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On the error term for some arithmetic functions in number fields

Sanying Shi

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Abstract Lü established asymptotic formulae for some arithmetic functions in any Galois extension $E/\mathbb{Q}$ of degree d. In this paper we obtain the sharp bound for the mean square of the error terms in their asymptotic formula.

Keywords Mean value · Number field · Dedekind zeta function

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1 Introduction

In number theory there are many arithmetic functions which play important role in arithmetics, number theory and discrete mathematics. We often try to study the asymptotic behavior, since their behavior is very irregular. One of the most famous examples is the k-dimensional divisor problem, which studies the behavior of the mean value of $d_k(n)$. Here $d_k(n)$ denotes the number of representations of n as a product of k natural numbers. Dirichlet first obtained that for integer $k \geq 2$, $\sum_{n\leq x} d_k(n) = xQ_k(\log x)+R_k(x)$, where $Q_k(t)$ is a polynomial in t of degree $k-1$, $R_k(x) = O\left(x^{\alpha_k+\varepsilon}\right)$, and $\alpha_k \leq 1-\frac{1}{k}$. Subsequently this problem has been extensively studied by many authors [5, 6, 9]. Here we take $k = 2$ for example. In [2], Cramér proved the classical mean-square result:

$$\int_1^X (R_2(x))^2\,dx = \frac{\zeta^4(3/2)}{6\pi^2\zeta(3)}X^{\frac{3}{2}} + O(X^{\frac{5}{4}+\epsilon}).$$

Later the mean square of the error term $R_2(x)$ was also studied in [10, 15, 16], and the higher-power moments of $R_2(x)$ were studied in [4, 7, 17–19].

There are similar problems in algebraic number fields. Let E be an algebraic number field of degree d over the rational field $\mathbb{Q}$, $\mathfrak{a}$ an integral ideal of the field E, and $N\mathfrak{a}$

Sanying Shi
School of Mathematics
Hefei University of Technology
Hefei, P.R. China
E-mail: vera123_99@hotmail.com

the absolute norm of $\mathfrak{a}$. The k-dimensional divisor problem in the field E is to study the mean value of the arithmetic function

$$D_k^E(n) = \sum_{N(\mathfrak{a}_1,\mathfrak{a}_2,\cdot,\mathfrak{a}_k)=n} 1. \tag{1.1}$$

Panteleeva [13] first considered the sum $\sum_{n\leq x} D_k^E(n), k\geq 1$ in the quadratic and cyclotomic fields (see [13] for details). Let C be a square-free number, and $C\leq \log^2 x$. Then it is proved in [13] that for the quadratic field $E=\mathbb{Q}(\sqrt{C})$,

$$\sum_{n\leq x} D_k^E(n) = xQ_k(\log x) + O(x^{1-\frac{10}{133}k^{-\frac{2}{3}}+\epsilon});$$

for the cyclotomic field $E=\mathbb{Q}(\zeta)$ $(\zeta^t=1)$,

$$\sum_{n\leq x} D_k^E(n) = xQ_k(\log x) + O(x^{1-\frac{1}{12}(\phi(t)k)^{-\frac{2}{3}}+\epsilon}),$$

where $Q_k(t)$ is a polynomial in t of degree $k-1$, $\phi(t)$ denotes the Euler's function.

In [14], Panteleeva further studied the asymptotic behavior of the product function of several multi-dimensional divisor functions, i.e.

$$D_{k_1}(n)D_{k_2}(n)\cdots D_{k_l}(n),$$

where $l\geq 1$, and $k_1,k_2,\cdots,k_l\geq 2$ are integers. Based on some deep results in analytic number theory, she proved that

$$\sum_{n\leq x} D_{k_1}(n)D_{k_2}(n)\cdots D_{k_l}(n) = xQ_m(\log x) + O\left(x^{1-\frac{2}{31}m^{-\frac{2}{3}}+\epsilon}\right), \tag{1.2}$$

where $m=k_1k_2\cdots k_l$, and $m\leq \log x$.

