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Sums and products of Darboux internally quasi-continuous functions

Paulina Szczuka

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Abstract In this paper we characterize the functions which can be written as the sum or the product of Darboux internally quasi-continuous functions, and find maximal additive and maximal multiplicative classes of the family of all Darboux internally quasi-continuous functions.

Keywords Darboux function · Quasi-continuous function · Internally quasi-continuous function · Sum of functions · Product of functions

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1 Preliminaries

The letters $\mathbb{R}$ and $\mathbb{N}$ denote the real line and the set of positive integers, respectively. The symbols $\mathrm{I}(a, b)$ and $\mathrm{I}[a, b]$ denote the open and the closed interval with endpoints a and b, respectively. For each $A \subset \mathbb{R}$ we use the symbols $\operatorname{int} A$, $\operatorname{cl} A$ and $\operatorname{bd} A$ to denote the interior, the closure and the boundary of A, respectively. We say that a set $A \subset \mathbb{R}$ is *simply open* [1], if it can be written as the union of an open set and a nowhere dense set.

The word function denotes a mapping from $\mathbb{R}$ into $\mathbb{R}$ unless otherwise explicitly stated. Set $c = \underline{\lim}_{t\to x^-} f(t)$ and $d = \overline{\lim}_{t\to x^-} f(t)$. We say that $x \in \mathbb{R}$ is a *Darboux point of f from the left*, if $c \le f(x) \le d$ and $f[(t, x)] \supset (c, d)$, for each $t < x$. Similarly we define the notion of a *Darboux point from the right*. We say that x is a *Darboux point of f*, if x is a Darboux point of f both from the left and from the right. Recall that f is a Darboux function[a] ($f \in \mathcal{D}$) iff each $x \in \mathbb{R}$ is a Darboux point of f (see, e.g., [2, Theorem 5.1]). We say that $f \in \mathcal{U}_0$ [3], if for all $a < b$ the set $f\big[[a, b]\big]$ is dense in $\mathrm{I}[f(a), f(b)]$.

The symbol $\mathcal{C}(f)$ stands for the set of all points of continuity of f. The set of all continuous functions we denote by $\mathcal{C}$. We say that f is *cliquish* (see [20]) ($f \in \mathcal{C}_q$), if

Paulina Szczuka
Kazimierz Wielki University
pl. Weyssenhoffa 11
85–072 Bydgoszcz, Poland
E-mail: paulinaszczuka@wp.pl

[a] We say that f is a *Darboux function* if it maps connected sets onto connected sets.

the set $\mathcal{C}(f)$ is dense in $\mathbb{R}$. We say that f is *internally cliquish* (see [19]) ($f \in \mathcal{C}_{qi}$), if the set $\operatorname{int}\mathcal{C}(f)$ is dense in $\mathbb{R}$. We say that f is *quasi-continuous* in the sense of KEMPISTY [9] ($f \in \mathcal{Q}$), if for all $x \in \mathbb{R}$ and open sets $U \ni x$ and $V \ni f(x)$, the set $\operatorname{int}(U \cap f^{-1}(V)) \neq \emptyset$. We say that f is *internally quasi-continuous* [14] ($f \in \mathcal{Q}_i$), if it is quasi-continuous and its set of points of discontinuity is nowhere dense; equivalently, f is internally quasi-continuous if $f \restriction \operatorname{int}\mathcal{C}(f)$ is dense in f. We say that x_0 is a point of internal quasi-continuity of f iff there is a sequence $(x_n) \subset \operatorname{int}\mathcal{C}(f)$ such that $x_n \to x_0$ and $f(x_n) \to f(x_0)$ (see [14]). The function f is *Darboux quasi-continuous* ($f \in \mathcal{DQ}$), if it is both Darboux and quasi-continuous. Analogously, the function f is *Darboux internally quasi-continuous* ($f \in \mathcal{DQ}_i$), if it is both Darboux and internally quasi-continuous. We say that f is constant ($f \in \mathcal{C}onst$) iff $f[\mathbb{R}]$ is a singleton.

