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Harmonic analysis on locally compact abelian groupoids

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Abstract For a locally compact abelian groupoid G with a fixed Haar system $\lambda = \{\lambda^u\}_{u\in G^0}$ and a quasi-invariant measure μ, we consider the irreducible representations of G and show that each irreducible representation is one-dimensional. We introduce a notion of duality for abelian groupoids and we show that the Pontrjagin duality holds if and only if G is a group bundle. Then we introduce the Fourier transform on the space of absolutely integrable functions $I(G, \lambda, \mu)$ and use to find the spectrum of $I(G, \lambda, \mu)$.

Keywords Abelian groupoid · Pontrjagin duality · Fourier transform · Irreducible representation

Mathematics Subject Classification (2010) 22A22 · 22A25

1 Introduction and preliminaries

The interesting problems and applications of the duality theory of locally compact abelian groups motivated us to study the representation theory of the locally compact abelian groupoids in order to get a duality theory in this setting. We start with some basic definitions from [7].

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A groupoid is a set G endowed with a product map: $G^2 \to G,\ (x,y) \mapsto xy$, where G^2 is a subset of $G \times G$ called the set of composable pairs, and an inverse map: $G \to G$, $x \mapsto x^{-1}$, such that for each $x, y, z \in G$,

(i) $(x^{-1})^{-1} = x$,
(ii) if $(x,y),(y,z) \in G^2$ then $(xy,z),(x,yz) \in G^2$ and $(xy)z = x(yz)$,
(iii) $(x^{-1},x) \in G^2$ and if $(x,y) \in G^2$ then $x^{-1}(xy) = y$,
(iv) $(x,x^{-1}) \in G^2$ and if $(z,x) \in G^2$ then $(zx)x^{-1} = z$.

For $x \in G$, $s(x) = x^{-1}x$ is called the source of x and $r(x) = xx^{-1}$ is called the range of x. The pair (x,y) is composable if and only if $s(x) = r(y)$. The set $G^0 = s(G) = r(G)$ is the unit space of G and its elements are called units in the sense that $xs(x) = x$ and $r(x)x = x$.

For $u, v \in G^0$, let $G^u = r^{-1}\{u\}$, $G_v = s^{-1}\{v\}$, $G^u_v = G^u \cap G_v$ and $G\{u\} = G^u_u$. The latter is a group, called the isotropy group at u. We put $u \sim v$ if $G^u_v \neq \emptyset$. Obviously $\sim$ is an equivalence relation on the unit space G^0. Its equivalence classes are called orbits, the orbit of u is denoted by $[u]$.

A topological groupoid consists of a groupoid G and a topology compatible with the groupoid structure, such that the composition map is continuous on G^2 in the induced product topology, and the inversion map is continuous on G. A locally compact groupoid is a topological groupoid G which satisfies the following conditions

(i) G^0 is locally compact and Hausdorff in the relative topology.
(ii) There is a countable family C of compact Hausdorff subsets of G whose interiors form a basis for the topology of G.

The space $C_c(G)$ for such a groupoid G is the complex vector space spanned by the functions f on G satisfying the following conditions:

1. f vanishes outside of a compact set K_f,
2. f is continuous on a Hausdorff neighbourhood of K_f.

A left Haar system for G consists of measures $\{\lambda^u, u \in G^0\}$ on G such that

(i) the support of λ^u is G^u,
(ii) (continuity) for each $f \in C_c(G)$, $u \mapsto \lambda(f)(u) = \int f d\lambda^u$ is continuous,
(iii) (left invariance) for any $x \in G$ and any $f \in C_c(G)$,

$$\int f(xy)d\lambda^{s(x)}(y) = \int f(y)d\lambda^{r(x)}(y). \tag{1.1}$$

For the rest of the paper, G is a locally compact, Hausdorff, second countable groupoid which admits a left Haar system $\lambda = \{\lambda^u\}$. Let μ be a measure on G^0. The measure on G induced by μ is $\nu = \int \lambda^u d\mu$, defined by $\int_G f d(\nu) = \int_{G_0} d\mu(u) \int_G f d\lambda^u$, for $f \in C_c(G)$. The measure on G^2 induced by μ is $\nu^2 = \int \lambda_u \times \lambda^u d\mu(u)$. The inversion of ν is $\nu^{-1} = \int \lambda_u d\mu(u)$, where λ_u is the inverse of λ^u, that is, $\lambda_u(E) = \lambda^u(E^{-1})$, for each measurable set E.

