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nse Characterization of some finite simple groups

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Abstract For a finite group G, let $\mathrm{nse}(G) = \{m_k \mid k \in \pi_e(G)\}$, where m_k is the number of elements of order k in G and $\pi_e(G)$ is the set of element orders of G. Let $2^n + 1 = p > 5$ be a prime number and G be a finite group such that $p \mid |G|$ and $p^2 \nmid |G|$. In this paper, we prove that $G \cong S$ if and only if $\mathrm{nse}(G) = \mathrm{nse}(S)$, where $S \in \{C_n(2), {}^2D_n(2), {}^2D_{n+1}(2)\}$. As a consequence of our result, we prove that if $\mathrm{nse}(S) = \mathrm{nse}(G)$ and $|G| = |S|$, then $G \cong S$.

Keywords Set of the number of elements of the same order · Classification of finite simple groups · Prime graph

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1 Introduction

The set of element orders of G is denoted by $\pi_e(G)$. For a group G and $i \in \pi_e(G)$, set $m_i(G) = |\{g \in G : \text{the order of } g \text{ is } i\}|$. In fact, $m_i(G)$ is the number of elements of order i in G and $\mathrm{nse}(G) := \{m_i(G) : i \in \pi_e(G)\}$. If there is no ambiguity, we write m_i instead of $m_i(G)$. Set $M_t(G) := \{g \in G : g^t = 1\}$. Groups G and H are called of the same order type if and only if $|M_t(G)| = |M_t(H)|$, $t = 1, 2, \ldots$. Thompson's problem asks:

Thompson's problem. Suppose that groups G and H are of the same order type. If G is solvable, is it true that H is also necessarily solvable?

Although, this problem has not been answered yet, but some authors focus on the analogous problem which asks about the structure of the groups with the given nse. We say that the group G is characterizable by the set of nse if every group H with $\mathrm{nse}(G) = \mathrm{nse}(H)$ is isomorphic to G. In [8] and [13], authors showed that the simple groups $PSL_2(q)$, where $q \in \{7, 8, 11, 13\}$ and A_5 are characterizable by the set nse.

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In [1] and [4], it is proved that the groups A_p, A_{p+1}, A_{p+2} and $PGL_2(p)$, where p is a prime, are characterizable by the set nse whose orders are divisible by p but p^2 does not divide their orders. In this paper, we are going to study the characterization of some finite simple groups of Lie type over the finite field of order 2 by nse. In fact, we prove the following theorem:

Main theorem. Let $n \geq 4$, $S \in \{C_n(2), {}^2D_n(2), {}^2D_{n+1}(2)\}$ and G be a finite group. Let $2^n + 1 = p$ be a prime divisor of $|G|$ but $p^2 \nmid |G|$. Then $G \cong S$ if and only if $\mathrm{nse}(G) = \mathrm{nse}(S)$.

As a consequence of our result, we show that if $\mathrm{nse}(S) = \mathrm{nse}(G)$ and $|G| = |S|$, where $n \geq 4$, $S \in \{C_n(2), {}^2D_n(2), {}^2D_{n+1}(2)\}$ and $2^n + 1 = p$ is prime, then $G \cong S$.

2 Notation and preliminary results

Throughout this article, we assume that p is a prime number and G is a finite group. For the natural number n, $\pi(n)$ is the set of prime divisors of n and $\pi(|G|)$ is denoted by $\pi(G)$. The prime graph $GK(G)$ of G is a graph whose vertex set is $\pi(G)$ and two distinct primes p and q are joined by an edge if and only if G contains an element of order pq. Let $t(G)$ be the number of connected components of $GK(G)$ and let $\pi_1, \pi_2, ..., \pi_{t(G)}$ be the connected components of $GK(G)$. If $2 \in \pi(G)$, then we always suppose that $2 \in \pi_1(G)$. $|G|$ can be expressed as a product of co-prime positive integers $OC_i,\ i = 1, 2, \ldots, t(G)$, where $\pi(OC_i) = \pi_i$. These OC_i's are called the order components of G and the set of order components of G will be denoted by $OC(G)$. Also we call $OC_2, ..., OC_{t(G)}$ the odd order components of G. The sets of order components of finite simple groups with disconnected prime graph can be obtained using [11] and [15]. Throughout this paper, we denote by ϕ the Euler's function. If q is a prime, then we denote by $S_q(G)$ a Sylow q-subgroup of G, by $\mathrm{Syl}_q(G)$ the set of Sylow q-subgroups of G and by $n_q(G)$ the number of Sylow q-subgroups of G, that is, $n_q(G) = |\mathrm{Syl_q(G)}|$. For a prime a and the natural number n, we use $|n|_a = a^e$, when $a^e \| n$, i.e., $a^e \mid n$ but $a^{e+1} \nmid n$. Also, $|cl_G(x)|$ denotes the order of the conjugacy class of x in G.