Remark 1.1 We can easily see that the following inclusions are satisfied:

$$\mathcal{DQ} \subset \mathcal{Q} \subset \mathcal{C}_q, \;\; \mathcal{DQ}_i \subset \mathcal{Q}_i \subset \mathcal{C}_{qi} \subset \mathcal{C}_q, \;\; \mathcal{Q}_i \subset \mathcal{Q}, \;\; \mathcal{DQ}_i \subset \mathcal{DQ} \subset \mathcal{D} \subset \mathcal{U}_0.$$

For a family of functions $\mathcal{F}$ we define its maximal additive class $\mathcal{M}_a(\mathcal{F})$ as the set of all functions g such that $f + g \in \mathcal{F}$ whenever $f \in \mathcal{F}$. The maximal multiplicative class $\mathcal{M}_m(\mathcal{F})$ is defined analogously, i.e., as the set of all functions g such that $fg \in \mathcal{F}$ whenever $f \in \mathcal{F}$. The symbol $\mathcal{M}_D$ denotes the family of Darboux functions with the following property: if x_0 is right-hand (left-hand) point of discontinuity of f, then $f(x_0) = 0$ and there is a sequence (x_n) converging to x_0 such that $x_n > x_0$ ($x_n < x_0$) and $f(x_n) = 0$, for each $n \in \mathbb{N}$ (see [6] and [8]). The symbol $\mathcal{M}$ denotes the class of all functions f such that f has a zero in each interval in which it takes on both positive and negative values (see [4]). The symbol $[f = a]$ stands for the set $\{x \in \mathbb{R} : f(x) = a\}$.

2 Introduction

It was remarked by LINDENBAUM [10] (without proof) that each function f can be written as the sum of two Darboux functions. The proof of this result can be found in a paper written by SIERPIŃSKI [18]. In 1995 MALISZEWSKI characterized the sum of two quasi-continuous functions [11, Theorem 4.2]). Directly from this characterization and from [13, Theorem 4] follows:

Theorem 2.1 *For each function f the following conditions are equivalent:*

1. *there are Darboux quasi-continuous functions g_1 and g_2 such that $f = g_1 + g_2$,*
2. *f is a finite sum of Darboux quasi-continuous functions,*
3. *f is cliquish.*

In 1960 MARCUS remarked that not every function is the product of Darboux functions [15]. The problem of characterizing of the class of the products of Darboux functions was solved by CEDER [4], [5]. In 1985 GRANDE constructed a nonnegative Baire one function which cannot be the product of a finite number of quasi-continuous functions and asked for characterization of such products (see [7]). An answer to this question is the following theorem.

Theorem 2.2 [12, Theorem III.2.1] *For each function f the following conditions are equivalent:*

1. *there are quasi-continuous functions g_1 and g_2 with $f = g_1 g_2$,*
2. *f is a finite product of quasi-continuous functions,*
3. *f is cliquish and the set $[f = 0]$ is simply open.*

Products of quasi-continuous functions have been characterized by NATKANIEC in [16]. (Actually Natkaniec proved that statements 2 and 3 from Theorem 2.2 are equivalent. More precisely, he proved that each function which satisfies 3 can be written as the product of eight quasi-continuous functions. Maliszewski showed that it can be written as the product of two quasi-continuous functions.) Moreover MALISZEWSKI [12] characterized the product of Darboux quasi-continuous functions in the following way:

Theorem 2.3 [12, Theorem III.3.1] *For each function f the following conditions are equivalent:*

1. *there are Darboux quasi-continuous functions g_1 and g_2 with $f = g_1 g_2$,*
2. *f is a finite product of Darboux quasi-continuous functions,*
3. *$f \in \mathcal{M}$, f is cliquish and the set $[f = 0]$ is simply open.*

In 2013 the sums and the products of internally quasi-continuous functions were characterized by MARCINIAK and SZCZUKA (see [14, Theorem 4.10] and [19, Theorem 4.2]).

