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A characterization of some linear groups by nse

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Abstract Let G be a group and $\omega(G)$ be the set of element orders of G. Let $k \in \omega(G)$ and s_k be the number of elements of order k in G. Let $\mathrm{nse}(G) = \{s_k \big| k \in \omega(G)\}$. Recently, Ahanjideh et al proved that let G be a finite group and p a prime such that $p \mid |G|$ and $p^2 \nmid |G|$, then $G \cong A_p$ if $\mathrm{nse}(G) = \mathrm{nse}(A_p)$. Let $L_n(q)$ denote the projective special linear groups of degree n over finite fields of order q. The groups $L_3(2)$, $L_4(2)$ and $L_5(2)$ are characterizable by nse. So, is there a group $L_n(q)$ characterizable by nse only? In this note, we will prove the following. Let p be a prime such that $p = 2^n - 1$ and assume that $p \| |G|$. If $\mathrm{nse}(G) = \mathrm{nse}(L_n(2))$, then G is isomorphic to $L_n(2)$.

Keywords Element order · Projective special linear group · Thompson's problem · Number of elements of the same order

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1 Introduction

For a finite group G, let $\omega(G)$ be the set of element orders of G. If $k \in \omega(G)$ and s_k be the number of elements of order k in G, then let $\mathrm{nse}(G) = \{s_k \big| k \in \omega(G)\}$. J.G. Thompson put forward a very interesting problem related to algebraic number fields as follows (see [18] or [19], for instance).

Thompson's Problem

Let $T(G) = \{(n, s_n) \big| n \in \omega(G) \text{ and } s_n \in \mathrm{nse}(G)\}$, where s_n is the number of elements with order n. Suppose that $T(G) = T(H)$. If G is a finite solvable group, is it true that H is also necessarily solvable?

If we consider the sizes of elements of same order but not the actual orders of elements, then it remains only $\mathrm{nse}(G)$, whether can it characterize finite simple groups? Recently, some results are gotten as follows.

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Theorem 1.1 *Let G be a group and H be one of the following groups.*

(1) $L_5(2)$ *(see [15]);*
(2) *PGL(2,p) (see [2]);*
(3) S_r*, where r is a prime, $r-2$ is a prime and $r < 5 \cdot 10^8$ (see [3]);*
(4) A_{11}*, A_p (see [21, 14]).*

Then, $\mathrm{nse}(G) = \mathrm{nse}(H)$ *if and only if* $G \cong H$*.*

Ahanjideh et al in [1] prove that let G be a finite group and p a prime such that $p \mid |G|$ and $p^2 \nmid |G|$, then $G \cong A_p$ if $\mathrm{nse}(G) = \mathrm{nse}(A_p)$. By [4], [15] and [17], $L_3(2) \cong L_2(7)$, $L_4(2) \cong A_8$ and $L_5(2)$ are characterizable by nse only. So in this note, we give a new characterization of $L_n(2)$ with $2^n - 1$ prime, by using nse only. In fact the following is proved.

Main theorem

Let p be a prime such that $p = 2^n - 1$ and assume that $p \| |G|$. If $\mathrm{nse}(G) = \mathrm{nse}(L_n(2))$, then G is isomorphic to $L_n(2)$.

We introduce some notation which will be needed in the proof of the main theorem. Let $GK(G)$ be a graph with vertex set $\pi(G)$ such that two primes p and q in $\pi(G)$ are joined by an edge if G has an element of order $p \cdot q$. We set $s(G)$ denote the number of connected components of the prime graph $GK(G)$ and let $\pi_1, \pi_2, \cdots, \pi_{s(G)}$ be the connected components of $GK(G)$. If $2 \in \pi(G)$, we assume that $2 \in \pi_1(G)$. $|G|$ can be expressed as a product of co-prime positive integers OC_i, $i = 1, 2, \cdots, s(G)$. The sets of order components of finite simple groups with disconnected prime graph can be obtained by [12] and [20]. Let $a \cdot b$ denote the products of an integer a by an integer b. Let r be a prime. Then, we denote the number of the Sylow r-subgroup G_r of G by $n_r(G)$ or n_r. Let S_n be the symmetric group of degree n. If n is an integer, then denote the r-part of n by $n_r = r^a$ or by $r^a \| n$, namely, $r^a \mid n$ but $r^{a+1} \nmid n$. The other symbols are standard (see [7], for instance).

2 Some preliminary results

In this section, we give some lemmas used to prove the main theorem.

Lemma 2.1 *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G \big| g^m = 1\}$, then $m \big| |L_m(G)|$.*

Proof. See [8]. □

Lemma 2.2 *Let G be a group containing more than two elements. If the maximal numbers of elements of the same order in G is finite, then G is finite and $|G| \leq s(s^2 - 1)$.*

Proof. See [17]. □

Lemma 2.3 *Let G be a group and P be a cyclic Sylow p-subgroup of G of order p^a. If there is a prime r such that $p^a r \in \omega(G)$, then $s_{p^a r} = s_r(C_G(P)) s_{p^a}$. In particular, $\phi(r) s_{p^a} \mid s_{p^a r}$, where $\phi(r)$ is the Euler function of r.*

Proof. See [16]. □

Lemma 2.4 *Let $q > 1$ be an integer, m be a nature number, and p be an odd prime. If p divides $q-1$, then $(q^m - 1)_p = m_p \cdot (q-1)_p$.*

Proof. See Lemma 8(1) of [9]. □

Lemma 2.5 *Let a, b and n be positive integers such that $(a, b) = 1$. Then, there exists a prime p with the following properties:*

– *p divides $a^n - b^n$,*
– *p does not divide $a^k - b^k$ for all $k < n$,*

with the following exceptions: $a = 2, b = 1$; $n = 6$ and $a + b = 2^k$; $n = 2$.

