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Pseudo-slant lightlike submanifolds of indefinite Sasakian manifolds

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Abstract In this paper, we introduce the notion of pseudo-slant lightlike submanifolds of indefinite Sasakian manifolds giving characterization theorem with some non-trivial examples of such submanifolds. Integrability conditions of distributions D_1, D_2 and $RadTM$ on pseudo-slant lightlike submanifolds of an indefinite Sasakian manifold have been obtained. We also obtain necessary and sufficient conditions for foliations determined by above distributions to be totally geodesic.

Keywords manifolds with indefinite metrics · Global submanifolds · Sasakian structure on manifolds

Mathematics Subject Classification (2010) 53C15 · 53C40 · 53C50

1 Introduction

In 1990, Chen defined slant immersions in complex geometry as a natural generalization of both holomorphic immersions and totally real immersions (see [4]). Further, Lotta introduced the concept of slant immersions of a Riemannian manifold into an almost contact metric manifold (see [8]). The geometry of slant and semi-slant submanifolds of Sasakian manifolds was studied by Cabrerizo, Carriazo, Fernandez and Fernandez in (see [2], [3]). Carriazo defined and studied bi-slant submanifolds of almost Hermitian and almost contact metric manifolds and further gave the notion of pseudo-slant submanifolds (see [1]). The theory of lightlike submanifolds of a semi-Riemannian manifold was introduced by Duggal and Bejancu (see [5]). A submanifold M of a semi-Riemannian manifold $\overline{M}$ is said to be lightlike submanifold if the induced metric g on M is degenerate, i.e. there exists a non-zero $X \in \Gamma(TM)$ such that $g(X, Y) = 0$, for all $Y \in \Gamma(TM)$. Lightlike geometry has its applications in general relativity, particularly in black hole theory, which gave impetus to study lightlike submanifolds of semi-Riemannian manifolds equipped with certain structures.

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The theory of slant, contact Cauchy-Riemann lightlike submanifolds has been studied in (see [7], [10]). The objective of this paper is to introduce the notion of pseudo-slant lightlike submanifolds of indefinite Sasakian manifolds. This new class of lightlike submanifolds of an indefinite Sasakian manifold includes slant, contact Cauchy-Riemann lightlike submanifolds as its sub-cases. The paper is arranged as follows. There are some basic results in section 2. In section 3, we study pseudo-slant lightlike submanifolds of an indefinite Sasakian manifold, giving some examples. Section 4 is devoted to the study of foliations determined by distributions on pseudo-slant lightlike submanifolds of indefinite Sasakian manifolds.

2 Preliminaries

A submanifold (M^m, g) immersed in a semi-Riemannian manifold $(\overline{M}^{m+n}, \overline{g})$ is called a lightlike submanifold [5] if the metric g induced from $\overline{g}$ is degenerate and the radical distribution $RadTM$ is of rank r, where $1 \leq r \leq m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $RadTM$ in TM, that is

$$TM = RadTM \oplus_{orth} S(TM). \tag{2.1}$$

Now consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $RadTM$ in $TM^{\perp}$. Since for any local basis $\{\xi_i\}$ of $RadTM$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$ and $\overline{g}(N_i, N_j) = 0$, it follows that there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$. Let $tr(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\overline{M}|_M$. Then:

$$tr(TM) = ltr(TM) \oplus_{orth} S(TM^{\perp}), \tag{2.2}$$

$$T\overline{M}|_M = TM \oplus tr(TM), \tag{2.3}$$

$$T\overline{M}|_M = S(TM) \oplus_{orth} [RadTM \oplus ltr(TM)] \oplus_{orth} S(TM^{\perp}). \tag{2.4}$$

Following are four cases of a lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$:

Case.1 r-lightlike if $r < \min(m, n)$,

Case.2 co-isotropic if $r = n < m$, $S\left(TM^{\perp}\right) = \{0\}$,

Case.3 isotropic if $r = m < n$, $S\left(TM\right) = \{0\}$,

Case.4 totally lightlike if $r = m = n$, $S(TM) = S(TM^{\perp}) = \{0\}$.

The Gauss and Weingarten formulae are given as

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V, \tag{2.6}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$, where $\nabla_X Y, A_V X$ belong to $\Gamma(TM)$ and $h(X, Y), \nabla^t_X V$ belong to $\Gamma(tr(TM))$. ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$ respectively. The second fundamental form h is a symmetric $F(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(tr(TM))$ and the shape operator A_V

is a linear endomorphism of $\Gamma(TM)$. From (2.5) and (2.6), for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, we have:

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{2.7}$$
$$\overline{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \tag{2.8}$$
$$\overline{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{2.9}$$

where $h^l(X, Y) = L(h(X, Y))$, $h^s(X, Y) = S(h(X, Y))$, $D^l(X, W) = L(\nabla^t_X W)$, $D^s(X, N) = S(\nabla^t_X N)$. L and S are the projection morphisms of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$ respectively. ∇^l and ∇^s are linear connections on $ltr(TM)$ and $S(TM^{\perp})$ called the lightlike connection and screen transversal connection on M respectively.