In this paper we examine smaller class of functions, namely, the family $\mathcal{DQ}_i$ of Darboux internally quasi-continuous functions. We characterize the sums and the products of $\mathcal{DQ}_i$. Results which we obtain are quite analogous to Theorems 2.1 and 2.3. Moreover we show that maximal additive and maximal multiplicative classes for $\mathcal{DQ}_i$ consist of constant functions only.

3 Auxiliary lemmas

Lemma 3.1 was proved by JASTRZĘBSKI, JĘDRZEJEWSKI and NATKANIEC [8, Lemma 2.2].

Lemma 3.1 *Let $\mathcal{F} \subset \mathcal{U}_0$. If $\mathcal{C}onst \subseteq \mathcal{M}_a(\mathcal{F})$ and $-1 \in \mathcal{M}_m(\mathcal{F})$ then $\mathcal{M}_a(\mathcal{F}) \subsetneq \mathcal{C}$. If moreover the class $\mathcal{F}$ fulfils the additional condition: if $f: \mathbb{R} \to (0,1)$ and $f \in \mathcal{F}$, then $1/f \in \mathcal{F}$, then also $\mathcal{M}_m(\mathcal{F}) \subseteq \mathcal{M}_D$.*

Lemmas 3.2 and 3.3 are due to MALISZEWSKI (see [12, Lemma III.1.10] and [12, Lemma III.1.1], respectively).

Lemma 3.2 *Let $I = (a,b)$, $\Gamma > 0$ be an extended real number and $k > 1$. There are functions $g_1, \dots, g_k$ such that $g_1 \dots g_k = 0$ on $\mathbb{R}$ and for $i \in \{1, \dots, k\}$: $\mathbb{R} \setminus \mathcal{C}(g_i) = \operatorname{bd} I$ and $g_i[(a,c)] = g_i[(c,b)] = (-\Gamma, \Gamma)$, for each $c \in I$.*

Lemma 3.3 *Let $A \subset \mathbb{R}$ be nowhere dense and closed and $\mathcal{I}$ be the family of all components of $\mathbb{R} \setminus A$. There are pairwise disjoint families $\mathcal{I}_1, \dots, \mathcal{I}_4 \subset \mathcal{I}$ such that for each $j \in \{1, \dots, 4\}$ and $x \in A$ if x is not isolated in A from the left (from the right), then there is a sequence $(I_{j,n}) \subset \mathcal{I}_j$ with $\inf I_{j,n} \nearrow x$ (with $\sup I_{j,n} \searrow x$, respectively).*

The proof of Lemma 3.4 can be found in [19, Lemma 3.3].

Lemma 3.4 *Let $I = (a,b)$ and the function $f: \operatorname{cl} I \to (0, +\infty)$ is continuous. There are continuous functions $\psi_1, \psi_2: I \to (0, +\infty)$ such that $f = \psi_1 \psi_2$ on I and $\psi_i[(a,c)] = \psi_i[(c,b)] = (0, +\infty)$, for each $i \in \{1, 2\}$ and $c \in I$.*

4 Sums of Darboux internally quasi-continuous functions

Remark 4.1 The sum of two internally cliquish functions is internally cliquish.

Proof. If the functions f and g are internally cliquish, then sets $\operatorname{int}\mathcal{C}(f)$ and $\operatorname{int}\mathcal{C}(g)$ are dense in $\mathbb{R}$. Hence the set $\operatorname{int}\mathcal{C}(f)\cap\operatorname{int}\mathcal{C}(g)$ is dense in $\mathbb{R}$, too. Moreover, $\operatorname{int}\mathcal{C}(f)\cap\operatorname{int}\mathcal{C}(g)\subset\operatorname{int}\mathcal{C}(f+g)$, which proves that the function $f+g$ is internally cliquish. □

Theorem 4.1 *For each function f the following conditions are equivalent:*

1. *there are Darboux internally quasi-continuous functions g_1 and g_2 with $f=g_1+g_2$,*
2. *f is a finite sum of Darboux internally quasi-continuous functions,*
3. *f is internally cliquish.*

Proof. The implication $1\Rightarrow 2$ is obvious and the implication $3\Rightarrow 1$ follows directly from [14, Theorem 4.10].