Deza and Varukhina [3] considered the generalized problems of (1.2) in number fields, i.e. $D_{k_1}^E(n)D_{k_2}^E(n)\cdots D_{k_l}^E(n)$. Let $m=k_1k_2\cdots k_l$ and $m\leq (\log x)^{-\frac{5}{6}}$, they similarly established the following formulas:

$$\sum_{n\leq x} D_{k_1}^E(n)D_{k_2}^E(n)\cdots D_{k_l}^E(n) = xQ_m(\log x) + O\left(x^{1-\frac{1}{15}m^{-\frac{2}{3}}+\epsilon}\right), \tag{1.3}$$

for the quadratic field $E=\mathbb{Q}(\sqrt{C})$; and

$$\sum_{n\leq x} D_{k_1}^E(n)D_{k_2}^E(n)\cdots D_{k_l}^E(n) = xQ_m(\log x) + O\left(x^{1-\frac{1}{13}(\phi(t)m)^{-\frac{2}{3}}+\epsilon}\right),$$

for the cyclotomic field $E=\mathbb{Q}(\zeta)$ $(\zeta^t=1)$, where $Q_m(t)$ is a polynomial in t of degree $m-1$.

Recently, Lü [11] generalized Deza and Varukhina's (see [3]) results to any Galois extension of the rational field. More precisely, in [11], he proved that

Theorem 1.1 *Let $E/\mathbb{Q}$ be a Galois extension of degree d. For any $l \geq 1$, we have*

$$\sum_{n\leq x} D^E_{k_1}(n)D^E_{k_2}(n)\cdots D^E_{k_l}(n) = xQ_m(\log x) + O\left(x^{1-\frac{3}{md+6}+\epsilon}\right),$$

where $k_1, k_2, \cdots, k_l \geq 2$ are integers, $m = k_1k_2\cdots k_l d^{l-1}$, $Q_m(t)$ is a polynomial in t of degree $m-1$, and $\epsilon > 0$ is an arbitrarily small constant.

In the present paper we consider similarly the problem of the error term for (1.2) in mean square, and shall prove the following result.

Theorem 1.2 *Subject to assumptions in Theorem* 1.1*, and define*

$$\Delta(x) := \sum_{n\leq x} D^E_{k_1}(n)D^E_{k_2}(n)\cdots D^E_{k_l}(n) - xQ_m(\log x).$$

Then we have $\int_1^X \Delta^2(x)dx \ll_\epsilon X^{3-\frac{6}{md+3}+\epsilon}$, for any given $\epsilon > 0$.

Notations The Vinogradov symbol $A \ll B$ means that B is positive and the ratio A/B is bounded. The letter ϵ denotes an arbitrary small positive number, not the same at each occurrence.

2 Proof of Theorem 1.2

To prove Theorem 1.2, we need the following lemmas.

Lemma 2.1 *Let K be a Galois extension of degree d over $\mathbb{Q}$, and $L^E_{k_1,k_2,\cdots,k_l}(s)$ be defined in* (2.7)*. Then we have $L^E_{k_1,k_2,\cdots,k_l}(s)=\zeta^m(s,K)U(s)$, where $m = k_1k_2\cdots k_l d^{l-1}$, $U(s)$ denotes a Dirichlet series, which is absolutely and uniformly convergent for $\mathfrak{Re}(s) > \frac{1}{2}$, and $\zeta(s,K)$ denotes the Dedekind zeta function of the field K.*

Proof. This follows immediately by the arguments (see pages 353-354) in [11]. □

Lemma 2.2 *Let K be an algebraic number field of degree m, Then for any fixed $\epsilon > 0$*

$$\zeta(s,K) \ll (1+|t|)^{\frac{m}{3}(1-\sigma)+\epsilon}, \text{ for } \frac{1}{2} \leq \sigma \leq 1+\epsilon. \tag{2.1}$$

Proof. By Lemma 2.5 in [12] and the Phragmén-Lindelöf principle for a strip (see e.g. Theorem 5.53 in Iwaniec and Kowalski [8]), this lemma follows. □

Now we begin to complete the proof of Theorem 1.2.