Now by using (2.5), (2.7)-(2.9) and metric connection $\overline{\nabla}$, we obtain:

$$\overline{g}(h^s(X, Y), W) + \overline{g}(Y, D^l(X, W)) = g(A_W X, Y), \tag{2.10}$$
$$\overline{g}(D^s(X, N), W) = \overline{g}(N, A_W X). \tag{2.11}$$

Denote the projection of TM on $S(TM)$ by $\overline{P}$. Then from the decomposition of the tangent bundle of a lightlike submanifold, for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, we have:

$$\nabla_X \overline{P}Y = \nabla^*_X \overline{P}Y + h^*(X, \overline{P}Y), \tag{2.12}$$
$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi. \tag{2.13}$$

By using above equations, we obtain:

$$\overline{g}(h^l(X, \overline{P}Y), \xi) = g(A^*_\xi X, \overline{P}Y), \tag{2.14}$$
$$\overline{g}(h^*(X, \overline{P}Y), N) = g(A_N X, \overline{P}Y), \tag{2.15}$$
$$\overline{g}(h^l(X, \xi), \xi) = 0, \quad A^*_\xi \xi = 0. \tag{2.16}$$

It is important to note that in general ∇ is not a metric connection. Since $\overline{\nabla}$ is metric connection, by using (2.7), we get

$$(\nabla_X g)(Y, Z) = \overline{g}(h^l(X, Y), Z) + \overline{g}(h^l(X, Z), Y). \tag{2.17}$$

A semi-Riemannian manifold $(\overline{M}, \overline{g})$ is called an ϵ-almost contact metric manifold (see [6]) if there exists a (1, 1) tensor field ϕ, a vector field V called characteristic vector field and a 1-form η, satisfying:

$$\phi^2 X = -X + \eta(X)V, \quad \eta(V) = \epsilon, \quad \eta \circ \phi = 0, \quad \phi V = 0, \tag{2.18}$$
$$\overline{g}(\phi X, \phi Y) = \overline{g}(X, Y) - \epsilon\eta(X)\eta(Y), \tag{2.19}$$

for all $X, Y \in \Gamma(T\overline{M})$, where $\epsilon = 1$ or -1. It follows that

$$\overline{g}(V, V) = \epsilon, \tag{2.20}$$
$$\overline{g}(X, V) = \eta(X), \tag{2.21}$$
$$\overline{g}(X, \phi Y) = -\overline{g}(\phi X, Y). \tag{2.22}$$

Then $(\phi, V, \eta, \overline{g})$ is called an ϵ-almost contact metric structure on $\overline{M}$.

An ϵ-almost contact metric structure $(\phi, V, \eta, \overline{g})$ is called an indefinite Sasakian structure if and only if

$$(\overline{\nabla}_X \phi)Y = \overline{g}(X, Y)V - \epsilon\eta(Y)X, \tag{2.23}$$

for all $X, Y \in \Gamma(T\overline{M})$, where $\overline{\nabla}$ is Levi-Civita connection with respect to $\overline{g}$.

A semi-Riemannian manifold endowed with an indefinite Sasakian structure is called an indefinite Sasakian manifold. From (2.23), for any $X \in \Gamma(T\overline{M})$, we get

$$\overline{\nabla}_X V = -\phi X. \tag{2.24}$$

Let $(\overline{M}, \overline{g}, \phi, V, \eta)$ be an ϵ-almost contact metric manifold. If $\epsilon = 1$, then $\overline{M}$ is said to be a spacelike ϵ-almost contact metric manifold and if $\epsilon = -1$, then $\overline{M}$ is called a timelike ϵ-almost contact metric manifold. In this paper, we consider indefinite Sasakian manifolds with spacelike characteristic vector field V.

3 Pseudo-slant lightlike submanifolds

In this section, we introduce the notion of pseudo-slant lightlike submanifolds of indefinite Sasakian manifolds. At first, we state the following Lemmas for later use:

Lemma 3.1 *Let M be a r-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ of index $2q$ with structure vector field tangent to M. Suppose that $\phi RadTM$ is a distribution on M such that $RadTM \cap \phi RadTM = \{0\}$. Then $\phi ltr(TM)$ is a subbundle of the screen distribution $S(TM)$ and $\phi RadTM \cap \phi ltr(TM) = \{0\}$.*

Lemma 3.2 *Let M be a q-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ of index $2q$ with structure vector field tangent to M. Suppose $\phi RadTM$ is a distribution on M such that $RadTM \cap \phi RadTM = \{0\}$. Then any complementary distribution to $\phi RadTM \oplus \phi ltr(TM)$ in $S(TM)$ is Riemannian.*

The proofs of Lemma 3.1 and Lemma 3.2 follow as in Lemma 3.1 and Lemma 3.2 respectively of [10], so we omit them.

Definition 3.3 *Let M be a q-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ of index $2q$ such that $q < dim(M)$ with structure vector field tangent to M. Then we say that M is a pseudo-slant lightlike submanifold of $\overline{M}$ if following conditions are satisfied:*

(i) *$\phi RadTM$ is a distribution on M such that $RadTM \cap \phi RadTM = \{0\}$,*
(ii) *there exists non-degenerate orthogonal distributions D_1 and D_2 on M such that $S(TM) = (\phi RadTM \oplus \phi ltr(TM)) \oplus_{orth} D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\}$,*
(iii) *the distribution D_1 is anti-invariant, i.e. $\phi D_1 \subseteq S(TM^\perp)$,*
(iv) *the distribution D_2 is slant with angle $\theta(\neq \pi/2)$, i.e. for each $x \in M$ and each non-zero vector $X \in (D_2)_x$, the angle θ between ϕX and the vector subspace $(D_2)_x$ is a constant$(\neq \pi/2)$, which is independent of the choice of $x \in M$ and $X \in (D_2)_x$.*

This constant angle θ is called slant angle of distribution D_2. A screen pseudo-slant lightlike submanifold is said to be proper if $D_1 \neq \{0\}$, $D_2 \neq \{0\}$ and $\theta \neq 0$.

From the above definition, we have the following decomposition

$$\begin{aligned} TM = RadTM \oplus_{orth} (\phi RadTM \oplus \phi ltr(TM)) \\ \oplus_{orth} D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\}. \end{aligned} \tag{3.1}$$

In particular, we have:

(i) if $D_1 = 0$, then M is a slant lightlike submanifold,
(ii) if $D_1 \neq 0$ and $\theta = 0$, then M is a contact CR-lightlike submanifold.

Thus above new class of lightlike submanifolds of an indefinite Sasakian manifold includes slant, contact Cauchy-Riemann lightlike submanifolds as its sub-cases which have been studied in (see [7], [10]).