Note that the measures ν, ν^2, ν^{-1} are Radon. A measure μ on G^0 is said to be quasi-invariant if its induced measure ν is equivalent to its inverse ν^{-1}. Let μ be a quasi-invariant measure on G^0. The Radon-Nikodym derivative $D = \frac{d\nu}{d\nu^{-1}}$ is called the modular function of μ. An equivalent definition of the modular function on G is given in [8, Definition 2.3], defining the modular function Δ as a strictly positive continuous homomorphism on G such that $\Delta|_{G^u_u}$ is modular function for G^u_u. A system of measures

$\{\lambda^u\}_{u\in G^0}$ is said to be complete if for each $u \in G^0$, λ^u is complete on its σ-algebra $\mathfrak{M}_{\lambda^u}$. Suppose that μ is a quasi-invariant probability measure on G^0, and ν is Radon measure induced by μ, then we have, $\nu_0 = D^{-1}\nu$, where D is the modular function of μ, $\|f\|_{I,r} = \sup_{u\in G^0} \int |f| d\lambda^u$, $\|f\|_{I,s} = \sup_{u\in G^0} \int |f| d\lambda_u$, and $\|f\|_{I,\mu} = \max(\|f\|_{I,r}, \|f\|_{I,s})$. The Banach space $I(G, \lambda, \mu)$ is defined by $I(G, \lambda, \mu) := \left\{f \in L^1(G, \nu_0),\ \|f\|_{I,\mu} < \infty\right\}$ and with the convolution product $(f * g)(x) = \int_{G^{r(x)}} f(y)g(y^{-1}x)d\lambda^{r(x)}(y)$, and with the involution $f^*(x) = \overline{f(x^{-1})}$ becomes a Banach $*$-algebra (see [2]).

Now let us briefly review some basic facts of representation theory on locally compact and Hausdorff groupoids. A representation of the locally compact groupoid G is a triple $(\mu, \{H_u\}_u, \pi)$ consisting of a Hilbert bundle $(G^0, \{H_u\}_{u\in G^0}, \mu)$, where μ is a quasi-invariant measure on G^0 (with associated Radon measures $\nu, \nu^{-1}, \nu^2, \nu_0$) and for each $x \in G$, a unitary element $\pi(x) \in \mathcal{B}(H_{s(x)}, H_{r(x)})$ such that

(i) $\pi(u)$ is the identity map on H_u, for all u,
(ii) $\pi(xy) = \pi(x)\pi(y)$ for ν^2 - a.e. $(x, y) \in G^2$,
(iii) $\pi(x^{-1}) = \pi(x)^{-1}$ for ν-a.e. $x \in G$,
(iv) for any $\xi, \eta \in L^2(G^0, \{H_u\}_u, \mu)$, the map

$$x \mapsto \langle \pi(x)\xi(s(x)), \eta(r(x))\rangle \tag{1.2}$$

is ν-measurable on G.

Let $(G^0, \{H_u\}_{u\in G^0}, \mu)$ be a Hilbert bundle. A family $M = \{M_u\}_{u\in G^0}$, where M_u is a closed subspace of $H_u (u \in G^0)$, is called a subbundle. A subbundle $\{M_u\}_{u\in G^0}$ is called non-trivial if $0 \neq M_u \neq H_u$ for some $u \in G^0$. For a representation π of a locally compact groupoid G associated with the Hilbert bundle $(G^0, \{H_u\}_{u\in G^0}, \mu)$, a subbundle $\{M_u\}_{u\in G^0}$ is called invariant if for each $x \in G$, $\pi(x)M_{s(x)} \subseteq M_{r(x)}$. Note that if M is an invariant subbundle and $0 \neq M_u \neq H_u$ for some $(u \in G^0)$, then $0 \neq M_w \neq H_w$ for every $w \in [u]$.