Let E be an algebraic number field of degree d, and $\zeta(s,E)$ be its Dedekind zeta-function. Thus for $\mathfrak{Re}(s) > 1$,

$$\zeta(s,E) = \sum_{\mathfrak{a}} \frac{1}{(N\mathfrak{a})^s}, \tag{2.2}$$

where $\mathfrak{a}$ runs over all integral ideals of the field E, and $N\mathfrak{a}$ is the absolute norm of $\mathfrak{a}$. If $a(n)$ denotes the number of integral ideals in E with norm n, then we have

$$\zeta(s,E)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}, \quad s=\sigma+it, \quad \sigma>1. \tag{2.3}$$

By Lemma 9 in [1], it is known that $a(n)$ is a multiplicative function and satisfies

$$a(n)\ll\tau(n)^{d-1}, \tag{2.4}$$

where $\tau(n)$ is the divisor function, and $d=[E:\mathbb{Q}]$.

From the definition of $D_k^E(n)$ in (1.1), we have that for $\mathfrak{Re}s>1$

$$\zeta(s,E)^k=\sum_{\mathfrak{a}_1}\sum_{\mathfrak{a}_2}\cdots\sum_{\mathfrak{a}_k}\left(\frac{1}{(N\mathfrak{a}_1)}\frac{1}{(N\mathfrak{a}_2)}\cdots\frac{1}{(N\mathfrak{a}_k)}\right)^s=\sum_{n=1}^{\infty}\frac{D_k^E(n)}{n^s}. \tag{2.5}$$

Then by (2.3),

$$D_k^E(n)=\sum_{n_1n_2\cdots n_k=n}a_{n_1}a_{n_2}\cdots a_{n_k}.$$

Hence $D_k^E(n)$ is also multiplicative and

$$D_k^E(n)\ll\sum_{n_1n_2\cdots n_k=n}\tau(n_1)^{d-1}\tau(n_2)^{d-1}\cdots\tau(n_k)^{d-1}\ll n^{\epsilon}. \tag{2.6}$$

Therefore we introduce the L-function associated to $D_{k_1}^E(n)D_{k_2}^E(n)\cdots D_{k_l}^E(n)$ in the half-plane $\mathfrak{Re}s>1$

$$L_{k_1,k_2,\cdots,k_l}^E(s)=\sum_{n=1}^{\infty}\frac{D_{k_1}^E(n)D_{k_2}^E(n)\cdots D_{k_l}^E(n)}{n^s}, \tag{2.7}$$

for it is absolutely convergent in this region.

Let

$$T=X^{\frac{3}{md+3}}. \tag{2.8}$$

From (2.6), (2.7) and Perron's formula (see Proposition 5.54 in [8]), we get

$$\sum_{n\le x}D_{k_1}^E(n)D_{k_2}^E(n)\cdots D_{k_l}^E(n)$$
$$=\frac{1}{2\pi i}\int_{1+\epsilon-iT}^{1+\epsilon+iT}L_{k_1,k_2,\cdots,k_l}^E(s)\frac{x^s}{s}ds+O\left(\frac{x^{1+\epsilon}}{T}\right). \tag{2.9}$$

By the property $L^E_{k_1,k_2,\cdots,k_l}(s)$ only has a simple pole at $s=1$ for $\mathfrak{Re}(s)>\frac{1}{2}$ and Cauchy's residue theorem, we have

$$\begin{aligned}\sum_{n\le x} r(n) &= \frac{1}{2\pi i}\left\{\int_{\frac{1}{2}+\epsilon-iT}^{\frac{1}{2}+\epsilon+iT}+\int_{\frac{1}{2}+\epsilon+iT}^{1+\epsilon+iT}+\int_{1+\epsilon-iT}^{\frac{1}{2}+\epsilon-iT}\right\} L^E_{k_1,k_2,\cdots,k_l}(s)\frac{x^s}{s}ds\\ &\quad+\operatorname*{\mathbf{Res}}_{s=1} L^E_{k_1,k_2,\cdots,k_l}(s)x + O\left(\frac{x^{1+\epsilon}}{T}\right)\\ &:= xQ_m(\log x)+I_1(x)+I_2(x)+I_3(x)+O\left(x^{1+\epsilon}T^{-1}\right)\\ &:= xQ_m(\log x)+\varDelta(x).\end{aligned}$$