Let $(\mathbb{R}^{2m+1}_{2q}, \overline{g}, \phi, \eta, V)$ denote the manifold $\mathbb{R}^{2m+1}_{2q}$ with its usual Sasakian structure given by

$$\eta = \frac{1}{2}(dz - \sum_{i=1}^{m} y^i dx^i), V = 2\partial z,$$

$$\overline{g} = \eta \otimes \eta + \frac{1}{4}(-\sum_{i=1}^{q} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i),$$

$$\phi(\sum_{i=1}^{m}(X_i\partial x_i + Y_i\partial y_i) + Z\partial z) = \sum_{i=1}^{m}(Y_i\partial x_i - X_i\partial y_i) + \sum_{i=1}^{m} Y_i y^i \partial z,$$

where (x^i, y^i, z) are the cartesian coordinates on $\mathbb{R}^{2m+1}_{2q}$. Now we construct some examples of pseudo-slant lightlike submanifolds of an indefinite Sasakian manifold.

Example 3.1 Let $(\mathbb{R}^{13}_2, \overline{g}, \phi, \eta, V)$ be an indefinite Sasakian manifold, where $\overline{g}$ is of signature $(-,+,+,+,+,+,-,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial z\}$.

Suppose M is a submanifold of $\mathbb{R}^{13}_2$ given by $x^1 = y^2 = u_1$, $x^2 = u_2$, $y^1 = u_3$, $x^3 = y^4 = u_4$, $x^4 = y^3 = u_5$, $x^5 = u_6 \cos u_7$, $y^5 = u_6 \sin u_7$, $x^6 = \cos u_6$, $y^6 = \sin u_6$, $z = u_8$.

The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\}$, where
$Z_1 = 2(\partial x_1 + \partial y_2 + y^1\partial z)$,
$Z_2 = 2(\partial x_2 + y^2\partial z)$,
$Z_3 = 2\partial y_1$,
$Z_4 = 2(\partial x_3 + \partial y_4 + y^3\partial z)$,
$Z_5 = 2(\partial x_4 + \partial y_3 + y^4\partial z)$,
$Z_6 = 2(\cos u_7\partial x_5 + \sin u_7\partial y_5 - \sin u_6\partial x_6 + \cos u_6\partial y_6 + \cos u_7 y^5\partial z - \sin u_6 y^6\partial z)$,
$Z_7 = 2(-u_6 \sin u_7\partial x_5 + u_6 \cos u_7\partial y_5 - u_6 \sin u_7 y^5\partial z)$,
$Z_8 = V = 2\partial z$.

Hence $RadTM = span\{Z_1\}$ and $S(TM) = span\{Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, V\}$.

Now $ltr(TM)$ is spanned by $N_1 = \partial x_1 - \partial y_2 + y^1\partial z$ and $S(TM^{\perp})$ is spanned by:
$W_1 = 2(\partial x_3 - \partial y_4 + y^3\partial z)$,
$W_2 = 2(\partial x_4 - \partial y_3 + y^4\partial z)$,
$W_3 = 2(\cos u_7\partial x_5 + \sin u_7\partial y_5 + \sin u_6\partial x_6 - \cos u_6\partial y_6 + \cos u_7 y^5\partial z + \sin u_6 y^6\partial z)$,
$W_4 = 2(u_6 \cos u_6\partial x_6 + u_6 \sin u_6\partial y_6 + u_6 \cos u_6 y^6\partial z)$.

It follows that $\phi Z_1 = Z_2 - Z_3$, which implies $\phi RadTM$ is a distribution on M. On the other hand, we can see that $D_1 = span\{Z_4, Z_5\}$ such that $\phi Z_4 = W_2$, $\phi Z_5 = W_1$, which implies D_1 is anti-invariant with respect to ϕ and $D_2 = span\{Z_6, Z_7\}$ is a slant distribution with slant angle $\pi/4$. Hence M is a pseudo-slant 2-lightlike submanifold of $\mathbb{R}^{13}_2$.

Example 3.2 Let $(\mathbb{R}^{13}_2, \overline{g}, \phi, \eta, V)$ be an indefinite Sasakian manifold, where $\overline{g}$ is of signature $(-,+,+,+,+,+,-,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial z\}$.

Suppose M is a submanifold of $\mathbb{R}^{13}_2$ given by $-x^1 = y^2 = u_1$, $x^2 = u_2$, $y^1 = u_3$, $x^3 = u_4 \cos\beta$, $y^3 = u_4 \sin\beta$, $x^4 = u_5 \sin\beta$, $y^4 = u_5 \cos\beta$, $x^5 = u_6$, $y^5 = u_7$, $x^6 = k\cos u_7$, $y^6 = k \sin u_7$, $z = u_8$, where k is any constant.

The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$, where

$Z_1 = 2(-\partial x_1 + \partial y_2 - y^1 \partial z)$,

$Z_2 = 2(\partial x_2 + y^2 \partial z)$,

$Z_3 = 2\partial y_1$,

$Z_4 = 2(\cos\beta \partial x_3 + \sin\beta \partial y_3 + y^3 \cos\beta \partial z)$,

$Z_5 = 2(\sin\beta \partial x_4 + \cos\beta \partial y_4 + y^4 \sin\beta \partial z)$,

$Z_6 = 2(\partial x_5 + y^5 \partial z)$,

$Z_7 = 2(\partial y_5 - k \sin u_7 \partial x_6 + k\cos u_7 \partial y_6 - k \sin u_7 y^6 \partial z)$,

$Z_8 = V = 2\partial z$.

Hence $RadTM = span\{Z_1\}$ and $S(TM) = span\{Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, V\}$.