Let π be a representation of a locally compact groupoid G associated with the Hilbert bundle $(G^0, \{H_u\}_{u\in G^0}, \mu)$. If $M = \{M_u\}_{u\in G^0}$ is an invariant subbundle, then so is $M^\perp = \left\{M_u^\perp\right\}_{u\in G^0}$. A representation π of a locally compact groupoid G is called reducible, if π admits a non-trivial invariant subbundle $M = \{M_u\}_{u\in G^0}$, otherwise π is called irreducible. In this case π^M by $\pi^M(x) = \pi(x)|_{M_{s(x)}} : M_{s(x)} \to M_{r(x)}$ is called a sub-representation of π. If π and π' are two representations of a locally compact groupoid G associated with two Hilbert bundles $(G^0, \{H_u\}_{u\in G^0}, \mu)$ and $(G^0, \{H'_u\}_{u\in G^0}, \mu')$, respectively, then we put

$$\mathcal{C}(\pi, \pi') = \left\{(T_u)_u \in \prod_{u\in G^0} \mathcal{B}(H_u, H'_u) : \pi'(x)T_{s(x)} = T_{r(x)}\pi(x)\ (x \in G)\right\}$$

and write $\mathcal{C}(\pi, \pi) = \mathcal{C}(\pi)$.

Two representations π and π' are called (unitarily) equivalent if $\mu \sim \mu'$ and there is $(T_u)_{u\in G^0} \in \mathcal{C}(\pi, \pi')$ such that T_u is a unitary operator for every $u \in G^0$. Note that if $(T_u)_{u\in G^0} \in \mathcal{C}(\pi)$ and T_u^* denotes the adjoint operator to T_u, then $(T_u^*)_{u\in G^0} \in \mathcal{C}(\pi)$. We observe that $\mathcal{C}(\pi)$ is a unital $*$-algebra, where the operations are defined pointwise (see [1]).

Following [1], for a representation π of a locally compact groupoid G associated with the Hilbert bundle $(G^0, \{H_u\}_{u\in G^0}, \mu)$ we put

$$\Lambda = \left\{ (c_u\pi(u))_u \in \prod_{u\in G^0} \mathcal{B}(H_u) : c_u \in \mathbb{C}, c_u = c_w \quad \text{whenever } u \sim w \right\}.$$

If $(c_u\pi(u))_u \in \mathcal{C}(\pi)$, then $c_u = c_w$ whenever $u \sim w$ that is $(c_u\pi(u))_u \in \Lambda$. Therefore, if $(T_u)_{u\in G^0} \in \mathcal{C}(\pi) \setminus \Lambda$ then there exists $u \in G^0$ with T_u not in $\mathbb{C}\pi(u)$. It is also obvious that $\Lambda \subseteq \mathcal{C}(\pi)$.

From the Schur's Lemma [1, Lemma 3.11], a representation π of a locally compact groupoid G is irreducible if and only if $\Lambda = \mathcal{C}(\pi)$. In particular, in the case where G is transitive, then π is irreducible if and only if $\mathcal{C}(\pi) = \mathbb{C}(\pi(u))_{u\in G^0}$.

2 Locally compact abelian groupoids

Locally compact abelian groupoids were studied by Myrnouri in [4] and Myrnouri and Amini in [5]. We start with a definition.

Definition 2.1 ([5], Definition 1.1) *A groupoid G is called abelian, if its isotropy groups are abelian, equivalently, for each $x, y \in G$, if $r(x) = s(x) = r(y) = s(y)$, then $xy = yx$.*

It is known that in the case where G is a locally compact abelian group, each irreducible representation of G is one dimensional. In the groupoid case we have a similar result.