Throughout this paper let $n \geq 4$ and $S \in \{C_n(2), {}^2D_n(2), {}^2D_{n+1}(2)\}$. Also, let $2^n + 1 = p$ be a prime number.

In the following lemmas, we are going to bring some useful results which will be used during the proof of the main theorem:

Definition 2.1 ([6]) Let a and n be integers greater than 1. Then, a Zsigmondy prime of $a^n - 1$ is a prime l such that $l \mid (a^n - 1)$ but $l \nmid (a^i - 1)$ for $1 \leq i < n$.

Lemma 2.2 ([6]) *If a and n are integers greater than 1, then there exists a Zsigmondy prime of $a^n - 1$, unless $(a, n) = (2, 6)$ or $n = 2$ and $a = 2^s - 1$ for some natural number s.*

Remark 2.1 If l is a Zsigmondy prime of $a^n - 1$, then Fermat's little theorem shows that $n \mid l - 1$. Put $Z_n(a) = \{l : l \text{ is a Zsigmondy prime of } a^n - 1\}$. If $r \in Z_n(a)$ and $r \mid a^m - 1$, then we can see at once that $n \mid m$.

Lemma 2.3 ([5]) *Let G be a Frobenius group of even order with kernel K and complement H. Then $t(G) = 2$, the prime graph components of G are $\pi(H)$ and $\pi(K)$ and the following assertions hold:*

(1) *K is nilpotent;*
(2) $|K| \equiv 1 \pmod{|H|}$.

Lemma 2.4 ([5]) *Let G be a 2-Frobenius group, i.e., G is a finite group and has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Then:*
(*a*) *$t(G) = 2$, $\pi_1 = \pi(G/K) \cup \pi(H)$ and $\pi_2 = \pi(K/H)$;*
(*b*) *G/K and K/H are cyclic, $|G/K| \mid (|K/H| - 1)$ and $G/K \leq \mathrm{Aut}(K/H)$.*

Lemma 2.5 ([15]) *If G is a finite group such that $t(G) \geq 2$, then G has one of the following structures:*
(*a*) *G is a Frobenius group or 2-Frobenius group;*
(*b*) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(H) \cup \pi(G/K) \subseteq \pi_1$ and K/H is a non-abelian simple group. In particular, H is nilpotent, $G/K \lesssim \mathrm{Out}(K/H)$ and the odd order components of G are the odd order components of K/H.*

Lemma 2.6 ([12]) *The equation $p^m - q^n = 1$, where p and q are primes and $m, n > 1$ has only solution, namely, $3^2 - 2^3 = 1$.*

Lemma 2.7 ([9]) *Let q be a prime power which is not of the form $3^r 2^s \pm 1$, where $r = 0, 1$ and $s \geq 1$. Let $M = C_n(q)$, where $n = 2^m$ $(m \geq 2)$ and $OC_2 = (q^n+1)/(2, q+1)$. If $x \in \pi_1(M)$, $x^\alpha \mid |M|$ and $x^\alpha - 1 \equiv 0 \pmod{OC_2}$, then $x^\alpha = q^{2kn}$, where $1 \leq k \leq n/2$. Also, if $x \in \pi_1(M)$, $x^\alpha \mid |M|$ and $x^\alpha + 1 \equiv 0 \pmod{OC_2}$, then $x^\alpha = q^{(2k+1)n}$, where $1 \leq k \leq n/2 - 1$.*

Corollary 2.8 *If $x \in \pi(S) - \{p\}$ such that $x^\alpha - 1 \equiv 0 \pmod{p}$, then either $x^\alpha \nmid |S|$ or $x = 2$.*