Now $ltr(TM)$ is spanned by $N_1 = \partial x_1 + \partial y_2 + y^1 \partial z$ and $S(TM^\perp)$ is spanned by

$W_1 = 2(\sin\beta \partial x_3 - \cos\beta \partial y_3 + y^3 \sin\beta \partial z)$,

$W_2 = 2(\cos\beta \partial x_4 - \sin\beta \partial y_4 + y^4 \cos\beta \partial z)$,

$W_3 = 2(k\cos u_7 \partial x_6 + k \sin u_7 \partial y_6 + k\cos u_7 y^6 \partial z)$,

$W_4 = 2(k^2 \partial y_5 + k \sin u_7 \partial x_6 - k\cos u_7 \partial y_6 + k \sin u_7 y^6 \partial z)$.

It follows that $\phi Z_1 = Z_2 + Z_3$, which implies $\phi RadTM$ is a distribution on M. On the other hand, we can see that $D_1 = span\{Z_4, Z_5\}$ such that $\phi Z_4 = W_1$, $\phi Z_5 = W_2$, which implies D_1 is anti-invariant with respect to ϕ and $D_2 = span\{Z_6, Z_7\}$ is a slant distribution with slant angle $\theta = 1/\sqrt{1+k^2}$. Hence M is a pseudo-slant 2-lightlike submanifold of $\mathbb{R}^{13}_2$.

Now, for any vector field X tangent to M, we put $\phi X = PX + FX$, where PX and FX are tangential and transversal parts of ϕX respectively. We denote the projections on $RadTM$, $\phi RadTM$, $\phi ltr(TM)$, D_1 and D_2 in TM by P_1, P_2, P_3, P_4, and P_5 respectively. Similarly, we denote the projections of $tr(TM)$ on $ltr(TM)$, $\phi(D_1)$ and D' by Q_1, Q_2 and Q_3 respectively, where D' is non-degenerate orthogonal complementary subbundle of $\phi(D_1)$ in $S(TM^\perp)$. Then, for any $X \in \Gamma(TM)$, we get:

$$X = P_1 X + P_2 X + P_3 X + P_4 X + P_5 X + \eta(X) V. \tag{3.2}$$

Now applying ϕ to (3.2), we have

$$\phi X = \phi P_1 X + \phi P_2 X + \phi P_3 X + \phi P_4 X + \phi P_5 X, \tag{3.3}$$

which gives

$$\phi X = \phi P_1 X + \phi P_2 X + \phi P_3 X + \phi P_4 X + f P_5 X + F P_5 X, \tag{3.4}$$

where fP_5X (resp. FP_5X) denotes the tangential (resp. transversal) component of ϕP_5X. Thus we get $\phi P_1X \in \phi RadTM, \phi P_2X \in RadTM, \phi P_3X \in ltr(TM), \phi P_4X \in \phi(D_1) \subseteq S(TM^{\perp}), fP_5X \in \Gamma(D_2)$ and $FP_5X \in \Gamma(D')$.
Also, for any $W \in \Gamma(tr(TM))$, we have

$$W = Q_1W + Q_2W + Q_3W. \tag{3.5}$$

Applying ϕ to (3.5), we obtain

$$\phi W = \phi Q_1W + \phi Q_2W + \phi Q_3W, \tag{3.6}$$

which gives

$$\phi W = \phi Q_1W + \phi Q_2W + BQ_3W + CQ_3W, \tag{3.7}$$

where BQ_3W (resp. CQ_3W) denotes the tangential (resp. transversal) component of ϕQ_3W. Thus we get $\phi Q_1W \in \Gamma(\phi ltr(TM)), \phi Q_2W \in \Gamma(D_1)$, $BQ_3W \in \Gamma(D_2)$ and $CQ_3W \in \Gamma(D')$.

Now, by using (2.23), (3.4), (3.7) and (2.7)-(2.9) and identifying the components on $RadTM$, $\phi RadTM$, $\phi ltr(TM)$, D_1, D_2, $ltr(TM)$, $\phi(D_1)$, D' and $\{V\}$, we obtain:

$$\begin{aligned} &P_1(\nabla_X\phi P_1Y) + P_1(\nabla_X\phi P_2Y) - P_1(A_{\phi P_4Y}X) + P_1(\nabla_X fP_5Y) \\ &\quad = P_1(A_{FP_5Y}X) + P_1(A_{\phi P_3Y}X) + \phi P_2\nabla_XY - \eta(Y)P_1X, \end{aligned} \tag{3.8}$$

$$\begin{aligned} &P_2(\nabla_X\phi P_1Y) + P_2(\nabla_X\phi P_2Y) - P_2(A_{\phi P_4Y}X) + P_2(\nabla_X fP_5Y) \\ &\quad = P_2(A_{FP_5Y}X) + P_2(A_{\phi P_3Y}X) + \phi P_1\nabla_XY - \eta(Y)P_2X, \end{aligned} \tag{3.9}$$

$$\begin{aligned} &P_3(\nabla_X\phi P_1Y) + P_3(\nabla_X\phi P_2Y) - P_3(A_{\phi P_4Y}X) + P_3(\nabla_X fP_5Y) \\ &\quad = P_3(A_{FP_5Y}X) + P_3(A_{\phi P_3Y}X) + \phi h^l(X,Y) - \eta(Y)P_3X, \end{aligned} \tag{3.10}$$

$$\begin{aligned} &P_4(\nabla_X\phi P_1Y) + P_4(\nabla_X\phi P_2Y) - P_4(A_{\phi P_4Y}X) + P_4(\nabla_X fP_5Y) \\ &\quad = P_4(A_{FP_5Y}X) + P_4(A_{\phi P_3Y}X) + \phi Q_2h^s(X,Y) - \eta(Y)P_4X, \end{aligned} \tag{3.11}$$

$$\begin{aligned} P_5(\nabla_X\phi P_1Y) + &P_5(\nabla_X\phi P_2Y) - P_5(A_{\phi P_4Y}X) + P_5(\nabla_X fP_5Y) \\ &= P_5(A_{FP_5Y}X) + P_5(A_{\phi P_3Y}X) + fP_5\nabla_XY \\ &+ BQ_3h^s(X,Y) - \eta(Y)P_5X, \end{aligned} \tag{3.12}$$