Then in order to prove Theorem 1.2, we shall prove the following results.

$$\int_1^X I_i^2(x)dx \ll_\epsilon X^{3-\frac{6}{md+3}+\epsilon},\ i=1,2,3 \tag{2.10}$$

and

$$\int_1^X \left(O\left(\frac{x^{1+\epsilon}}{T}\right)\right)^2 dx \ll_\epsilon X^{3-\frac{6}{md+3}+\epsilon}. \tag{2.11}$$

From (2.8), it is easy to get

$$\int_1^X \left(O\left(\frac{x^{1+\epsilon}}{T}\right)\right)^2 dx = O\left(\frac{X^{3+\epsilon}}{T^2}\right) \ll X^{3-\frac{6}{md+3}+\epsilon}. \tag{2.12}$$

For $I_1(x)$, we get

$$I_1(x)=\frac{1}{2\pi}\int_{-T}^{T} L^E_{k_1,\cdots,k_l}\left(\frac{1}{2}+\epsilon+it\right)\frac{x^{\frac{1}{2}+\epsilon+it}}{\frac{1}{2}+\epsilon+it}dt.$$

Then we have

$$\begin{aligned}\int_1^X I_1^2(x)dx &= \frac{1}{4\pi^2}\int_1^X\left(\int_{-T}^{T} L^E_{k_1,\cdots,k_l}\left(\frac{1}{2}+\epsilon+it_1\right)\frac{x^{\frac{1}{2}+\epsilon+it_1}}{\frac{1}{2}+\epsilon+it_1}dt_1\right.\\ &\quad\times\left.\int_{-T}^{T}\overline{L^E_{k_1,\cdots,k_l}\left(\frac{1}{2}+\epsilon+it_2\right)}\frac{x^{\frac{1}{2}+\epsilon-it_2}}{\frac{1}{2}+\epsilon-it_2}dt_2\right)dx\\ &= \frac{1}{4\pi^2}\int_{-T}^{T}\int_{-T}^{T}\frac{L^E_{k_1,\cdots,k_l}\left(\frac{1}{2}+\epsilon+it_1\right)\overline{L^E_{k_1,\cdots,k_l}\left(\frac{1}{2}+\epsilon+it_2\right)}}{(\frac{1}{2}+\epsilon+it_1)(\frac{1}{2}+\epsilon-it_2)}\\ &\quad\times\left(\int_1^X x^{1+2\epsilon+i(t_1-t_2)}dx\right)dt_1dt_2\\ &\ll X^{2+2\epsilon}\int_{-T}^{T}dt_1\end{aligned}$$

$$
\begin{aligned}
&\times \int_{-T}^{T} \frac{|L_{k_1,\cdots,k_l}^{E}\left(\frac{1}{2}+\epsilon+it_1\right)||L_{k_1,\cdots,k_l}^{E}\left(\frac{1}{2}+\epsilon+it_2\right)|}{(1+|t_1|)(1+|t_2|)(1+|t_1-t_2|)} dt_2 \\
&\ll X^{2+2\epsilon} \int_{-T}^{T} dt_1 \int_{-T}^{T} \left(\frac{|L_{k_1,\cdots,k_l}^{E}\left(\frac{1}{2}+\epsilon+it_1\right)|^2}{(1+|t_1|)^2} \right. \\
&\left. + \frac{|L_{k_1,k_2,\cdots,k_l}^{E}\left(\frac{1}{2}+\epsilon+it_2\right)|^2}{(1+|t_2|)^2} \right) \frac{dt_2}{1+|t_1-t_2|} \\
&\ll X^{2+2\epsilon} \int_{-T}^{T} \frac{|L_{k_1,\cdots,k_l}^{E}\left(\frac{1}{2}+\epsilon+it_1\right)|^2}{(1+|t_1|)^2} dt_1 \int_{-T}^{T} \frac{dt_2}{1+|t_1-t_2|}.
\end{aligned}
$$