Proof. Let $x^\alpha \mid |S|$. We are going to complete the proof in the following cases:
i. If $S = C_n(2)$, then since $OC_2(S) = p$, $x^\alpha \mid |S|$ and $x^\alpha - 1 \equiv 0 \pmod{p}$, Lemma 2.7 completes the proof.
ii. If $S = {}^2D_n(2)$, then $|S| \mid |C_n(2)|$ and $OC_2(C_n(2)) = p$. Thus Lemma 2.7 completes the proof.
iii. Let $S = {}^2D_{n+1}(2)$. If $x \notin \pi(2^{n+1} + 1) \cup \{2\}$, then $|S|_x = |C_n(2)|_x$. Thus since $OC_2(C_n(2)) = p$, Lemma 2.7 completes the proof. Now let $x \in \pi(2^{n+1} + 1)$. Then without loss of generality, we can assume that $p \in Z_\alpha(x)$. Thus Remark 2.1 shows that $\alpha \mid p - 1 = 2^n$ and hence, for some natural number m, $\alpha = 2^m$. On the other hand, $p \mid x^\alpha - 1$ and hence $p \mid x^{2^{m-1}} + 1$. Now comparing $|C_n(2)|_x$ and $|{}^2D_{n+1}(2)|_x$ shows that either $x \in Z_{2(n+1)}(2)$ or $x^{2^{m-1}} \mid |C_n(2)|$, which in the latter case, Lemma 2.7 leads us to get a contradiction and if $x \in Z_{2(n+1)}(2)$, then $|S|_x \leq (2^{n+1}+1)/3 < p$, which is a contradiction again. This shows that $x = 2$, as claimed. □

Note that since $2^n + 1$ is prime, we deduce that n is a power of 2. Thus the following lemma may be concluded from [2, Corollary 2.2] and [3, Corollary 3.7]:

Lemma 2.9 *If $r \in Z_{2n}(2)$ and x is an element of $S - \{1\}$, then either $|C_S(x)|_r = 1$ or x is a p-element and $|cl_S(x)| = |S|/p$.*

Lemma 2.10 ([7]) *Let t be a positive integer dividing $|G|$. If $M_t(G) = \{g \in G : g^t = 1\}$, then $t \mid |M_t(G)|$.*

From Lemma 2.10, it may be concluded that:

Corollary 2.11 *For the finite group G:*

(i) $n|\sum_{s|n} m_s$ *if* $n \mid |G|$;
(ii) *if* $n \in \pi_e(G)$, *then* $m_n = \phi(n)k$, *where* k *is the number of cyclic subgroups of order* n *in* G. *In particular,* $\phi(n) \mid m_n$;
(iii) *if* $P \in \mathrm{Syl}_p(G)$ *is cyclic of prime order* p, *then* $m_p = \phi(p)n_p(G)$;
(iv) *if* $P \in \mathrm{Syl}_p(G)$ *is cyclic of prime order* p *and* $r \in \pi(G) - \{p\}$, *then* $m_{rp} = n_p(G)(p-1)(r-1)k$, *where* k *is the number of cyclic subgroups of order* r *in* $C_G(P)$.

Lemma 2.12

$$n_p(S) = \begin{cases} \dfrac{|S|}{np}, & \text{if } S = {}^2D_n(2) \\ \dfrac{|S|}{2np}, & \text{otherwise.} \end{cases}$$