Lemma 2.2 *Let π be a representation of a locally compact groupoid G associated with the Hilbert bundle $(G^0, \{H_u\}_{u\in G^0}, \mu)$. Let $M = \{M_u\}_{u\in G^0}$ be a subbundle of $H = \{H_u\}_{u\in G^0}$. If $(P_{M_u})_{u\in G^0} \in \mathcal{C}(\pi)$, then $\pi(x)M_u \subseteq M_u$ for all x. Conversely, if there exists $x \in G$ such that for each u, $\pi(x)M_u \subseteq M_u$ then $(P_{M_u})_{u\in G^0} \in \mathcal{C}(\pi)$.*

Proof. Suppose $u \in G^0$ and $\nu \in M_u$, then $\pi(x)\nu = \pi(x)P_{M_u}\nu = P_{M_u}\pi(x)\nu \in M_u$. Conversely, $\pi(x)P_{M_u}\nu = \pi(x)\nu = P_{M_u}\pi(x)\nu$ for $\nu \in M_u$ and $\pi(x)P_{M_u}\nu = 0 = P_{M_u}\pi(x)\nu$ for $\nu \in M_u^{\perp}$. Thus $\pi(x)P_{M_u} = P_{M_u}\pi(x)$ so $(P_{M_u})_{u\in G^0} \in \mathcal{C}(\pi)$. □

Remark 2.1 If $(c_u\pi(u))_u \in \mathcal{C}(\pi)$ and $x \in G_w^u$, then $c_u = c_w$ because if $x \in G_w^u$, then $r(x) = us(x) = w$ and since $(c_u\pi(u))_u \in \mathcal{C}(\pi)$, we have $\pi(x)c_{s(x)}\pi(s(x)) = c_{r(x)}\pi(r(x))\pi(x)$, hence $c_u\pi(x) = c_w\pi(x)$ and so $c_u = c_w$. Therefore, if $(T_u)_{u\in G^0} \in \mathcal{C}(\pi) \setminus \Lambda$, then there exists $u \in G^0$ such that T_u is not in $\mathbb{C}\pi(u)$.

Lemma 2.3 *Let $(T_u)_{u\in G^0} \in \mathcal{C}(\pi)$. If $x \in G_w^u$, then $\sigma(T_u) = \sigma(T_w)$.*

Proof. Suppose $x \in G_w^u$ then $r(x) = u, s(x) = w$. Since $\pi(x)T_{s(x)} = T_{r(x)}\pi(x)$, for every complex number c we have $T_u - c\pi(u) = \pi(x)T_w\pi(x^{-1}) - c\pi(u) = \pi(x)[T_w - c\pi(w)]\pi(x^{-1})$. Now $c \in \sigma(T_w)$ if and only if $c \in \sigma(T_u)$. □

Proposition 2.4 *If G is a locally compact abelian groupoid and π is an irreducible representation of G, then for each $u \in G^0$, $dimH_u = 1$.*

Proof. Suppose $u \in G^0$, since $u \sim u$, $G_u^u \neq \emptyset$. Hence there exist $x, y \in G_u^u$ (at least, for $y = x^{-1}$). Then $xy = yx$ and $\pi(x)\pi(y) = \pi(y)\pi(x)$. Now let $T_u = T_{r(y)} = \pi(y)$. If $w \neq u, T_w = \pi(w)$, then $(T_u)_{u\in G^0} \in \mathcal{C}(\pi)$, and since π is irreducible, $(T_u)_{u\in G^0} \in \Lambda$, thus for each $u \in G^0$ we have $T_u = c_u\pi(u)$, therefore $\pi(y) = c_y\pi(r(y))$, where $c_y \in \mathbb{C}$. Now if $dim H_u \neq 1$, then there is a non-trivial closed subspace $M_{r(y)}$ of $H_{r(y)}$ such that $M_{r(y)}$ is invariant under $\pi(y)$, which is impossible by the above observation. Hence $dim H_u = 1$. □

Remark 2.2 If G is an abelian groupoid with a Haar system $\lambda = \{\lambda^u\}_u$, then for each $u \in G^0$, the modular function $D \equiv 1$ on G_u^u, because in this case $\lambda_u = \lambda^u$.