$$\begin{aligned} &h^l(X,\phi P_1Y) + h^l(X,\phi P_2Y) + D^l(X,\phi P_4Y) + h^l(X,fP_5Y) \\ &\quad = \phi P_3\nabla_XY - \nabla^l_X\phi P_3Y - D^l(X,FP_5Y), \end{aligned} \tag{3.13}$$

$$\begin{aligned} &Q_2h^s(X,\phi P_1Y) + Q_2h^s(X,\phi P_2Y) + Q_2\nabla^s_X\phi P_4Y + Q_2h^s(X,fP_5Y) \\ &\quad = Q_2\nabla^s_XFP_5Y - Q_2D^s(X,\phi P_3Y) + \phi P_4\nabla_XY, \end{aligned} \tag{3.14}$$

$$\begin{aligned} &Q_3h^s(X,\phi P_1Y) + Q_3h^s(X,\phi P_2Y) + Q_3\nabla^s_X\phi P_4Y + Q_3h^s(X,fP_5Y) \\ &\quad = CQ_3h^s(X,Y) - Q_3\nabla^s_XFP_5Y - Q_3D^s(X,\phi P_3Y) + FP_5\nabla_XY, \end{aligned} \tag{3.15}$$

$$\begin{aligned} &\eta(\nabla_X\phi P_1Y) + \eta(\nabla_X\phi P_2Y) - \eta(A_{\phi P_4Y}X) + \eta(\nabla_X fP_5Y) \\ &\quad = \eta(A_{\phi P_3Y}X) + \eta(A_{FP_5Y}X) + \overline{g}(X,Y)V. \end{aligned} \tag{3.16}$$

Theorem 3.4 *Let M be a q-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ of index $2q$ with structure vector field tangent to M. Then M is a pseudo-slant lightlike submanifold of $\overline{M}$ if and only if:*

(i) $\phi RadTM$ is a distribution on M such that $RadTM \cap \phi RadTM = \{0\}$,
(ii) the distribution D_1 is an anti-invariant distribution, i.e. $\phi D_1 \subseteq S(TM^{\perp})$,
(iii) there exists a constant $\lambda \in (0,1]$ such that $P^2X = -\lambda X$.

Moreover, there also exists a constant $\mu \in [0,1)$ such that $BFX = -\mu X$, for all $X \in \Gamma(D_2)$, where D_1 and D_2 are non-degenerate orthogonal distributions on M such that $S(TM) = (\phi RadTM \oplus \phi ltr(TM)) \oplus_{orth} D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\}$ and $\lambda = \cos^2\theta$, θ is slant angle of D_2.

Proof. Let M be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then distribution D_1 is anti-invariant with respect to ϕ and $\phi RadTM$ is a distribution on M such that $RadTM \cap \phi RadTM = \{0\}$.

Now for any $X \in \Gamma(D_2)$, we have $|PX| = |\phi X| \cos\theta$, which implies

$$\cos\theta = \frac{|PX|}{|\phi X|}. \tag{3.17}$$

In view of (3.17), we get

$$\cos^2\theta = \frac{|PX|^2}{|\phi X|^2} = \frac{g(PX,PX)}{g(\phi X,\phi X)} = \frac{g(X,P^2X)}{g(X,\phi^2X)},$$

which gives

$$g(X,P^2X) = \cos^2\theta\, g(X,\phi^2X). \tag{3.18}$$

Since M is pseudo-slant lightlike submanifold and $\cos^2\theta = \lambda$ (constant) $\in (0,1]$ therefore from (3.18), we get $g(X,P^2X) = \lambda g(X,\phi^2X) = g(X,\lambda\phi^2X)$, which implies

$$g(X,(P^2 - \lambda\phi^2)X) = 0. \tag{3.19}$$

Since X is non-null vector, we have $(P^2 - \lambda\phi^2)X = 0$, which implies

$$P^2X = \lambda\phi^2X = -\lambda X. \tag{3.20}$$

Now, for any vector field $X \in \Gamma(D_2)$, we have

$$\phi X = PX + FX, \tag{3.21}$$

where PX and FX are tangential and transversal parts of ϕX respectively.

Applying ϕ to (3.21) and taking tangential component we get

$$-X = P^2X + BFX. \tag{3.22}$$

From (3.20) and (3.22), we get

$$BFX = -\sin^2\theta\, X, \tag{3.23}$$

where $\sin^2\theta = 1 - \lambda = \mu$ (constant) $\in [0,1)$. This proves (iii).

Conversely suppose that conditions (i), (ii) and (iii) are satisfied. From (3.22), for any $X \in \Gamma(D_2)$, we get

$$-X = P^2 X - \mu X, \tag{3.24}$$

which implies

$$P^2 X = -\cos^2\theta\, X, \tag{3.25}$$

where $\cos^2\theta = 1 - \mu = \lambda$ (constant)$\in (0, 1]$.

Now

$$\cos\theta = \frac{g(\phi X, PX)}{|\phi X||PX|} = \frac{g(X, \phi PX)}{|\phi X||PX|} = \frac{g(X, P^2 X)}{|\phi X||PX|}$$
$$= \lambda \frac{g(X, \phi^2 X)}{|\phi X||PX|} = \lambda \frac{g(\phi X, \phi X)}{|\phi X||PX|}.$$

From above equation, we get

$$\cos\theta = \lambda \frac{|\phi X|}{|PX|}. \tag{3.26}$$

Therefore (3.17) and (3.26) give $\cos^2\theta = \lambda$ (constant). □

Hence M is a pseudo-slant lightlike submanifold.