$2\Rightarrow 3$. Assume that there is $k\in\mathbb{N}$ and there are Darboux internally quasi-continuous functions $g_1,\ldots,g_k$ such that $f=g_1+\cdots+g_k$. Since each internally quasi-continuous function is internally cliquish, using Remark 4.1 we obtain that the function f is internally cliquish. □

It is well known that the maximal additive class of the family of all Darboux quasi-continuous functions consists of constant functions only [17, Theorem 2]. We will prove an analogous theorem. We start with the following assertion.

Proposition 4.2 *Let $I=[a,b]$ be a closed interval. For every non-constant, continuous function $f\colon I\to\mathbb{R}$ there is a Darboux internally quasi-continuous function $g\colon I\to\mathbb{R}$ such that $f+g$ does not have the Darboux property.*

Proof. The proof is a repetition of that of [17, Theorem 1] given by Natkaniec [17]. For fixed non-constant, continuous function he constructed a Darboux quasi-continuous function such that their sum does not have the Darboux property. In fact, such function is also internally quasi-continuous. □

Theorem 4.3 $\mathcal{M}_a(\mathcal{DQ}_i)=\mathcal{C}onst$.

Proof. We can easily see that $\mathcal{C}onst\subset\mathcal{M}_a(\mathcal{DQ}_i)$. Thus we need to prove only the opposite inclusion. Observe that all assumptions of Lemma 3.1 are fulfilled. So, $\mathcal{M}_a(\mathcal{DQ}_i)\subseteq\mathcal{C}$. Moreover, if $f\in\mathcal{C}\setminus\mathcal{C}onst$, there is $a<b$ such that f is continuous and non-constant on $[a,b]$. By Proposition 4.2, there is a Darboux internally quasi-continuous function $\bar{g}\colon[a,b]\to\mathbb{R}$ such that $f+\bar{g}$ does not have the Darboux property on $[a,b]$. Define the function $g\colon\mathbb{R}\to\mathbb{R}$ as follows:

$$g(x)=\begin{cases}\bar{g}(a), & \text{if } x\le a,\\ \bar{g}(x), & \text{if } x\in[a,b],\\ \bar{g}(b), & \text{if } x\ge b.\end{cases}$$

Then clearly $g\in\mathcal{DQ}_i$ and $f+g\notin\mathcal{D}$. This proves that $f\notin\mathcal{M}_a(\mathcal{DQ}_i)$. □

5 Products of Darboux internally quasi-continuous functions

Theorem 5.1 *For each function f the following conditions are equivalent:*

1. *there are Darboux internally quasi-continuous functions g_1 and g_2 with $f = g_1 g_2$,*
2. *f is a finite product of Darboux internally quasi-continuous functions,*
3. *$f \in \mathcal{M}$, f is internally cliquish and the set $[f = 0]$ is simply open.*

Proof. The implication $1 \Rightarrow 2$ is obvious.

$2 \Rightarrow 3$. Since $\mathcal{DQ}_i \subset \mathcal{DQ}$, by Theorem 2.3, $f \in \mathcal{M}$ and the set $[f = 0]$ is simply open. Moreover, since $\mathcal{DQ}_i \subset \mathcal{Q}_i$, by [19, Theorem 4.2] f is internally cliquish.