Proof. See [22]. □

Remark 2.1 If $b = 1$, the prime p is called the Zsigmondy prime. If p is a Zsigmnody of $a^n - 1$, then Fermat's little theorem shows that $n \mid p - 1$. Put $Z_n(a) = \{p : p$ is a Zsigmondy prime of $a^n - 1\}$. If $r \in Z_n(a)$ and $r \mid a^m - 1$, then $n \mid m$.

Lemma 2.6 *Let G be a Frobenius group of even order with kernel K and complement H. Then, $s(G) = 2$, the prime graph components of G are $\pi(H)$ and $\pi(K)$ and the following assertions hold:*

(1) *K is nilpotent;*
(2) *$|K| \equiv 1 (mod \quad |H|)$.*

Proof. See [6]. □

Lemma 2.7 *Let G be a 2-Frobenius group, i.e., G is a finite group and has a normal series $1 \leq H \leq K \leq G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Then:*

(1) *$t(G) = 2$, $\pi_1 = \pi(G/K) \cup \pi(H)$ and $\pi_2 = \pi(K/H)$;*
(2) *G/K and K/H are cyclic, $|G/K| \mid (|K/H| - 1)$ and $G/K \leq \mathrm{Aut}(K/H)$.*

Proof. See [6]. □

Lemma 2.8 *If G is a finite group such that $t(G) \geq 2$, then G has one of the following structures:*

(1) *G is a Frobenius group or 2-Frobenius group;*
(2) *G has a normal series $1 \leq H \leq K \leq G$ such that $\pi(G/K) \cup \pi(H) \subseteq \pi_1$ and K/H is a non-abelian simple group. In particular, H is nilpotent, $G/K \lesssim \mathrm{Out}(K/H)$ and the odd order components of G are the odd order components of K/H.*

Proof. See [20]. □

Lemma 2.9

(1) *$n_p(L_n(2)) = \frac{|L_n(2)|}{n \cdot p}$;*
(2) *Let S be a Sylow 2-subgroup of $L_n(2)$, then $n_2(L_n(2)) = \frac{|L_n(2)|}{|S|}$.*

Proof. (1) Remind that under assumption $2^n - 1 = p$. Let $R \in \mathrm{Syl}_p(L_n(2))$. Since $|GL_n(2)| = |SL_n(2)| = |L_n(2)|$ by [7, p. x], we can assume that $R \in \mathrm{Syl}_p(GL_n(2))$ and hence, $Z_p.Z_n = N_{GL_n(2)}(R)$. Therefore

$$np \mid |N_{GL_n(2)}(R)|. \tag{2.1}$$

Considering the maximal torus of $GL_n(2)$ shows that $C_{GL_n(2)}(R) \cong Z_{2^n-1} = Z_p$ and [11, Chapter 2, Satz 7.3] implies that $N_{GL_n(2)}(R)/C_{GL_n(2)}(R) \cong Z_n$. Thus

$$|N_{GL_n(2)}(R)| \mid np. \tag{2.2}$$

Equ. (2.1) and (2.2) force $N_{GL_n(2)}(R) = Z_p.Z_n$. Since $|GL_n(2)| = |SL_n(2)| = |L_n(2)|$, then by Frattini's arguments, $n_p(L_n(2)) = \frac{|L_n(2)|}{n\cdot p}$.

(2) Let S be a Sylow 2-subgroup of $L_n(2)$. Then, by Corollary of [13], $N_{L_n(2)}(S) = S$. It follows from Frattini's arguments, that $n_2(L_n(2)) = \frac{|L_n(2)|}{|S|}$.

This completes the proof. □

3 Proof of the main theorem

Let G be a group such that $\mathrm{nse}(G) = \mathrm{nse}(L_n(2)$, with $2^n - 1 = p$, and s_n be the number of elements of order n. By Lemma 2.2 we have that G is finite. We note that $s_n = k\phi(n)$, where k is the number of cyclic subgroups of order n. Also we note that if $n > 2$, then $\phi(n)$ is even. If $m \in \omega(G)$, then by Lemma 2.1 and the above discussion, we have

$$\begin{cases} \phi(m) \mid s_m \\ m \mid \sum_{d \mid m} s_d. \end{cases} \tag{3.1}$$

Recall that $L_2(2) \cong S_3$, $L_3(2) \cong L_2(7)$ and $L_4(2) \cong A_8$. By [12], we have $s(L_3(2)) = 3$, $s(L_4(2)) = 2$, and $s(L_n(2)) = \begin{cases} 1, & \text{if } n \neq p, p+1 \\ 2, & \text{if } n = p, p+1 \end{cases}$, where $p \geq 5$ is a prime number. More precisely, when $n = p$ or $p+1$, $L_n(2)$ has two connected components, one of them is $GK_1(L_n(2))$ with $\pi(L_p(2)) = \pi(2\sum_{i=1}^{p-1}(2^i-1))$, (resp. $\pi_1(L_{p+1}(2)) = \pi(2(2^{p+1} - 1)\sum_{i=1}^{p-1}(2^i-1)))$ and the other in both cases is $GK_2(L_n(2))$ with $\pi_2 = \pi(2^p-1)$, while if $n \neq p, p+1$, then $\pi_1(L_n(2)) = \pi(L_n(2))$. If $2^n - 1$ is prime, then n is prime by [10]. The orders of finite simple groups under discussion are $L_n(2) = 2^{\frac{n(n-1)}{2}} \sum_{i=2}^{n}(2^i - 1)$.

In the proof of main theorem, we use the order components of finite simple groups that are listed as in Tables 1, 2 and 3 (see [12] and [20]).

Proof of the main theorem

Proof. In the following, we always assume that $2^n - 1 = p$ is a prime and $L = L_n(2)$.

Since $L_3(2)$, $L_4(2)$ and $L_5(2)$ are characterizable by nse, then we only consider that $n \geq 7$. We will divide the proof into the following series of Lemmas.