Corollary 3.5 *Let M be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with slant angle θ, then for any $X, Y \in \Gamma(D_2)$, we have:*

(i) $g(PX, PY) = \cos^2\theta\,(g(X, Y) - \eta(X)\eta(Y))$,
(ii) $g(FX, FY) = \sin^2\theta\,(g(X, Y) - \eta(X)\eta(Y))$.

The proof of above Corollary follows by using similar steps as in proof of Corollary 3.2 of [9].

Lemma 3.6 *Let M be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then for any $X, Y \in \Gamma(TM) - \{V\}$, we have:*

(i) $g(\nabla_X Y, V) = \overline{g}(Y, \phi X)$,
(ii) $g([X, Y], V) = 2\overline{g}(X, \phi Y)$.

Proof. Let M be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Since $\overline{\nabla}$ is a metric connection, from (2.7) and (2.24), for any $X, Y \in \Gamma(TM) - \{V\}$, we have:

$$g(\nabla_X Y, V) = \overline{g}(Y, \phi X). \tag{3.27}$$

From (2.22) and (3.27), we have $g([X, Y], V) = 2\overline{g}(X, \phi Y)$. □

Theorem 3.7 *Let M be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then $RadTM$ is integrable if and only if:*

(i) $P_1(\nabla_X \phi P_1 Y) = P_1(\nabla_Y \phi P_1 X)$, $P_5(\nabla_X \phi P_1 Y) = P_5(\nabla_Y \phi P_1 X)$ *and* $h^l(Y, \phi P_1 X) = h^l(X, \phi P_1 Y)$,
(ii) $Q_2 h^s(Y, \phi P_1 X) = Q_2 h^s(X, \phi P_1 Y)$ *and* $Q_3 h^s(Y, \phi P_1 X) = Q_3 h^s(X, \phi P_1 Y)$, *for all* $X, Y \in \Gamma(RadTM)$.

Proof. Let M be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$.

Let $X, Y \in \Gamma(RadTM)$. From (3.8), we have $P_1(\nabla_X \phi P_1 Y) = \phi P_2 \nabla_X Y$, which gives $P_1(\nabla_X \phi P_1 Y) - P_1(\nabla_Y \phi P_1 X) = \phi P_2[X, Y]$. In view of (3.12), we obtain $P_5(\nabla_X \phi P_1 Y) = fP_5 \nabla_X Y + Bh^s(X, Y)$, which implies $P_5(\nabla_X \phi P_1 Y) - P_5(\nabla_Y \phi P_1 X) = fP_5[X, Y]$. From (3.13), we have $h^l(X, \phi P_1 Y) = \phi P_3 \nabla_X Y$, which gives $h^l(X, \phi P_1 Y) - h^l(Y, \phi P_1 X) = \phi P_3[X, Y]$.

Also from (3.14), we get $Q_2 h^s(X, \phi P_1 Y) = \phi P_4 \nabla_X Y$, which gives $Q_2 h^s(X, \phi P_1 Y) - Q_2 h^s(Y, \phi P_1 X) = \phi P_4[X, Y]$. In view of (3.15), we obtain $Q_3 h^s(X, \phi P_1 Y) = Ch^s(X, Y) + FP_5 \nabla_X Y$, which implies $Q_3 h^s(X, \phi P_1 Y) - Q_3 h^s(Y, \phi P_1 X) = FP_5[X, Y]$. This concludes the theorem. □

Theorem 3.8 *Let M be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then D_1 is integrable if and only if:*

(i) $P_1(A_{\phi P_4 Y} X) = P_1(A_{\phi P_4 X} Y)$, $P_2(A_{\phi P_4 Y} X) = P_2(A_{\phi P_4 X} Y)$ *and* $P_5(A_{\phi P_4 Y} X) = P_5(A_{\phi P_4 X} Y)$,

(ii) $D^l(Y, \phi P_4 X) = D^l(X, \phi P_4 Y)$ *and* $Q_3 \nabla^s_Y \phi P_4 X = Q_3 \nabla^s_X \phi P_4 Y$, *for all* $X, Y \in \Gamma(D_1)$.

Proof. Let M be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Let $X, Y \in \Gamma(D_1)$. From (3.8), we have $P_1(A_{\phi P_4 Y} X) + \phi P_2 \nabla_X Y = 0$, which gives $P_1(A_{\phi P_4 X} Y) - P_1(A_{\phi P_4 Y} X) = \phi P_2[X, Y]$. From (3.9), we get $P_2(A_{\phi P_4 Y} X) + \phi P_1 \nabla_X Y = 0$, which gives $P_2(A_{\phi P_4 X} Y) - P_2(A_{\phi P_4 Y} X) = \phi P_1[X, Y]$. In view of (3.12), we obtain $P_5(A_{\phi P_4 Y} X) + fP_5 \nabla_X Y + BQ_3 h^s(X, Y) = 0$, which implies $P_5(A_{\phi P_4 X} Y) - P_5(A_{\phi P_4 Y} X) = fP_5[X, Y]$. From (3.13), we get $D^l(X, \phi P_4 Y) = \phi P_3 \nabla_X Y$, which gives $D^l(X, \phi P_4 Y) - D^l(Y, \phi P_4 X) = \phi P_3[X, Y]$. Also from (3.15), we have $Q_3 \nabla^s_X \phi P_4 Y = CQ_3 h^s(X, Y) + FP_5 \nabla_X Y$, which implies $Q_3 \nabla^s_X \phi P_4 Y - Q_3 \nabla^s_Y \phi P_4 X = FP_5[X, Y]$. This completes the proof. □

Theorem 3.9 *Let M be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then $D_2 \oplus \{V\}$ is integrable if and only if:*

(i) $P_1(\nabla_X fP_5 Y - \nabla_Y fP_5 X) = P_1(A_{FP_5 Y} X - A_{FP_5 X} Y)$,

(ii) $P_2(\nabla_X fP_5 Y - \nabla_Y fP_5 X) = P_2(A_{FP_5 Y} X - A_{FP_5 X} Y)$,

(iii) $h^l(X, fP_5 Y) - h^l(Y, fP_5 X) = D^l(Y, FP_5 X) - D^l(X, FP_5 Y)$,

(iv) $Q_2(\nabla^s_X FP_5 Y - \nabla^s_Y FP_5 X) = Q_2(h^s(X, fP_5 Y) - h^s(Y, fP_5 X))$, *for all* $X, Y \in \Gamma(D_2 \oplus \{V\})$.