$3 \Rightarrow 1$. Assume that $f \in \mathcal{M}$, the set $[f = 0]$ is simply open and the function f is internally cliquish. Hence $\operatorname{int} \mathcal{C}(f)$ is dense in $\mathbb{R}$. Define $C := \operatorname{bd}[f = 0]$. Observe that the set C closed and and since $[f = 0]$ is simply open, C is nowhere dense. Let $\mathcal{I}$ be the family of all components of $\mathbb{R} \setminus C$. By Lemma 3.3, there are pairwise disjoint families $\mathcal{I}_1, \ldots, \mathcal{I}_4 \subset \mathcal{I}$ such that for each $j \in \{1, \ldots, 4\}$ and $x \in C$ if x is not isolated in C from the left (from the right), then there is a sequence $(I_{j,n}) \subset \mathcal{I}_j$ with $\inf I_{j,n} \nearrow x$ (with $\sup I_{j,n} \searrow x$, respectively). Without loss of generality we can assume that if $I \in \mathcal{I} \setminus \bigcup_{j=1}^{3} \mathcal{I}_j$, then $I \in \mathcal{I}_4$. Observe that, since $[f = 0]$ is simply open, we have only $I \cap [f = 0] = \emptyset$ or $I \subset [f = 0]$, for each $I \in \mathcal{I}$. So, if there is $I \in \mathcal{I}$ such that $I \cap [f = 0] = \emptyset$, then $f > 0$ on I or $f < 0$ on I.

Fix an interval $I \in \mathcal{I}$ and let $I = (a_I, b_I)$. If $I \subset [f = 0]$, then by Lemma 3.2 applied for $\Gamma = +\infty$ and $k = 2$, there are continuous functions $g_{1I}, g_{2I} \colon I \to \mathbb{R}$ such that $0 = f{\restriction}I = g_{1I} g_{2I}$ and for $i \in \{1, 2\}$

$$g_{iI}[(a_I, c)] = g_{iI}[(c, b_I)] = \mathbb{R}, \quad \text{for each } c \in I. \tag{5.1}$$

If $I \cap [f = 0] = \emptyset$, then $|f| > 0$ on I. Write the set $I \cap \operatorname{int} \mathcal{C}(f)$ as the union of family $\mathcal{J}$ consisting of the non-overlapping compact intervals, such that for each $x \in I \cap \operatorname{int} \mathcal{C}(f)$, there are $J_1, J_2 \in \mathcal{J}$ with $x \in \operatorname{int}(J_1 \cup J_2)$. Moreover, observe that since $\operatorname{int} \mathcal{C}(f)$ is open and dense in $\mathbb{R}$, the set $I \setminus \operatorname{int} \mathcal{C}(f)$ is nowhere dense.

Fix an interval $J \in \mathcal{J}$ and let $J = [a_J, b_J]$. Since $|f{\restriction}J| > 0$ and $J \subset \operatorname{int} \mathcal{C}(f)$, by Lemma 3.4, there are continuous functions $\psi_{1J}, \psi_{2J} \colon (a_J, b_J) \to (0, +\infty)$ such that $|f| = \psi_{1J} \psi_{2J}$ on (a_J, b_J) and for $i \in \{1, 2\}$

$$\psi_{iJ}[(a_J, c)] = \psi_{iJ}[(c, b_J)] = (0, +\infty), \quad \text{for each } c \in (a_J, b_J). \tag{5.2}$$

Now define functions $\psi_{1I}, \psi_{2I} \colon I \to \mathbb{R}$ as follows:

$$\psi_{1I}(x) = \begin{cases} \psi_{1J}(x) & \text{if } x \in \operatorname{int} J \text{ and } J \in \mathcal{J}, \\ 1 & \text{otherwise,} \end{cases}$$

$$\psi_{2I}(x) = \begin{cases} \psi_{2J}(x) & \text{if } x \in \operatorname{int} J \text{ and } J \in \mathcal{J}, \\ |f(x)| & \text{otherwise.} \end{cases}$$

Then clearly $|f{\restriction}I| = \psi_{1I} \psi_{2I}$. Moreover, using (5.2) and since the set $I \setminus \operatorname{int} \mathcal{C}(f)$ is nowhere dense and $|f{\restriction}I| > 0$, for each $i \in \{1, 2\}$ and each $c \in I$ holds:

$$\psi_{iI}[(a_I, c) \cap \operatorname{int} \mathcal{C}(\psi_{iI})] = \psi_{iI}[(c, b_I) \cap \operatorname{int} \mathcal{C}(\psi_{iI})] = (0, +\infty). \tag{5.3}$$

Now we will show that functions ψ_{1I} and ψ_{2I} are Darboux internally quasi - continuous on I. Fix $x \in I$. We consider two cases.

