(1+\alpha_j)S_j(f;x,y)$ for $f\in L_{p,\,\Omega}(\mathbb{R}^2)$ and $j\in\mathbb{N}$. By some simple calculations, we obtain

$$\left\|C_k\left(1;x,y\right)-1\right\|_{p,\Omega}=\left|\frac{1}{k}\sum_{j=1}^{k}\alpha_j\right|\left\|1\right\|_{p,\Omega},$$

$$\left\|C_k\left(u;x,y\right)-x\right\|_{p,\Omega}\leq\frac{1}{k}\sum_{j=1}^{k}\frac{b_j}{j}\left\|1\right\|_{p,\Omega}+\left|\frac{1}{k}\sum_{j=1}^{k}\alpha_j\right|\left\|x\right\|_{p,\Omega},$$

$$\left\|C_k\left(v;x,y\right)-y\right\|_{p,\Omega}\leq\frac{1}{k}\sum_{j=1}^{k}\frac{b_j}{j}\left\|1\right\|_{p,\Omega}+\left|\frac{1}{k}\sum_{j=1}^{k}\alpha_j\right|\left\|y\right\|_{p,\Omega}$$

$$\left\|C_k\left(u^2+v^2;x,y\right)-\left(x^2+y^2\right)\right\|_{p,\Omega}\leq\frac{4}{k}\sum_{j=1}^{k}\frac{b_j}{j}\left\|x+y\right\|_{p,\Omega}$$
$$+\frac{4}{3k}\sum_{j=1}^{k}\frac{b_j^2}{j^2}\left\|1\right\|_{p,\Omega}+\left|\frac{1}{k}\sum_{j=1}^{k}\alpha_j\right|\left\|x^2+y^2\right\|_{p,\Omega},$$

where $C_k(f;x,y)=\sum_{j=1}^{\infty}c_{kj}L_j\left(f(u,v);x,y\right)$. Also,

$$\sup_k\left\|C_k\right\|_{L_{p,\Omega}\to L_{p,\Omega}}=\sup_k\sup_{\|f\|_{p,\Omega}=1}\left\|C_k(f;x,y)\right\|_{p,\Omega}<\infty.$$

Hence, (2.9), (2.10) conditions are provided which means that for any function $f\in L_{p,\Omega}(\mathbb{R}^2)$, we have $st-\lim_k\|C_kf-f\|_{p,\Omega}=0$.

Acknowledgements The authors are thankful to referee(s) for making valuable suggestions leading to a better presentation of the paper.

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