Lemma 3.1 *Let $p \neq r \in \pi(L)$ such that $r \mid 2^{n_i} - 1$ for some $2 \leq n_i < n$. Then, $p \mid s_r(L)$.*

Proof. By [5, Proposition 7], the maximal torus T of $L_n(q)$ has the order

$$\frac{1}{(n, q-1)(q-1)}(q^{n_1} - 1)(q^{n_2} - 1) \cdots (q^{n_k} - 1),$$

for an appropriate $n_1 + n_2 + \cdots + n_k = n$. Then, there is an element $x \in L$ and some torus T such that $|x| = r$ and $T \leq C_L(x)$ for some T. It follows that $n_i \geq 2$ for $i = 1, 2 \cdots, n-1$ and so $|x^G|$ is the multiple of $\frac{|L|}{|T|}$ for some T. But $s_r(L) = \sum_{|x|=r, x\neq 1} |x^G|$. Hence $p \mid s_r(L)$. $\square$

Lemma 3.2 *For $u \in \omega(L)$, either $p \mid s_u(L)$ or $u = p$ and $s_p(L) = \frac{(2^n-2)|L|}{np}$.*

Proof. By Lemma 3.1, for $p \neq u \in \omega(L)$, $p \mid s_u(L)$. Since $2^n - 1 = p$ is a prime and$|L_p| = p$, then L_p is cyclic. By equ. (3.1) and Lemma 2.9, $s_p(L) = \phi(p)n_p(L) = \frac{(2^n-2)|L|}{np}$. $\square$

Lemma 3.3 *$s_2(G) = s_2(L)$. In particular, $p \mid s_2$.*

Proof. We know that if $2 < n \in \omega(G)$, then s_n is even. By Lemma 2.1, $2 \mid 1 + s_2(L)$. On the other hand, in G, the only odd number in $\mathrm{nse}(G)\backslash\{1\}$ is $s_2(G)$. Hence we have $s_2(G) = s_2(L)$. By Lemma 3.1, $p \mid s_2$. $\square$

Lemma 3.4 *For $u \in \omega(G)$, $p \nmid s_u(G)$ if and only if $s_u(G) = s_p(G)$. In particular $s_p(G) = s_p(L)$.*

Proof. By Lemma 3.2, $s_u(G) = s_p(L)$ if and only if $p \nmid s_u(G)$. By Lemma 2.1, $p \mid 1 + s_p(G)$ and so $p \nmid s_p(G)$. Therefore, we have $s_p(G) = s_p(L)$. $\square$

Lemma 3.5 *Let $r \in \pi(G) - \{p\}$. Then, $r \cdot p \notin \omega(G)$.*

Proof. If $2 \cdot p \in \omega(G)$, then by Lemma 2.1, $2 \cdot p \mid 1 + s_2 + s_p + s_{2\cdot p}$ and also $p \mid 1 + s_p$.

By Lemma 3.3, $p \mid s_2$. It follows that $p \mid s_{2\cdot p}$.

By hypothesis, we have that $|G_p| = |G|_p = p$. Then, by Lemma 2.3, $s_{2\cdot p} = s_p \cdot t$ for some integer t and so $p \mid t$.

Let $t = pk$. Then, $(k, p) = 1$ (by hypothesis $|G_p| = p$), $s_{2\cdot p} = s_p \cdot pk = \frac{2(2^{n-1}-1)|L|}{n}k$. We know that $|L| = \sum_{n\in \mathrm{nse}(G)} n$ and so $s_{2\cdot p} > 2|L|$, a contradiction. It follows that $s_{2\cdot p} = s_p \cdot pk \notin \mathrm{nse}(G)$. Therefore, $s_{2\cdot p} = s_p$, also a contradiction since $p \nmid s_p$.

Therefore $2 \cdot p \notin \omega(G)$. Similarly, we can prove that $r \cdot p \notin \omega(G)$, for $r \in \pi(G)\backslash\{2, p\}$. $\square$

Lemma 3.6 *$|G| \mid \frac{(2^n-2)|L|}{n}$.*

Proof. By hypothesis, G_p is cyclic. Equ. (3.1) implies that $s_p(G) = \phi(p)n_p(G)$. By Lemma 3.4, $\phi(p)n_p(L) = s_p(L) = s_p(G) = \phi(p)n_p(G)$. By Lemma 3.5, for $r \in \pi(G) - \{p\}$, $r \cdot p \notin \omega(G)$. Let G_r be a Sylow r-subgroup of G, then the Sylow r-subgroup of G acts fixed point freely on the set of elements of order p. It follows that $|G_r| \mid s_p$. Hence $|G| \mid \frac{(2^n-2)|L|}{n}$. $\square$

Lemma 3.7 *There is a normal series* $1 \leq H \leq K \leq G$ *such that* K/H *is isomorphic to* $L_n(2)$.

Proof. By Lemma 3.5, $s(G) \geq 2$. Then, by Lemma 2.8, we have the following:

(1) G is a Frobenius group or 2-Frobenius group;
(2) G has a normal series $1 \leq H \leq K \leq G$ such that $\pi(G/K) \cup \pi(H) \subseteq \pi_1$ and K/H is a non-abelian simple group. In particular, H is nilpotent, $G/K \lesssim Out(K/H)$ and the odd order components of G are the odd order components of K/H.

So in the following, we divide the proof into four steps.

Step 1: G is neither a Frobenius group nor a 2-Frobenius group.