Case 1. $x \in I \cap \operatorname{int} \mathcal{C}(f)$

Then there are $J_1, J_2 \in \mathcal{J}$ such that $x \in \operatorname{int}(J_1 \cup J_2)$. If $x \in \operatorname{int} J_1 \cup \operatorname{int} J_2$, then since ψ_{1J_i} and ψ_{2J_i} are continuous on $\operatorname{int} J_i$ for $i \in \{1,2\}$, functions ψ_{1I} and ψ_{2I} are Darboux internally quasi-continuous at x. In the other case $x \in \operatorname{bd} J_1 \cap \operatorname{bd} J_2$. Since ψ_{1I} and ψ_{2I} are positive on $J_1 \cup J_2$ and continuous on $\operatorname{int} J_1 \cup \operatorname{int} J_2$, using (5.2) we clearly obtain that ψ_{1I} and ψ_{2I} are Darboux internally quasi-continuous at x.

Case 2. $x \in I \setminus \operatorname{int} \mathcal{C}(f)$

Fix $i \in \{1,2\}$. Since $I \setminus \operatorname{int} \mathcal{C}(f)$ is nowhere dense, by condition (5.2), for each $n \in \mathbb{N}$ there is $J_n \in \mathcal{J}$ such that $J_n \subset (x, x+\frac{1}{n})$ and there is $x_n \in J_n \cap \operatorname{int} \mathcal{C}(\psi_{iI})$ with $\psi_{iI}(x_n) = \psi_{iI}(x)$. Hence there is a sequence $(x_n) \subset \operatorname{int} \mathcal{C}(\psi_{iI})$ such that $x_n \to x$ and $\psi_{iI}(x_n) \to \psi_{iI}(x)$. Consequently, the function ψ_{iI} is internally quasi-continuous at x. Moreover, $\psi_{iI}(x) > 0$ and, by (5.2), $\psi_{iI}[J_n] = (0, \infty)$, for each $n \in \mathbb{N}$. So, it is easy to see that x is a Darboux point of ψ_{iI} from the right. Similarly we can show that x is a Darboux point of ψ_{iI} from the left. Consequently, x is a Darboux point of ψ_{iI}.

Further we define functions $g_1, g_2 \colon \mathbb{R} \to \mathbb{R}$ as follows:

$$g_1(x) = \begin{cases} \psi_{1I}(x), & \text{if } x \in I,\ I \cap [f=0] = \emptyset,\ I \in \mathcal{I}_1, \\ -\psi_{1I}(x), & \text{if } x \in I,\ I \cap [f=0] = \emptyset,\ I \in \mathcal{I}_2, \\ -1, & \text{if } f(x) \neq 0,\ I \cap [f=0] = \emptyset,\ x \in \operatorname{bd} I,\ I \in \mathcal{I}_2, \\ \psi_{1I}(x) \operatorname{sgn} f(x), & \text{if } x \in I,\ I \cap [f=0] = \emptyset,\ I \in \mathcal{I}_3, \\ \operatorname{sgn} f(x), & \text{if } f(x) \neq 0,\ I \cap [f=0] = \emptyset,\ x \in \operatorname{bd} I,\ I \in \mathcal{I}_3, \\ -\psi_{1I}(x) \operatorname{sgn} f(x), & \text{if } x \in I,\ I \cap [f=0] = \emptyset,\ I \in \mathcal{I}_4, \\ -\operatorname{sgn} f(x), & \text{if } f(x) \neq 0,\ I \cap [f=0] = \emptyset,\ x \in \operatorname{bd} I,\ I \in \mathcal{I}_4, \\ g_{1I}(x), & \text{if } x \in I,\ I \subset [f=0],\ I \in \mathcal{I}, \\ 0, & \text{if } f(x) = 0,\ x \in C, \\ 1, & \text{otherwise,} \end{cases}$$