Proof. Let $R \in \mathrm{Syl}_p(S)$, where $S \in \{{}^2D_n(2), C_n(2)\}$. Considering the order of S leads us to see that R is cyclic and hence, there exists $x \in S$ such that $R = \langle x \rangle$. Now [14, P.16, L.-17,-18] shows that $N_{GO^-_{2n}(2)}(R) = N_{Sp_{2n}(2)}(R) = Z_p.Z_{2n}$. But ${}^2D_n(2)$ is a normal subgroup of $GO^-_{2n}(2)$ of index 2 and hence, Frattini's argument shows that $[N_{GO^-_{2n}(2)}(R) : N_{{}^2D_n(2)}(R)] = 2$. Also, $Sp_{2n}(2) = C_n(2)$ and hence, the result follows for $S \in \{{}^2D_n(2),\ C_n(2)\}$. Now let $S = {}^2D_{n+1}(2)$ and $R_1 \in \mathrm{Syl}_p({}^2D_{n+1}(2))$. Then considering the order of S and [10, P.88] allow us to assume that $R_1 = \left\{ \begin{pmatrix} I_2 & 0 \\ 0 & x \end{pmatrix} : I_2 \in GO^+_2(2),\ x \in R \right\}$, where $R \in \mathrm{Syl}_p({}^2D_n(2))$. But R is an irreducible subgroup of $GO^-_{2n}(2)$ and hence, Schur's lemma forces $N_{GO^-_{2(n+1)}(2)}(R_1) = \left\{ \begin{pmatrix} A_0 & 0 \\ 0 & B_0 \end{pmatrix} : A_0 \in GO^+_2(2),\ B_0 \in N_{GO^-_{2n}(2)}(R) \right\}$. Thus the above statements show that $|N_{GO^-_{2(n+1)}(2)}(R_1)| = |GO^+_2(2)||N_{GO^-_{2n}(2)}(R)| = 4np$. Also ${}^2D_{n+1}(2)$ is a normal subgroup of $GO^-_{2(n+1)}(2)$ of index 2 and hence, Frattini's argument shows that $[N_{GO^-_{2(n+1)}(2)}(R_1) : N_{{}^2D_{n+1}(2)}(R_1)] = 2$. Thus the result follows. □

3 Proof of the main theorem

In this section, let G be a finite group such that $p \| |G|$ and $\mathrm{nse}(G) = \mathrm{nse}(S)$.

Lemma 3.1 *For* $u \in \pi_e(S)$, *either* $p \mid m_u(S)$ *or* $u = p$ *and*

$$m_p(S) = \begin{cases} \dfrac{2^n|S|}{np}, & \text{if } S = {}^2D_n(2) \\ \dfrac{2^n|S|}{2np}, & \text{otherwise.} \end{cases}$$

Proof. Since $|S_p| = |S|_p = p$, we deduce that S_p is cyclic. Thus Corollary 2.11 (iii) forces $m_p(S) = \phi(p)n_p(S)$ and hence, Lemma 2.12 shows that

$$m_p(S) = \begin{cases} \dfrac{2^n|S|}{np}, & \text{if } S = {}^2D_n(2) \\ \dfrac{2^n|S|}{2np}, & \text{otherwise.} \end{cases}$$

On the other hand, $m_u(S) = \sum_{\text{for some } y\in S \text{ with } O(y)=u} |cl_S(y)|$ and hence, Lemma 2.9 completes the proof. □

Lemma 3.2 *For every $u \in \pi_e(G)$, $p \nmid m_u(G)$ if and only if $m_u(G) = m_p(S)$. In particular, $m_p(G) = m_p(S)$.*

Proof. Since $m_u(G) \in \text{nse}(S)$, Lemma 3.1 completes the proof. Also, Corollary 2.11 (i) forces $p \mid 1 + m_p(G)$ and hence, $p \nmid m_p(G)$. Thus $m_p(G) = m_p(S)$, as desired. □

Lemma 3.3 $m_2(G) = m_2(S)$.

Proof. By Corollary 2.11 (ii), for every $u \in \pi_e(S)$, $\phi(u) \mid m_u(S)$. Thus if $u > 2$, then m_u is even. On the other hand, $2 \mid 1 + m_2(S)$ and hence, $m_2(S)$ is odd. Applying the same reasoning shows that the only odd number in nse(G) is $m_2(G)$ and hence, $m_2(G) = m_2(S)$, as desired. □

Hereafter, set

$$d_S = \begin{cases} 1, & \text{if } S = {}^2D_n(2) \\ 2, & \text{otherwise.} \end{cases}$$

Lemma 3.4 *For every $r \in \pi_e(G) - \{p\}$, $rp \notin \pi_e(G)$.*

Proof. Suppose on the contrary $rp \in \pi_e(G)$. Since $p^2 \nmid |G|$, we deduce that $S_p(G)$ is cyclic and hence, Corollary 2.11 (iv) forces $m_{pr}(G) = m_p(G)\phi(r)k$, for some natural number k. Thus $m_{pr}(G) = \frac{2^n|S|\phi(r)k}{d_S np}$ and hence, one of the following holds:

- Let $p \mid m_{rp}(G)$. Then $p \mid \phi(r)k$ and hence, $m_{rp}(G) \geq \frac{2^n|S|}{d_S n} > |S|$. Thus $m_{rp}(G) \notin$ nse(S), which is a contradiction.
- Let $p \nmid m_{rp}(G)$. Then Lemma 3.1 shows that $m_{rp}(G) = m_p(G)$ and hence, Corollary 2.11 (iv) forces $r = 2$. Thus Lemma 3.3 implies that $m_2(G) = m_2(S)$ and hence by Lemma 3.1, $p \mid m_2(G)$. On the other hand, Corollary 2.11 (i) forces $2p \mid (1 + m_p + m_2 + m_{2p})$ and $p \mid (1 + m_p)$. Therefore, $p \mid m_{2p} = m_p$, which is a contradiction. □

Corollary 3.5 *p is an odd order component of G.*

Proof. It follows immediately from Lemma 3.4. □

Lemma 3.6(i) *$n_p(G) = n_p(S)$.*
(ii) *$|G| \mid \frac{2^n|S|}{n}$.*

Proof. Since $S_p(G)$ is cyclic, Corollary 2.11 (iii) forces $m_p(G)=\phi(p)n_p(G)$. Thus by Lemma 3.2, $\phi(p)n_p(S) = m_p(S) = m_p(G) = \phi(p)n_p(G)$ and hence (i) follows. Now let $r \in \pi(G) - \{p\}$. By Lemma 3.4, $S_r(G)$ acts fixed point freely on the set of elements of order p in G and hence, $|G|_r \mid m_p(G) = \phi(p)n_p(G)$. Also, $|G|_p = p$. Thus by (i) and Lemma 2.12, $|G| \mid \frac{2^n|S|}{n}$. □

Proof of the main theorem. If $G \cong S$, then it is obvious that $\mathrm{nse}(G) = \mathrm{nse}(S)$. Now we assume that $\mathrm{nse}(G) = \mathrm{nse}(S)$. We are going to prove the main theorem in the following steps:

Step 1. G is neither a Frobenius group nor a 2-Frobenius group.

Proof. Suppose on the contrary, G is a Frobenius group with kernel K and complement H. By Lemma 2.3 and Corollary 3.5, $p \in \{\pi(H), \pi(K)\}$, so $\pi(K) = \{p\}$ or $\pi(H) = \{p\}$. If $\pi(K) = \{p\}$, then since $K \trianglelefteq G$ and $p \| |G|$, we deduce that $S_p(G) = K$ is a normal and cyclic subgroup of G. Thus $m_p(G) = p - 1$, which is a contradiction with Lemmas 3.1 and 3.2. Now, let $\pi(H) = \{p\}$. By Lemmas 2.9 and 3.5, $\frac{|S|}{d_S np} = n_p(G) \mid |G|$ and $|G| \mid \frac{2^n|S|}{n}$. Thus Lemma 2.2 forces to exist $r \in Z_{n-1}(2) \cap \pi(G)$. It follows immediately that $|G|_r = |S|_r = |2^{n-1} - 1|_r$. But $\{\pi(K), \pi(H)\} = \{\pi_1(G), \pi_2(G)\}$. Thus $r \in \pi(K)$. Since K is nilpotent, $S_r(G)$ is a normal subgroup of G, so $S_p(G)$ acts fixed point freely on $S_r(G) - \{1\}$ and hence, $p \mid |S_r(G)| - 1$. This guarantees that $p \leq |S_r(G)| \leq 2^{n-1} - 1 < p$, which is a contradiction. If G is a 2-Frobenius group, then Lemma 2.4 implies that there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a cyclic group of order p and $|G/K| \mid (p-1)$. Now applying the previous argument for the Frobenius group K with kernel H and complement K/H leads us to get a contradiction. □

Step 2. There exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a simple group and p is an odd order component of K/H.

Proof. It follows immediately from Lemma 2.5 and Step 1. □

Step 3. $\pi(H) \subseteq \{2\}$.