Let G be a Frobenius group of even order with kernel H and complement K. Then, by Lemma 2.6, $s(G) = 2$, $\pi(G) = \{\pi(H), \pi(K)\}$. By Lemma 3.4, $p = \pi(H)$ or $\{p\} = \pi(K)$. If $\pi(H) = \{p\}$, then since H is nilpotent, H_p is characteristic in H. By hypothesis, we have that $H_p^g = G_p$ for some g and hence, G_p is normal in G. It follows that $s_p(G) = p - 1$, contradicting Lemma 3.4. Now let $\pi(K) = \{p\}$. By Lemmas 2.9 and 3.4, $\frac{|L|}{np} = n_p \mid |G|$ and by Lemma 3.6, $|G| \mid \frac{(2^n-2)|L|}{n}$. So by Lemma 2.5, there is a prime $r \in Z_{n-1}(2) \cap \pi(G)$ and so $|L_r| = |L|_r = |2^{n-1} - 1|_r \leq |G_r|$. Since $\pi(G) = \pi(K) \cup \pi(H) = \pi_1(G) \cup \pi_2(G)$, then $r \in \pi(H)$. Since H is nilpotent, G_r is normal in G. It follows that the Sylow p-subgroup of G acts fixed point freely on the set of elements of order r and so $p \mid |L_r(G)| - 1$. Thus $p \leq |L_r(G)| \leq (2^{\frac{n-1}{2}} - 1)_r^2 < 2^n - 1 = p$ (in fact, since $2^n - 1$ is a prime, by [10], n is a prime). Thus we have that $n - 1$ is even and so let $n - 1 = 2t$ for some integer $t \geq 1$, $r \mid 2^{\frac{n-1}{2}} - 1$ or $r \mid 2^{\frac{n-1}{2}} + 1$. On the other hand, by binomial theorem and since $n \geq 7$, $(2^{\frac{n-1}{2}} - 1)^2 = 2^{n-1} - 2 \cdot 2^{\frac{n-1}{2}} + 1 \leq 2^{n-1} - 1$. Similarly, we have $(2^{\frac{n-1}{2}} + 1)^2 < 2^n - 1$, a contradiction.

Let G be a 2-Frobenius group. Then, G has a normal series $1 \leq H \leq K \leq G$ such that $\pi(G/K) \cup \pi(H) \subseteq \pi_1$ and K/H is a cyclic group of order p and $|G/K| \mid (p - 1)$. Similarly, as the argument for the Frobenius group, we get a contradiction.

Therefore G has a normal series $1 \leq H \leq K \leq G$ such that $\pi(G/K) \cup \pi(H) \subseteq \pi_1$ and K/H is a non-abelian simple group. In particular, H is nilpotent, $G/K \lesssim Out(K/H)$ and the odd order components of G are the odd order components of K/H.

Step 2: K/H is isomorphic to $L_n(2)$.

According to classification theorem of finite simple groups, K/H is an alternating group, sporadic group or simple group of Lie type. By Lemma 3.5, $s(K/H) \geq 2$.

Let $K/H \cong A_m$ with $m \geq 5$, then since $2^n - 1 = p$ is a prime and $p \in \pi(K/H)$, $m \geq 2^n - 1$. Thus there is a prime $u \in \pi(A_m) \cap \pi(G)$ such that $\frac{p+1}{2} < u < p$. By Lemma 2.9, $u \mid \frac{|L|}{np}$. Hence, there exists $t \in \{2i, i : 1 \leq i \leq n - 1\} \cup \{n\}$ such that $u \in Z_t(2)$. Obviously $u > \frac{2^n-1+1}{2} = 2^{n-1}$ and so $u = 2^{n-1} + 1$. Since n, u are primes, then $n - 1 = 2^k + 1$ and so $n = 2$, a contradiction.

Let K/H be sporadic simple groups, we can rule out this case by considering their odd order component since the odd components of K/H is $p = 2^n - 1$.

Therefore, K/H is isomorphic to a simple group of Lie type. We consider the following cases.

Case 1: Let $s(K/H) = 2$. Then, we have that $OC_2(K/H) = p = 2^n - 1$.

1.1. Let $K/H \cong C_m(q)$, where $m = 2^u > 2$, then $\frac{q^m+1}{(2,q-1)} = 2^n - 1$. If q is odd, then $q^m - 1 = 2^{n+1} - 4$. On the other hand, $q^m - 1 = 2^2(2^{n-1} - 1)$. Since $2 \mid q-1$ and $m = 2^u$, then by Lemma 2.4, $(q^m - 1)_2 = (q-1)_2 \cdot m_2 = 2^2$. But $m \geq 3$, $(q^m - 1)_2 \cdot m_2 \geq 2^3$, a contradiction. If q is even, then $q^{m-1} = 2^{n-1} - 1$ and hence, $m = 1, n = 1$, a contradiction. Similarly, we can rule out these cases "$K/H \cong B_m(q)$ or $K/H \cong C_m(q)$ with $m = 2^u \geq 4$".

1.2. Let $K/H \cong C_r(3)$ or $B_r(3)$, then $\frac{3^r-1}{2} = 2^n - 1$. Thus $3^r = 2^{n+1} - 1$, which contradicts Lemma 2.5. Similarly, we can rule out these cases "$K/H \cong D_r(3)$ or $K/H \cong D_{r+1}(3)$".

1.3. Let $K/H \cong C_r(2)$, then $2^r - 1 = 2^n - 1$ and so $n = r$. Therefore, by Lemma 3.6, $2^n + 1 \mid |G| \mid \frac{(2^n-2)|L|}{n}$, a contradiction. Similarly, we can rule out these cases "$K/H \cong D_r(2)$ and $K/H \cong D_{r+1}(2)$".

1.4. Let $K/H \cong D_r(5)$ where $r \geq 5$, then $\frac{5^r-1}{4} = 2^n - 1$, namely $5^r = 2^{n+2} - 3$ and so $r = 1 = n$ or $r = 3, n = 5$, a contradiction.

1.5. Let $K/H \cong^2 D_m(3)$, where $9 \leq m = 2^r + 1$ and m is not a prime, then $\frac{3^{m-1}+1}{2} = 2^n - 1$ and hence, $3(3^{m-2} - 1) = 2^n$, a contradiction. Also we can rule out $K/H \cong^2 D_{m+1}(2)$.