3 Duality

In the proof of the main result of the last section, the isotropy groups G_u^u play an important role. In this section we dualize these abelian groups and consider the associated group bundle as the dual object of the underlying abelian groupoid. This may seem to be too restrictive, but the logic behind it should become clear after we prove our duality theorem. Indeed, unlike the group case, commutativity of isotropy groups do not force the convolution algebra of absolutely integrable functions on the groupoid to be a commutative algebra, unless the groupoid is a group bundle (see [6]). We show that the latter condition is exactly what is needed for the duality to hold.

Definition 3.1 *Suppose G is a locally compact abelian groupoid. We define the dual object of G denoting $\widehat{G}$ as*

$$\widehat{G} := \bigcup_{u\in G^0} \widehat{G_u^u}. \tag{3.1}$$

We denoted the elements of $\widehat{G}$ by ξ, η, etc. and call them characters.

Remark 3.1 Since each G_u^u is a locally compact abelian group so is $\widehat{G_u^u}$ and since G_u^u's are disjoint, $\widehat{G}$ is an abelian group bundle with the following product map. If $\xi, \eta \in \widehat{G}$, then there exists exactly one $u \in G^0, v \in G^0$ such that $\xi = \xi_u \in \widehat{G_u^u}, \eta = \eta_v \in \widehat{G_v^v}$. If $u = v$, we define the product by

$$(\xi.\eta)(x) = \xi(x)\eta(x), \qquad (x \in G_u^u), \tag{3.2}$$

otherwise, it is undefined.

Remark 3.2 In fact, each $\xi \in \widehat{G}$ gives a $*$-homomorphism from G_u^u to $\mathbb{T}$ with $\xi(u) = 1$. Also one can define the source and range maps $\widehat{r}, \widehat{s} : \widehat{G} \to (\widehat{G})^0$ by $\widehat{r}(\xi) = \xi\xi^{-1}, \widehat{s}(\xi) = \xi^{-1}\xi$.

Theorem 3.2 (Pontrjagin Duality Theorem) *Suppose G is a locally compact abelian groupoid. The following are equivalent:*

(*i*) *$I(G, \lambda, \mu)$ is commutative,*
(*ii*) *G is a group bundle,*

(iii) $\widehat{\widehat{G}} \cong G$.

Proof. $(i \to ii)$ If $I(G,\lambda,\mu)$ is commutative, then $C_c(G)$ is commutative, and so is the enveloping C^*-algebra $C^*(G)$. Thus from [3, Proposition 1.5.7], G is an abelian group bundle.

$(ii \to i)$ Conversely, if G is an abelian group bundle and $f, g \in I(G,\lambda,\mu)$, $x \in G$,

$$\begin{aligned} f * g(x) &= \int_{G^{r(x)}} f(y)g(y^{-1}x)d\lambda^{r(x)}(y) \\ &= \int_{G^{r(x)}} f(y^{-1})g(yx)d\lambda^{r(x)}(y^{-1}) \\ &= \int_{G^{r(x)}} f(xy^{-1})g(y)d\lambda^{r(x)}(xy^{-1}) \\ &= \int_{G^{r(x)}} f(y^{-1}x)g(y)d\lambda^{r(x)}(y^{-1}) \\ &= \int_{G^{r(x)}} f(y^{-1}x)g(y)d\lambda^{r(x)}(y) = g * f(x). \end{aligned}$$

Thus $I(G,\lambda,\mu)$ is commutative.