$$g_2(x) = \begin{cases} \psi_{2I}(x) \operatorname{sgn} f(x), & \text{if } x \in I,\ I \cap [f=0] = \emptyset,\ I \in \mathcal{I}_1, \\ -\psi_{2I}(x) \operatorname{sgn} f(x), & \text{if } x \in I,\ I \cap [f=0] = \emptyset,\ I \in \mathcal{I}_2, \\ -f(x), & \text{if } f(x) \neq 0,\ I \cap [f=0] = \emptyset,\ x \in \operatorname{bd} I,\ I \in \mathcal{I}_2, \\ \psi_{2I}(x), & \text{if } x \in I,\ I \cap [f=0] = \emptyset,\ I \in \mathcal{I}_3, \\ |f(x)|, & \text{if } f(x) \neq 0,\ I \cap [f=0] = \emptyset,\ x \in \operatorname{bd} I,\ I \in \mathcal{I}_3, \\ -\psi_{2I}(x), & \text{if } x \in I,\ I \cap [f=0] = \emptyset,\ I \in \mathcal{I}_4, \\ -|f(x)|, & \text{if } f(x) \neq 0,\ I \cap [f=0] = \emptyset,\ x \in \operatorname{bd} I,\ I \in \mathcal{I}_4, \\ g_{2I}(x), & \text{if } x \in I,\ I \subset [f=0],\ I \in \mathcal{I}, \\ f(x), & \text{otherwise.} \end{cases}$$

Observe that, if x is bilaterally isolated in C and $f(x) \neq 0$, then there are $I, I' \in \mathcal{I}$ such that $x \in \operatorname{cl} I \cap \operatorname{cl} I'$ and $I \subset [f=0]$ or $I' \subset [f=0]$. Thus functions g_1 and g_2 are well defined. Clearly $f = g_1 g_2$. To complete the proof we must show that functions g_1 and g_2 are Darboux internally quasi-continuous.

Fix $i \in \{1,2\}$ and let $x \in \mathbb{R}$. First assume that there is $I \in \mathcal{I}$ such that $x \in I$. Since g_{iI} is continuous on I and ψ_{iI} is Darboux internally quasi-continuous on I, the function g_i is Darboux internally quasi-continuous at x. So, let $x \in C$. We consider two cases.

Case 1. x is isolated in C from the left.

Then there is $I \in \mathcal{I}$ such that $x = \sup I$. If $I \subset [f = 0]$, then using condition (5.1) we clearly obtain that g_i is internally quasi-continuous at x and x is a Darboux point from the left of g_i. So, we can assume that $I \cap [f = 0] = \emptyset$. Observe that, if $f(x) \neq 0$, then $\operatorname{sgn} g_i(x) = \operatorname{sgn} g_i(z)$, for each $z \in I$. So, since $g_i \restriction I$ is internally quasi-continuous, by (5.3), we clearly obtain that g_i is internally quasi-continuous at x. Moreover, it is easy to see that x is a Darboux point from the left of g_i.

Case 2. x is not isolated in C from the left.

Since for each $j \in \{1, \dots, 4\}$ there is a sequence $(I_{j,n}) \subset \mathcal{I}_j$ with $\inf I_{j,n} \nearrow x$, by (5.1), (5.3) and the definition of g_i, for each $n \in \mathbb{N}$ we have

$$\begin{aligned} g_1[(I_{2,n} \cup I_{1,n}) \cap \operatorname{int} \mathcal{C}(g_1)] \supset (-\infty, 0) \cup (0, +\infty) \quad \text{and} \\ g_2[(I_{4,n} \cup I_{3,n}) \cap \operatorname{int} \mathcal{C}(g_2)] \supset (-\infty, 0) \cup (0, +\infty). \end{aligned} \tag{5.4}$$