Proof. Let M be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Let $X, Y \in \Gamma(D_2 \oplus \{V\})$. From (3.8), we have $P_1(\nabla_X fP_5 Y) - P_1(A_{FP_5 Y} X) = \phi P_2 \nabla_X Y$, which gives $P_1(\nabla_X fP_5 Y - \nabla_Y fP_5 X) - P_1(A_{FP_5 Y} X - A_{FP_5 X} Y) = \phi P_2[X, Y]$. From (3.9), we get $P_2(\nabla_X fP_5 Y) - P_2(A_{FP_5 Y} X) = \phi P_1 \nabla_X Y$, which gives

$$P_2(\nabla_X fP_5 Y - \nabla_Y fP_5 X) - P_2(A_{FP_5 Y} X - A_{FP_5 X} Y) = \phi P_1[X, Y].$$

In view of (3.13), we obtain $h^l(X, fP_5 Y) + D^l(X, FP_5 Y) = \phi P_3 \nabla_X Y$, which implies $h^l(X, fP_5 Y) - h^l(Y, fP_5 X) + D^l(X, FP_5 Y) - D^l(Y, FP_5 X) = \phi P_3[X, Y]$. From (3.14), we have

$$Q_2 h^s(X, fP_5 Y) - Q_2 \nabla^s_X FP_5 Y = \phi P_4 \nabla_X Y,$$

which gives $Q_2(\nabla^s_Y FP_5X - \nabla^s_X FP_5Y) + Q_2(h^s(X, fP_5Y) - h^s(Y, fP_5X)) = \phi P_4[X, Y]$. Thus, the theorem is completed. □

Theorem 3.10 *Let M be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field V tangent to M. Then induced connection ∇ is not a metric connection.*

Proof. Let M be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Suppose that the induced connection is a metric connection. Then $\nabla_X \phi P_2 Y \in \Gamma(RadTM)$ and $h^l(X, Y) = 0$ for all $X, Y \in \Gamma(TM)$. Thus for any $Z \in \phi RadTM$ and $W \in \phi ltr(TM)$, from (2.23), we have

$$\overline{\nabla}_W \phi Z - \phi \overline{\nabla}_W Z = \overline{g}(Z, W)V. \tag{3.28}$$

In view of (2.7), (3.28) and taking tangential components, we get

$$\begin{aligned} \nabla_W \phi Z - \phi P_1 \nabla_W Z - \phi P_2 \nabla_W Z - \phi Q_2 h^s(Z, W) \\ = fP_5 \nabla_W Z + BQ_3 h^s(Z, W) + \overline{g}(Z, W)V. \end{aligned} \tag{3.29}$$

Since $TM = RadTM \oplus_{orth} (\phi RadTM \oplus \phi ltr(TM)) \oplus_{orth} D_1 \oplus_{orth} D_2 \oplus_{orth} V$, from (3.29), we obtain

$$\nabla_W \phi Z - \phi P_2 \nabla_W Z = 0, \ \phi P_1 \nabla_W Z = 0, \ \phi Q_2 h^s(Z, W) = 0, \tag{3.30}$$

$$fP_5 \nabla_W Z - BQ_3 h^s(Z, W) = 0, \quad \overline{g}(Z, W)V = 0. \tag{3.31}$$

Now taking $W = \phi N$ and $Z = \phi \xi$ in (3.31), we get $\overline{g}(N, \xi)V = 0$.

Thus $V = 0$, which is a contradiction. Hence M does not have a metric connection. □

4 Foliations determined by distributions

In this section, we obtain necessary and sufficient conditions for foliations determined by distributions on a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold to be totally geodesic.

Definition 4.1 *A pseudo-slant lightlike submanifold M of an indefinite Sasakian manifold $\overline{M}$ is said to be mixed geodesic if its second fundamental form h satisfies $h(X, Y) = 0$, for all $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$. Thus M is mixed geodesic pseudo-slant lightlike submanifold if $h^l(X, Y) = 0$ and $h^s(X, Y) = 0$, for all $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$.*

Theorem 4.2 *Let M be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then $RadTM$ defines a totally geodesic foliation if and only if $\nabla_X \phi P_2 Z + \nabla_X fP_5 Z = A_{\phi P_3 Z} X + A_{\phi P_4 Z} X + A_{FP_5 Z} X$, for all $X \in \Gamma(RadTM)$ and $Z \in \Gamma(S(TM))$.*

Proof. Let M be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. It is easy to see that $RadTM$ defines a totally geodesic foliation if and only if $\nabla_X Y \in RadTM$, for all $X, Y \in \Gamma(RadTM)$. Since $\overline{\nabla}$ is metric connection, from (2.7), (2.19), (2.23) and (3.4), for any $X, Y \in \Gamma(RadTM)$ and $Z \in \Gamma(S(TM))$, we get $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\overline{\nabla}_X(\phi P_2 Z + \phi P_3 Z + \phi P_4 Z + fP_5 Z + FP_5 Z), \phi Y)$, which implies $\overline{g}(\nabla_X Y, Z) = \overline{g}(A_{\phi P_3 Z} X + A_{\phi P_4 Z} X + A_{FP_5 Z} X - \nabla_X \phi P_2 Z - \nabla_X fP_5 Z, \phi Y)$. This proves the theorem. □

Theorem 4.3 *Let M be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then D_1 defines a totally geodesic foliation if and only if:*

(i) $\nabla^s_X FZ = -h^s(X, fZ)$,
(ii) $h^s(X, \phi N) = 0$ *and* $D^s(X, \phi W) = 0$, *for all* $X \in \Gamma(D_1)$, $Z \in \Gamma(D_2)$, $N \in \Gamma(ltr(TM))$ *and* $W \in \phi ltr(TM)$.