1.6. Let $K/H \cong^2 D_m(2)$, where $m = 2^r + 1 \geq 5$, then $2^{m-1} + 1 = 2^n - 1$ and hence, $2^{m-2} = 2^{n-1} - 1$, a contradiction.

1.7. Let $K/H \cong^2 D_r(3)$, where $r \neq 2^m + 1 \geq 5$, then $\frac{3^r+1}{4} = 2^n - 1$ and hence, $3^r = 2^{n+2} - 5$. Thus $n = 3$ and $r = 3$, a contradiction.

1.8. Let $K/H \cong G_2(q)$, where $2 < q \equiv \epsilon \mod 3$ and $\epsilon = \pm 1$, then $q^2 - \epsilon q + 1 = 2^n - 1$ and hence, $(q-2)(q+1) = 2^n$ or $(q+2)(q-1) = 2^n$ and hence, $q = 2$ or 3 and $n = 2$, a contradiction.

1.9. Let $K/H \cong^2 F_4(2)$, then $|^2F_4(2)| = 2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$ and hence, $2^n - 1 = 13$, a contradiction. Also we can rule out "$K/H \cong^2 A_3(2)$".

1.10. Let $K/H \cong L_r(q)$, where $(r, q) \neq (3, 2), (3, 4)$. Since $\frac{q^r-1}{(q-1)(r,q-1)} = 2^n - 1$, then $r = n$ and $q = 2$, as desired.

1.11. Let $K/H \cong U_r(q)$, then $\frac{q^r+1}{(q+1)(r,q+1)} = 2^n - 1$.

(1) Let q is odd. If $(r, q+1) = 1$, then $q^{r-1} - q^{r-2} + \cdots - q + 1 = 2^n - 1$ and hence, $q^{r-1} - q^{r-2} + \cdots - q = 2(2^{n-1} - 1)$. It follows that $q = 2$, a contradiction. If $(r, q+1) = r$, then $r = 2$ or $3 \leq r$ is a prime. If $r = 2$, then $q + 1 \nmid q^2 + 1$, a contradiction. Thus $r \geq 3$ and so $r \mid q + 1 \mid q^r + 1$. It follows that $r \mid q^{2r} - 1$. Then, by Fremat's little theorem, $r \mid \phi(2r) = r - 1$, and so $r = 1$, a contradiction.

(2) Let q is even. If $(r, q+1) = 1$, then $q^{r-1} - q^{r-2} + \cdots - q + 1 = 2^n - 1$ and hence, $q^{r-1} - q^{r-2} + \cdots - q = 2(2^{n-1} - 1)$. It follows that $q = 2$ and $r = 2$, a contradiction. If $(r, q+1) = r$, then $3 \leq r$ is a prime and so $r \mid q + 1 \mid q^r + 1$. It follows that $r \mid q^{2r} - 1$. Then, by Fremat's little theorem $r \mid \phi(2r) = r - 1$, and so $r = 1$, a contradiction.

1.12. Let $K/H \cong L_{r+1}(q)$, with $q - 1 \mid r + 1$. Since $\frac{q^r-1}{(r,q-1)} = p$, $p \in Z_r(q)$ and hence $r \mid p - 1 = 2^n - 2 = 2(2^{n-1} - 1)$. It follows that $r = 2$ or $r \in Z_{n-1}(2)$.

(1) Let $r = 2$. Then, $q = 4$ or 2. If $q = 4$, then $p = 15$, a contradiction. Hence $q = 2$, $p = 3$ and $n = 2$. It follows that K/H is isomorphic to $L_3(2)$, as desired.
(2) Let $r \in Z_{n-1}(2)$. Then, $q-1 \mid 2^{n-1}$ and so q is a Mersenne prime. Order consideration also rules out this case.

Similarly, we can rule out the case "$K/H \cong U_{r+1}(q)$".

1.13. Let $K/H \cong E_6(q)$, where $q = u^a$, then $\frac{q^6+q^3+1}{(3,q-1)} = p = 2^n - 1$. Thus $p \in Z_9(q)$ and hence, $9 \mid 2^n - 2 = 2(2^{n-1} - 1)$. It follows that $9 \mid 2^{n-1} - 1$ and so $2 \mid n - 1$ or $4 \mid n - 1$. Thus $n = 2t + 1$ or $n = 4t + 1$.

(1) Let $n = 2t + 1$. Then, $p = 2^{2t} - 1 = (2^t - 1)(2^t + 1)$ and so $t = 1$, $p = 3$ and $n = 2$. Hence $\frac{q^6+q^3+1}{(3,q-1)} = 3$, but the equation has no solution in $\mathbb{N}$.
(2) Let $n = 4t + 1$. Then, $p = (2^{2t} - 1)(2^{2t} + 1)$, the equation has no solution in $\mathbb{N}$.

Similarly, we can rule out "$K/H \cong^2 E_6(q)$".

Case 2: Let $s(L/K) = 3$. Then, $2^n - 1 \in \{OC_2(K/H), OC_3(K/H)\}$.

2.1. Let $K/H \cong L_2(q)$, where $4 \mid q + 1$. Then, $\frac{q-1}{2} = 2^n - 1$ or $q = 2^n - 1$. If the former, then $4 \mid q + 1 = 2^n$ and so $n \geq 2$. If $n \geq 3$, then order consideration rules out this case. If $n = 2$, then $q = 7$ and so $|K/H| \mid |L_2(7)| \mid \frac{(2^2-2)|L_2(2)|}{2} = |L_2(2)|$, contradiction. If $q = 2^n - 1 = p$ and so $n \geq 2$, similarly, we can rule out this case.

2.2. Let $K/H \cong L_2(q)$, where $4 \mid q - 1$. Then, $q = p$ or $\frac{q+1}{2} = p$. If $q = p$, then $4 \mid 2^n-2 = 2(2^{n-1}-1)$, a contradiction. If $q = 2p-1$, then $4 \mid 2^{n+1}-3$, a contradiction.