$(ii \to iii)$ If G is a group bundle, then $G = \bigcup_{u\in G^0} G_u^u$ and since $\widehat{G} = \bigcup_{u\in G^0} \widehat{G_u^u}$ therefore

$$\widehat{\widehat{G}} = \bigcup_{u\in G^0} \widehat{\widehat{G_u^u}} \cong \bigcup_{u\in G^0} G_u^u = G.$$

$(iii \to ii)$ Conversely, if $\widehat{\widehat{G}} \cong G$ since $\widehat{G}$ is group bundle, $G = \bigcup \widehat{\widehat{G_u^u}}$. By the classical Pontrjagin duality theorem we have, $G_u^u = \widehat{\widehat{G_u^u}}$, thus $G = \bigcup_{u\in G^0} G_u^u$, and so G is group bundle. □

4 The Fourier transform and spectrum of $I(G,\lambda,\mu)$

In this section we define the Fourier transform on $I(G,\lambda,\mu)$ and show that, as in the group case, the spectrum of this algebra is the dual groupoid of G, when G is a group bundle. If G is not a group bundle, this algebra is not commutative and may have a trivial (even empty) spectrum. Throughout the rest of the paper, G is an abelian group bundle and for some technical reasons, we put $\xi_u = 0$ on $G^u \setminus G_u^u$, for each $u \in G^0$.

Definition 4.1 *The Fourier transform is the map* $\mathfrak{F} : I(G,\lambda,\mu) \longrightarrow C_0(\widehat{G})$; $f \longmapsto \mathfrak{F}(f)$ *defined by*

$$\mathfrak{F}(f)(\xi) = \widehat{f}(\xi) = \int_{G^u} \overline{\xi(x)} f(x) d\lambda^u(x), \tag{4.1}$$

where $\xi = \xi_u$, *for some* $u \in G^0$.

Lemma 4.2 *The map* $\mathfrak{F}$ *is a norm decreasing* $*$*-homomorphism from* $I(G,\lambda,\mu)$ *to* $C_0(\widehat{G})$ *with a dense range.*

Proof. For $f, g \in I(G,\lambda,\mu)$ and $\xi \in \widehat{G}$, we have

$$\begin{aligned}(f*g)\hat{}(\xi) &= \int_{G^u} \overline{\xi(x)} f*g(x)d\lambda^u(x) \\ &= \int_{G^u}\int_{G^u} \overline{\xi(yx)} f(y)g(x)d\lambda^u(y)d\lambda^u(x) \\ &= \int_{G^u}\int_{G^u} \overline{\xi(yx)} f(y)g(x)d\lambda^u(y)d\lambda^u(x) \\ &= \int_{G^u} \overline{\xi(y)} f(y)d\lambda^u(y) \int_{G^u} \overline{\xi(x)} g(x)d\lambda^u(x) = (f)\hat{}(\xi)(g)\hat{}(\xi),\end{aligned}$$

and

$$\begin{aligned}(f^*)\hat{}(\xi) &= \int_{G^u} \overline{\xi(x)} f^*(x)d\lambda^u(x) = \int_{G^u} \overline{\xi(x) f(x^{-1})} D(x^{-1})d\lambda^u(x) \\ &= \overline{\int_{G^u} \overline{\xi(x)} f(x^{-1})d\lambda^u(x)} = \overline{\hat{f}}(\xi).\end{aligned}$$

Moreover,

$$|\hat{f}(\xi)| = |\int_{G^u} \overline{\xi(x)}(f)(x)d\lambda^u(x)| \leq \int_{G_u^u} |f(x)|d\lambda^u(x),$$

and a similar inequality holds for the range replaced by the source. Thus $\|\hat{f}\|_\infty \leq \|f\|_{I,\mu}$.