To go further, we get

$$
\begin{aligned}
\int_{-T}^{T} \frac{dt_2}{1+|t_1-t_2|} &\ll \int_{t_1-1}^{t_1+1} dt_2 + \left(\int_{t_1+1}^{T} + \int_{-T}^{t_1-1} \right) \frac{dt_2}{|t_1-t_2|} \\
&\ll 1 + \int_{t_1+1}^{T} \frac{dt_2}{|t_1-t_2|} \ll \int_{1}^{T+|t_1|} \frac{dt}{t} \ll \log(2T).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\int_1^X I_1^2(x)dx &\ll X^{2+2\epsilon} \log 2T \int_{-T}^{T} \frac{|L_{k_1,k_2,\cdots,k_l}^{E}\left(\frac{1}{2}+\epsilon+it_1\right)|^2}{(1+|t_1|)^2} dt_1 \ll X^{2+3\epsilon} \\
&+ X^{2+3\epsilon} \int_1^T \left| \zeta^m\left(\frac{1}{2}+\epsilon+it, E\right) U\left(\frac{1}{2}+\epsilon+it\right) \right|^2 t^{-2} dt \qquad (2.13) \\
&\ll X^{2+3\epsilon} + X^{2+3\epsilon} \int_1^T \left| \zeta^m\left(\frac{1}{2}+\epsilon+it, E\right) \right|^2 t^{-2} dt \\
&\ll X^{2+3\epsilon} + X^{2+3\epsilon} \int_1^T \left(t^{\frac{md}{6}+\epsilon} \right)^2 t^{-2} dt \\
&\ll X^{2+3\epsilon} + X^{2+5\epsilon} T^{\frac{md}{3}-1} \ll X^{3-\frac{6}{md+3}+\epsilon},
\end{aligned}
$$

where we have used Lemmas 2.1 and Lemma 2.2.

By Lemma 2.2, we have

$$
\begin{aligned}
I_2(x)+I_3(x) &\ll \int_{\frac{1}{2}+\epsilon}^{1+\epsilon} x^{\sigma} |\zeta^m(\sigma+iT,E)| T^{-1} d\sigma \\
&\ll \max_{\frac{1}{2}+\epsilon\le\sigma\le 1+\epsilon} x^{\sigma} T^{\frac{md}{3}(1-\sigma)+\epsilon} T^{-1} \\
&= \max_{\frac{1}{2}+\epsilon\le\sigma\le 1+\epsilon} \left(\frac{x}{T^{md/3}} \right)^{\sigma} T^{\frac{md}{3}-1+\epsilon} \ll \frac{x^{1+\epsilon}}{T} + x^{\frac{1}{2}+\epsilon} T^{\frac{md}{6}-1+\epsilon},
\end{aligned}
$$

which yields

$$\begin{aligned}\int_1^X (I_2(x)+I_3(x))^2dx &\ll \int_1^X \left(\frac{x^{1+\epsilon}}{T}+x^{\frac{1}{2}+\epsilon}T^{\frac{md}{6}-1+\epsilon}\right)^2 dx\\ &\ll \int_1^X \left(\frac{x^{1+\epsilon}}{T}\right)^2 dx+\int_1^X \left(x^{\frac{1}{2}+\epsilon}T^{\frac{md}{6}-1+\epsilon}\right)^2 dx \qquad (2.14)\\ &\ll \frac{X^{3+\epsilon}}{T^2}+X^{2+2\epsilon}T^{\frac{md}{3}-2+2\epsilon} \ll X^{3-\frac{6}{md+3}+\epsilon}.\end{aligned}$$

The inequalities (2.10) and (2.11) immediately follow from (2.12), (2.13) and (2.14). That is, we have

$$\int_1^X \Delta^2(x)dx \ll_\epsilon X^{3-\frac{6}{md+3}+\epsilon}.$$

Then the proof of Theorem 1.2 is completed.

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