2.3. Let $K/H \cong L_2(q)$, where $q > 2$ and q is even. Then, $p \in \{q - 1, q + 1\}$. If $p = q - 1$, then $q = p + 1 = 2^n$ and so $2^n + 1 \mid |G|$, a contradiction. If $p = q + 1$, then $q = p - 1 = 2(2^{n-1} - 1)$ and so $n = 2$, a contradiction.

2.4. Let $K/H \cong U_6(2)$. Then, $|K/H| = 2^{15} \cdot 3^6 \cdot 7 \cdot 11$ and so $2^n - 1 = 11$, a contradiction.

2.5. Let $K/H \cong L_3(2)$. Then, $|K/H| = 2^3 \cdot 3 \cdot 7$ and $2^n - 1 = 7$. Thus $n = 3$, which is the desired result.

2.6. Let $K/H \cong^2 D_r(3)$, where $r = 2^t+1 \geq 5$. Then, $\frac{3^r+1}{4} = 2^n-1$ or $\frac{3^{r-1}+1}{2} = 2^n-1$. If the former, then $3(3^{r-1} - 1) = 2^3(2^{n-1} - 1)$. Since $2 \mid 3 - 1$, then by Lemma 2.4, $r - 1 = 4$ and so $t = 2$. It follows that $30 = 2^{n-1} - 1$, a contradiction. If the latter $3^{r-1} = 2(2^{n-1} - 1)$, a contradiction.

2.7. Let $K/H \cong G_2(q)$, where $q \equiv 0 \mod 3$. Then, $q^2-q+1 = 2^n-1$ or $q^2+q+1 = 2^n - 1$. If the former, then $q(q - 1) = 2(2^{n-1} - 1)$ and so $q = 3$ and $n = 3$. Order consideration rules out this case. If the latter, $q(q+1) = 2(2^{n-1}-1)$. But the equation has no solution in $\mathbb{N}$. Similarly, we can rule out "$K/H \cong^2 G_2(q)$".

2.8. Let $K/H \cong F_4(q)$, where q is even. Then, $q^4+1 = 2^n-1$ or $q^4-q^2+1 = 2^n-1$. If the former, $q^4 = 2(2^{n-1}-1)$, a contradiction. If the latter, $q^2(q^2-1) = 2(2^{n-1}-1)$, but the equation has no solution in $\mathbb{N}$.

2.9. Let $K/H \cong^2 F_4(q)$, where $q = 2^{2t+1} > 2$. Then, $q^2 + \sqrt{2q^3} + q + \sqrt{2q} + 1 = 2^n - 1$ or $q^2 - \sqrt{2q^3} + q - \sqrt{2q} + 1 = 2^n - 1$. It is easy to get that the equations $q^2 + \sqrt{2q^3} + q + \sqrt{2q} = 2(2^{n-1} - 1)$ and $q^2 - \sqrt{2q^3} + q - \sqrt{2q} = 2(2^{n-1} - 1)$ have no solution in $\mathbb{N}$.

2.10. Let $K/H \cong E_7(2)$, then $2^n - 1 \in \{73, 127\}$ and so $n = 8$. Order consideration rules out this case.

2.11. Let $K/H \cong E_7(3)$, then $2^n - 1 \in \{757, 1093\}$, which is impossible.

Case 3: $s(K/H) \in \{4, 5\}$. Then, $p = 2^n - 1 \in \{OC_2(K/H), OC_3(K/H), OC_4(K/H), OC_5(K/H)\}$.

3.1. Let $K/H \cong L_3(4)$ or ${}^2E_6(2)$. Then, $2^n - 1 = 7$ or $2^n - 1 = 19$. If the former, $n = 3$, then order consideration rules out. If the latter, it is impossible.

3.2. Let $K/H \cong {}^2 B_2(q)$, where $q = 2^{2t+1}$ and $t \geq 1$. Then, $2^n - 1 \in \{q - 1, q \pm \sqrt{2q} + 1\}$. If the former, then $n = 2t + 1$, order consideration rules out. If the latter, then $2(2^{n-1} - 1) = 2^{t+1}(2^t - 1)$, the equation has no solution in $\mathbb{N}$.

Hence $K/H \cong L_n(2)$ with $2^n - 1$ prime.

Step 3: $\pi(H) \subsetneqq \{2\}$.

Let $r \in \pi(H)$. Then, $r \neq p$ and by Lemma 2.8, H is nilpotent. It follows that H_r is normal in G. By Lemma 3.5, $r \cdot p \notin \omega(G)$ and hence, the Sylow p-subgroup of G acts fixed point freely on the set of elements of order r. Thus $p \mid |H_r| - 1$. If $r \neq 2$, then by Lemma 3.6, $|H_r| \mid (2^{n-1} - 1)_r |L|_r$. By the proof of Step 1, we know that $r \nmid 2^{n-1} - 1$ and hence, $|H_r| \mid |L|_r$. On the other hand, $H_r \leq 2^t - 1$ for some integer $t \in \{i : 1 \leq i \leq n - 1\}$ and so $p \leq |H_r| \leq 2^{n-1} - 1 \leq 2^n - 1 = p$, a contradiction. Thus $r = 2$.

Step 4: G is isomorphic to $L_n(2)$.

By Step 2, we have that K/H is isomorphic to $L_n(2)$. We will prove that $H = 1$. Assume the contrary, then by Step 3, H is a 2-group. By Lemma 3.6, $|H| = [\frac{2^n-2}{n}]_2 = 2$. Thus by Lemma 3.5, the Sylow p-subgroup of G acts fixed point freely on $|H| - 1 = 1$, a contradiction. Hence $H = 1$. It follows that $K \cong L_n(2)$ and $G \leq \mathrm{Aut}(L_n(2)) \cong SL_n(2) \cong L_n(2)$. Thus G is isomorphic to $L_n(2)$. □

This completes the proof. □

4 Some applications

On Thompson's conjecture, if G and H are of the same order type, then $\mathrm{nse}(G) = \mathrm{nse}(H)$ and $|G| = |H|$.