Proof. Let M be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. The distribution D_1 defines a totally geodesic foliation if and only if $\nabla_X Y \in D_1$, for all $X, Y \in \Gamma(D_1)$. Since $\overline{\nabla}$ is metric connection, using (2.7), (2.19) and (2.23), for any $X, Y \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$, we get $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\overline{\nabla}_X \phi Z, \phi Y)$, which gives $\overline{g}(\nabla_X Y, Z) = \overline{g}(\nabla^s_X FZ + h^s(X, fZ), \phi Y)$. In view of (2.7), (2.19) and (2.23), for any $X, Y \in \Gamma(D_1)$ and $N \in \Gamma(ltr(TM))$, we obtain $\overline{g}(\nabla_X Y, N) = -\overline{g}(\phi Y, \overline{\nabla}_X \phi N)$, which implies $\overline{g}(\nabla_X Y, N) = -\overline{g}(\phi Y, h^s(X, \phi N))$. Now, from (2.7), (2.19) and (2.23), for any $X, Y \in \Gamma(D_1)$ and $W \in \phi ltr(TM)$, we have $\overline{g}(\nabla_X Y, W) = -\overline{g}(\phi Y, \overline{\nabla}_X \phi W)$, which gives $\overline{g}(\nabla_X Y, W) = \overline{g}(\phi Y, D^s(X, \phi W))$. Thus, we obtain the required results. □

Theorem 4.4 *Let M be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then $D_2 \oplus \{V\}$ defines a totally geodesic foliation if and only if:*

(i) $h^s(X, \phi N) = 0$, $\nabla^s_X \phi Z = 0$ *and* $D^s(X, \phi W) = 0$,
(ii) $A_{\phi Z} X$, $\nabla_X \phi N$ *and* $A_{\phi W} X$, *have no component in* $D_2 \oplus \{V\}$, *for all* $X \in \Gamma(D_2 \oplus \{V\})$, $Z \in \Gamma(D_1)$, $N \in \Gamma(ltr(TM))$ *and* $W \in \phi ltr(TM)$.

Proof. Let M be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. The distribution $D_2 \oplus \{V\}$ defines a totally geodesic foliation if and only if $\nabla_X Y \in D_2 \oplus \{V\}$, for all $X, Y \in \Gamma(D_2 \oplus \{V\})$. Since $\overline{\nabla}$ is metric connection, using (2.7), (2.19) and (2.23), for any $X, Y \in \Gamma(D_2 \oplus \{V\})$ and $Z \in \Gamma(D_1)$, we have $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\overline{\nabla}_X \phi Z, \phi Y)$, which gives $\overline{g}(\nabla_X Y, Z) = \overline{g}(A_{\phi Z} X, fY) - \overline{g}(\nabla^s_X \phi Z, FY)$. In view of (2.7), (2.19) and (2.23), for any $X, Y \in \Gamma(D_2 \oplus \{V\})$ and $N \in \Gamma(ltr(TM))$, we obtain $\overline{g}(\nabla_X Y, N) = -\overline{g}(\phi Y, \overline{\nabla}_X \phi N)$, which implies $\overline{g}(\nabla_X Y, N) = -\overline{g}(fY, \nabla_X \phi N) - \overline{g}(FY, h^s(X, \phi N))$. Now, from (2.7), (2.19) and (2.23), for any $X, Y \in \Gamma(D_2 \oplus \{V\})$ and $W \in \phi ltr(TM)$, we get $\overline{g}(\nabla_X Y, W) = -\overline{g}(\phi Y, \overline{\nabla}_X \phi W)$, which gives $\overline{g}(\nabla_X Y, W) = \overline{g}(fY, A_{\phi W} X) - \overline{g}(FY, D^s(X, \phi W))$. This concludes the theorem. □

Theorem 4.5 *Let M be a mixed geodesic pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then D_1 defines a totally geodesic foliation if and only if:*

(i) $\nabla^s_X FZ = 0$,
(ii) $h^s(X, \phi N) = 0$ *and* $D^s(X, \phi W) = 0$, *for all* $X \in \Gamma(D_1)$, $Z \in \Gamma(D_2)$, $N \in \Gamma(ltr(TM))$ *and* $W \in \phi ltr(TM)$.

Proof. Let M be a mixed geodesic pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then $h(X, Y) = 0$, for all $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$. The distribution D_1 defines a totally geodesic foliation if and only if $\nabla_X Y \in D_1$, for all $X, Y \in \Gamma(D_1)$. Since $\overline{\nabla}$ is metric connection, using (2.7), (2.19) and (2.23), for any

$X, Y \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$, we get $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\overline{\nabla}_X \phi Z, \phi Y)$, which gives $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\nabla^s_X FZ + h^s(X, fZ), \phi Y)$. In view of (2.7), (2.19) and (2.23), for any $X, Y \in \Gamma(D_1)$ and $N \in \Gamma(ltr(TM))$, we obtain $\overline{g}(\nabla_X Y, N) = -\overline{g}(\phi Y, \overline{\nabla}_X \phi N)$, which implies $\overline{g}(\nabla_X Y, N) = -\overline{g}(\phi Y, h^s(X, \phi N))$. Now, from (2.7), (2.19) and (2.23), for any $X, Y \in \Gamma(D_1)$ and $W \in \phi ltr(TM)$, we have $\overline{g}(\nabla_X Y, W) = -\overline{g}(\phi Y, \overline{\nabla}_X \phi W)$, which gives $\overline{g}(\nabla_X Y, W) = -\overline{g}(\phi Y, D^s(X, \phi W))$. This completes the proof. □

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