So, it is easy to see that the function g_i is internally quasi-continuous at x. Moreover, since x is not isolated in $C = \operatorname{bd}[f = 0]$ from the left, $(x - \delta, x) \cap [f = 0] \neq \emptyset$, for each $\delta > 0$. Therefore, $g_i\big[C \cap [f = 0]\big] = \{0\}$, which implies that

$$(x - \delta, x) \cap [g_i = 0] \neq \emptyset, \quad \text{for each } \delta > 0.$$

Hence and by (5.4), $g_i[((x-\delta, x)] = \mathbb{R}$ for each $\delta > 0$, which proves that x is a Darboux point from the left of g_i.

Similarly we can show that if $x \in C$, then x is a Darboux point from the right of g_i. So, for each $x \in \mathbb{R}$ the function g_i is internally quasi-continuous at x and x is a Darboux point of g_i. This completes the proof. □

It is well-known that the maximal multiplicative class of the family of all Darboux quasi-continuous functions contains constant functions only [17, Theorem 2]. Next two assertions show that the family of all Darboux internally quasi-continuous functions behave the same like the family of all Darboux quasi-continuous functions.

Proposition 5.2 *Let $I = [a, b]$ be a closed interval. For every non-constant, continuous function $f\colon I \to \mathbb{R}$ there is a Darboux internally quasi-continuous function $g\colon I \to \mathbb{R}$ such that fg does not have the Darboux property.*

Proof. The proof is a repetition of that of [17, Theorem 1] given by Natkaniec. For fixed non-constant, continuous function he constructed a Darboux quasi-continuous function such that their product does not have the Darboux property. In fact, such function is also internally quasi-continuous. □

Theorem 5.3 $\mathcal{M}_m(\mathcal{DQ}_i) = \mathcal{C}onst.$

Proof. We can easily see that $\mathcal{C}onst \subset \mathcal{M}_m(\mathcal{DQ}_i)$. So, we need to prove only the opposite inclusion. First we will show that $\mathcal{M}_m(\mathcal{DQ}_i) \subseteq \mathcal{C}$. Fix $f \in \mathcal{M}_m(\mathcal{DQ}_i)$ and suppose that f is not continuous. Observe that all assumptions of Lemma 3.1 are fulfilled. So, $\mathcal{M}_m(\mathcal{DQ}_i) \subseteq \mathcal{M}_D$. Since $f \in \mathcal{M}_D$, the set $\mathbb{R} \setminus \mathcal{C}(f)$ is nowhere dense, $f(x) = 0$ for each $x \in \operatorname{cl}(\mathbb{R} \setminus \mathcal{C}(f))$ and f is continuous on every component of the set $\mathbb{R} \setminus \operatorname{cl}(\mathbb{R} \setminus \mathcal{C}(f))$. Moreover, the condition $f \notin \mathcal{C}$ implies that $f \notin \mathcal{C}onst$. Since $f \in \mathcal{D}$, the range of f has the cardinality equals the continuum and consequently, there is a component I of $\mathbb{R} \setminus \operatorname{cl}(\mathbb{R} \setminus \mathcal{C}(f))$ such that $f \restriction I$ is continuous and not constant. So, there are $a, b \in I$ such that $a < b$, $f(a) \neq f(b)$ and $f \restriction [a, b] \in \mathcal{C}$. By Proposition 5.2, there is a Darboux internally quasi-continuous function $\bar{g} \colon [a, b] \to \mathbb{R}$ such that $f\bar{g}$ does not have the Darboux property on $[a, b]$. Define the function $g \colon \mathbb{R} \to \mathbb{R}$ as follows:

$$g(x) = \begin{cases} \bar{g}(a), & \text{if } x \leq a, \\ \bar{g}(x), & \text{if } x \in [a, b], \\ \bar{g}(b), & \text{if } x \geq b. \end{cases}$$

Then clearly $g \in \mathcal{DQ}_i$ and $fg \notin \mathcal{D}$. This proves that $f \notin \mathcal{M}_m(\mathcal{DQ}_i)$. □

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