Corollary 4.1 *Let $p = 2^n - 1$ be a prime and $p \| |G|$. Then, $G \cong L_n(2)$ if and only if $|G| = |L_n(2)|$ and $\mathrm{nse}(G) = \mathrm{nse}(L_n(2))$ where $2^n - 1$ is a prime.*

Shi gave the following conjecture.

Conjecture ([18]) *Let G be a group and H a finite simple group. Then, $G \cong H$ if and only if (a) $\omega(G) = \omega(H)$ and (b) $|G| = |H|$.*

Then, we have the following corollary.

Corollary 4.2 *Let $p = 2^n - 1$ be a prime and $p \| |G|$. Then, $G \cong L_n(2)$ if and only if $\omega(G) = \omega(L_n(2))$ and $|G| = |L_n(2)|$.*

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Table 1. The order components of finite simple groups L with $s(L) = 2$

L	Restrictions of L	m_1	m_2
A_n	$6 < n = p, p+1, p+2$ one of $n, n-2$ is not a prime	$n!/2p$	p
$A_{p-1}(q)$	$(p,q) \neq (3,2),(3,4)$	$q^{p(p-1)/2}\prod_{i=1}^{p-1}(q^i-1)$	$\frac{(q^p-1)}{(q-1)(p,q-1)}$
$A_p(q)$	$(q-1) \mid (p+1)$	$q^{p(p+1)/2}(q^{p+1}-1)\prod_{i=2}^{p-1}(q^i-1)$	$\frac{q^p-1}{q-1}$
${}^2A_{p-1}(q)$		$q^{p(p-1)/2}\prod_{i=1}^{p-1}(q^i-(-1)^i)$	$\frac{(q^p+1)}{(q+1)(p,q+1)}$
${}^2A_p(q)$	$(q+1) \mid (p+1)$ $(p,q) \neq (3,3),(5,2)$	$q^{p(p+1)/2}(q^{p+1}-1)\prod_{i=2}^{p-1}(q^i-1)$	$\frac{q^p+1}{q+1}$
${}^2A_3(2)$		$2^6 \cdot 3^4$	5
$B_n(q)$	$n = 2^m \geq 4$, q odd	$q^{n^2}(q^n-1)\prod_{i=1}^{n-1}(q^{2i}-1)$	$\frac{q^n+1}{2}$
$B_p(3)$		$3^{p^2}(3^p+1)\prod_{i=1}^{p-1}(3^{2i}-1)$	$\frac{3^p-1}{2}$
$C_n(q)$	$n = 2^m \geq 2$, q odd	$q^{n^2}(q^n-1)\prod_{i=1}^{n-1}(q^{2i}-1)$	$\frac{q^n+1}{(2,q-1)}$
$C_p(q)$	$q = 2,3$	$q^{p^2}(q^p+1)\prod_{i=1}^{p-1}(q^{2i}-1)$	$\frac{q^p-1}{(2,q-1)}$
$D_p(q)$	$p \geq 5, q = 2,3,5$	$q^{p(p-1)}\prod_{i=1}^{p-1}(q^{2i}-1)$	$\frac{q^p-1}{q-1}$
$D_{p+1}(q)$	$q = 2,3$	$q^{p(p+1)}(q^p+1)(q^{p+1}-1)$ $\prod_{i=1}^{p-1}(q^{2i}-1)/(2,p-1)$	$\frac{q^p-1}{(2,q-1)}$
${}^2D_n(q)$	$n = 2^m \geq 4$	$q^{n(n-1)}\prod_{i=1}^{n-1}(q^{2i}-1)$	$\frac{q^n+1}{(2,q+1)}$
${}^2D_n(2)$	$n = 2^m+1 \geq 5$	$2^{n(n-1)}(2^n+1)(2^{n-1}-1)$ $\prod_{i=1}^{n-2}(2^{2i}-1)$	$2^{n-1}+1$
${}^2D_p(3)$	$5 \leq p \neq 2^m+1$	$3^{p(p-1)}\prod_{i=1}^{p-1}(3^{2i}-1)$	$\frac{3^p+1}{4}$
${}^2D_n(3)$	$9 \leq 2^m+1 \neq p$	$3^{n(n-1)}(3^n+1)(3^{n-1}-1)$ $\prod_{i=1}^{n-1}(3^{2i}-1)/2$	$\frac{3^{n-1}}{2}$
$G_2(q)$	$2 < q \equiv \epsilon \mod 3, \epsilon = \pm 1$	$q^6(q^3-\epsilon)(q^2-1)(q+\epsilon)$	$q^2-\epsilon q+1$
${}^3D_4(q)$		$q^{12}(q^6-1)(q^2-1)(q^4+q^2+1)$	$q^4-q^2=1$
$F_4(q)$	q odd	$q^{24}(q^8-1)(q^6-1)^2(q^4-1)$	q^4-q^2+1
${}^2F_4(2)'$		$2^{11}\cdot 3^3 \cdot 5^2$	13
$E_6(q)$		$q^{36}(q^{12}-1)(q^8-1)(q^6-1)$ $(q^5-1)(q^3-1)(q^2-1)$	$(q^6+q^3+1)/(3,q-1)$
${}^2E_6(q)$	$q > 2$	$q^{36}(q^{12}-1)(q^8-1)(q^6-1)$ $(q^5+1)(q^3+1)(q^2-1)$	$(q^6-q^3+1)/(3,q+1)$
M_{12}		$2^6 \cdot 3^3 \cdot 5$	5
J_2		$2^7 \cdot 3^3 \cdot 5^2$	7
Ru		$2^{14}\cdot 3^3\cdot 5^3\cdot 7\cdot 13$	29
He		$2^{10}\cdot 3^3\cdot 5^2\cdot 7^3$	17
McL		$2^7\cdot 3^6\cdot 5^3\cdot 7$	11
Co_1		$2^{21}\cdot 3^9\cdot 5^4\cdot 7^2\cdot 11\cdot 13$	23
Co_3		$2^{10}\cdot 3^7\cdot 5^3\cdot 7\cdot 11$	23
Fi_{22}		$2^{17}\cdot 3^9\cdot 5^2\cdot 7\cdot 11$	13
HN		$2^{14}\cdot 3^6\cdot 5^6\cdot 7\cdot 11$	19