The fact that $\mathfrak{F}$ is a $*$-homomorphism means that $I(G,\lambda,\mu)$ is symmetric, and by the Stone-Weierstrass theorem, the range of $\mathfrak{F}$ is a dense subset of $C_0(\widehat{G})$. The fact that $\mathfrak{F}(I(G,\lambda,\mu)) \subseteq C_0(\widehat{G})$ follows form the Riemann-Lebesgue lemma. □

If $\mu \in M(G)$ one may define the Fourier-Stieltjes transform on $M(G)$ by $\widehat{\mu}(\xi) = \int_G \overline{\xi(x)} d\mu(x)$. The map $\widehat{\mu} \mapsto \varphi_{\widehat{\mu}} : M(\widehat{G}) \to C_b(G)$ defined by

$$\varphi_{\widehat{\mu}}(x) = \int_{\widehat{G}} \overline{\xi(x)} d\widehat{\mu}(\xi)$$

is a norm decreasing linear injection, because if $\varphi_{\widehat{\mu}} = 0$, then for each $f \in I(G,\lambda,\mu)$,

$$0 = \int_{G^u}\int_{\widehat{G}} f(x)\overline{\xi(x)} d\widehat{\mu}(\xi)d\lambda^u(x) = \int_{\widehat{G}} \widehat{f}(\xi)d\widehat{\mu}(\xi).$$

Since $\mathfrak{F}(I(G,\lambda,\mu))$ is dense in $C_0(\widehat{G})$, we get $\widehat{\mu} = 0$. Indeed from the Pontrjagin duality theorem, $\varphi_{\widehat{\mu}}$ is the Fourier-Stieltjes transform on $M(\widehat{G})$ and is one to one. Therefore the Fourier-Stieltjes transform on $M(G)$ is one to one.

Now let $\widehat{f} = 0$, then for each $\xi \in \widehat{G}$,

$$0 = \widehat{f}(\xi) = \int_G \overline{\xi(x)} f(x)d\lambda^u(x) = \int_G \overline{\xi(x)} d\mu(x) = \widehat{\mu}(\xi),$$

thus $\widehat{\mu} = 0$. Therefore from the injectivity of Fourier-Stieltjes transform on $M(G)$, $\mu = 0$ and since $\|\mu\| = \|f\|_{I,\mu}$, we also have $f = 0$. This shows that the Fourier transform is one to one.

Proposition 4.3 $\sigma(I(G,\lambda,\mu)) = \widehat{G}$.

Proof. $\mathfrak{F}$ is the same as the Gelfand transform, thus its range is dense in $C_0(\sigma(I(G, \lambda, \mu)))$. Also from the previous lemma its range is dense in $C_0(\widehat{G})$. We show that, the supremum norm on $C_0(\sigma(I(G, \lambda, \mu)))$ by $\|.\|_\infty$ and the supremum norm on $C_0(\widehat{G})$ by $\|.\|^\infty$. Consider the map $id : Im\mathfrak{F} \subseteq C_0(\sigma(I(G,\lambda,\mu))) \to Im\mathfrak{F} \subseteq C_0(\widehat{G})$. Since $\widehat{G} \subseteq \sigma(I(G,\lambda,\mu))$, id is a continuous homomorphism. If $f \in I(G,\lambda,\mu)$ and $\widehat{f}|_{\widehat{G}} = 0$ then $\widehat{f}|_{\sigma(I(G,\lambda,\mu))} = 0$. Indeed, if $h \in \sigma(I(G,\lambda,\mu))$, then by definition of $\sigma(I(G,\lambda,\mu))$, one may choose an element $f \neq 0$ in $I(G,\lambda,\mu); \widehat{f}(h) \neq 0$. Thus there exists $\xi \in \widehat{G}; \widehat{f}(\xi) \neq 0$ (by the way of contraction, if $\widehat{f}(\xi) = 0$ for each ξ, then from the injectivity of the Fourier transform, $f = 0$). This means that id is injective, therefore id has an extension to a map $id : C_0(\sigma(I(G,\lambda,\mu))) \to C_0(\widehat{G})$ which is an injective continuous homomorphism, and so an isometry. Therefore its range is closed and dense, that is, id is surjective. Thus $C_0(\widehat{G}) \cong C_0(\sigma(I(G,\lambda,\mu)))$, and therefore $\sigma(I(G,\lambda,\mu)) \simeq \widehat{G}$. □

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