Table 2. The order components of finite simple groups L with $s(L) = 3$

L	Restrictions of L	m_1	m_2	m_3
A_n	$6 < n = p, p-2$ are primes	$\frac{n!}{2n(n-2)}$	p	$p-2$
$A_1(q)$	$4 \mid q+1$	$q+1$	q	$\frac{q-1}{2}$
$A_1(q)$	$4 \mid q-1$	$q-1$	q	$\frac{q+1}{2}$
$A_1(q)$	$2 \mid q$	q	$q+1$	$q-1$
$A_2(2)$		8	3	7
${}^2A_5(2)$		$2^{15} \cdot 3^6 \cdot 5$	7	11
${}^2D_p(3)$	$5 \leq p = 2^m + 1$	$2 \cdot 3^{p(p-1)}(3^{p-1}+1)$ $\prod_{i=1}^{p-2}(3^{2i}-1)$	$\frac{3^{p-1}}{2}$	$\frac{3^p+1}{4}$
${}^2D_{p+1}(2)$	$n \geq 2, p = 2^m - 1$	$2^{p(p-1)}(2^p-1)$ $\prod_{i=1}^{p-1}(2^{2i}-1)$	2^p+1	$2^{p+1}+1$
$G_2(q)$	$q \equiv 0 \mod 3$	$q^6(q^2-1)^3$	q^2-q+1	q^2+q+1
${}^2G_2(q)$	$q = 3^{2m+1} > 3$	$q^3(q^2-1)$	$q-\sqrt{3q}+1$	$q+\sqrt{3q}+1$
$F_4(q)$	q even	$q^{24}(q^6-1)^2(q^4-1)^2$	q^4+1	q^4-q^2+1
${}^2F_4(q)$	$q = 2^{2m+1} > 2$	$q^{12}(q^4-1)(q^3+1)$	$q^2-\sqrt{2q^3}+$ $q-\sqrt{2q}+1$	$q^2+\sqrt{2q^3}+$ $q+\sqrt{2q}+1$
$E_7(2)$		$2^{36} \cdot 3^{11} \cdot 5^2 \cdot 7^3 \cdot 11$ $13 \cdot 17 \cdot 19 \cdot 43$	73	127
$E_7(3)$		$2^{23} \cdot 3^{63} \cdot 5^2 \cdot 7^3 \cdot 11^2$ $13^2 \cdot 19 \cdot 37 \cdot 41 \cdot 61 \cdot 73 \cdot 547$	757	1093
M_{11}		$2^4 \cdot 3^2$	5	11
M_{23}		$2^7 \cdot 3^2 \cdot 5 \cdot 7$	11	23
M_{24}		$2^{10} \cdot 3^3 \cdot 5 \cdot 7$	11	13
J_3		$2^7 \cdot 3^5 \cdot 5$	17	19
HS		$2^9 \cdot 3^2 \cdot 5^3$	7	11
Suz		$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7$	11	13
Co_2		$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7$	11	23
Fi_{23}		$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	17	23
F_3		$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13$	19	31
F_2		$2^{24} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot$ $11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$	31	47

Table 3. The order components of finite simple groups L with $s(L) > 3$

L	Restrictions of L	m_1	m_2	m_3	m_4	m_5	m_6
$A_2(4)$		2^6	3	5	7		
${}^2B_2(q)$	$q = 2^{2m+1} > 2$	q^2	$q-1$	$q-\sqrt{2q}+1$	$q^2+\sqrt{2q}+1$		
${}^2E_6(2)$		$2^{36}\cdot 3^9\cdot 5^2\cdot 7^2\cdot 11$	13	17	19		
$E_8(q)$	$q \equiv 2,3 \mod 5$	$q^{120}(q^{20}-1)(q^{18}-1)$ $(q^{14}-1)(q^{12}-1)$ $(q^{10}-1)(q^8-1)$ $(q^4+1)(q^4+q^2+1)$	$\frac{q^{10}-q^5+1}{q^2-q+1}$	$\frac{q^{10}+q^5+1}{q^2+q+1}$	q^8-q^4+1		
M_{22}		$2^7\cdot 3^2$	5	7	11		
J_1		$2^3\cdot 3\cdot 5$	7	11	19		
ON		$2^9\cdot 3^4\cdot 5\cdot 7^3$	11	19	31		
LyS		$2^8\cdot 3^7\cdot 5^6\cdot 7\cdot 11$	31	37	67		
Fi'_{24}		$2^{21}\cdot 3^{16}\cdot 5^2\cdot 7^3\cdot 11\cdot 13$	17	23	29		
F_1		$2^{46}\cdot 3^{20}\cdot 5^9\cdot 7^6\cdot 11^2\cdot 13^3$ $17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 47$	41	59	71		
$E_8(q)$	$q \equiv 0,1,4 \mod 5$	$q^{120}(q^{18}-1)(q^{14}-1)$	$\frac{q^{10}-q^5+1}{q^2-q+1}$	$\frac{q^{10}+q^5+1}{q^2+q+1}$	q^8-q^4+1	$\frac{q^{10}+1}{q^2+1}$	
		$(q^{12}-1)^2(q^{10}-1)^2$ $(q^8-1)^2(q^4+q^2+1)$					
J_4		$2^{21}\cdot 3^3\cdot 5\cdot 7\cdot 11^3$	23	29	31	37	43

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