Proof. Let $r \in \pi(H)$. Then $r \neq p$ and since Lemma 2.5 forces H to be nilpotent, we deduce that $S_r(H) \trianglelefteq G$ and hence, $S_p(G)$ acts fixed point freely on $S_r(H) - \{1\}$. Thus $p \mid |S_r(H)| - 1$. If $r \neq 2$, then Lemma 3.6 (ii) shows that $|S_r(H)| \mid |S|_r$ and hence, Corollary 2.8 leads us to get a contradiction. Thus $r = 2$, as claimed. □

Step 4. K/H is not a sporadic simple group.

Proof. Suppose that K/H is a sporadic simple group. Since one of the odd order components of K/H is $p = 2^n + 1$, we get a contradiction by considering the odd order components of sporadic simple groups. □

Step 5. K/H can not be an alternating group $\mathbb{A}_m$, where $m \geq 5$.

Proof. If $K/H \cong \mathbb{A}_m$, then since $2^n+1 = p \in \pi(K/H)$, $m \geq 2^n+1$. Thus there exists a prime number $u \in \pi(A_m) \subseteq \pi(G)$ such that $\frac{(p-1)}{2} < u < p$ and hence, Lemma 3.6 (ii) forces $u \mid \frac{2^n|S|}{n}$. Thus there exists $t \in \{2i, i : 1 \leq i \leq n-1\} \cup \{n\}$ such that $u \in Z_t(2)$. But $u > 2^{n-1}$ and hence, $u = 2^{n-1}+1$ or 2^n-1. But n is a power of 2 and hence, $3 \mid 2^{n-1}+1$ and 2^n-1. Thus $3 \mid u$. This implies that $u = 3$ ad hence, $n = 2$, which is a contradiction. $\square$

Step 6. $K/H \in \{C_n(2), {}^2D_n(2), {}^2D_{n+1}(2)\}$ (under isomorphism).

Proof. By Steps 4 and 5, and the classification theorem of finite simple groups, K/H is a simple group of Lie type such that $t(K/H) \geq 2$ and $p \in OC(K/H)$. Thus K/H is isomorphic to one of the group (note that in the following cases, r is an odd prime number):

Case 1. Let $t(K/H) = 2$. Then $OC_2(K/H) = 2^n+1$. Then we have:

1.1. If $K/H \cong C_{n'}(q)$, where $n' = 2^u > 2$, then $\frac{q^{n'}+1}{(2,q-1)} = 2^n+1$. If q is odd, then $q^{n'} = 2^{n+1}+1$, which is a contradiction with Lemma 2.6. Thus $q = 2^t$ and hence, $q^{n'} = 2^n$. But $p \in Z_{2n}(2)$ and $p \in Z_{2n't}(2)$. Thus Remark 2.1 forces $n't = n$. We claim that $t = 1$. If not, then $Z_{n-1}(2) \cap \pi(K/H) = \emptyset$. But Lemma 2.2 forces $Z_{n-1}(2) \neq \emptyset$ and hence, Lemma 3.6 (ii) shows that $\pi(G)$ contains a prime $r \in Z_{n-1}(2)$. Since $r \nmid |\mathrm{Out}(K/H)|$ and $G/K \lesssim \mathrm{Out}(K/H)$, we deduce that $r \mid |H|$. Thus Step 3 shows that $r = 2$, which is a contradiction. Thus $t = 1$ and hence, $K/H \cong C_n(2)$. The same reasoning completes the proof in the case when either $S \cong B_{n'}(q)$ or $S \cong {}^2D_{n'}(q)$, where $n' = 2^u \geq 4$.

1.2. If $K/H \cong C_r(3)$ or $B_r(3)$, then $\frac{3^r-1}{2} = 2^n+1$. Thus $2^{n+1} = 3^r-3$, which is a contradiction. The same reasoning rules out the case when $S \cong D_r(3)$ or $S \cong D_{r+1}(3)$.

1.3. If $K/H \cong C_r(2)$, then $2^r-1 = 2^n+1$ and hence, $2^r = 2^n+2$, which is a contradiction. The same reasoning rules out the case when $S \cong D_r(2)$ or $S \cong D_{r+1}(2)$.

1.4. If $K/H \cong D_r(5)$, where $r \geq 5$, then $(5^r-1)/4 = (2^n+1)$. Thus $5^r-5 = 2^{n+2}$, which is contradiction.

1.5. If $K/H \cong {}^2D_{n'}(3)$, where $9 \leq n' = 2^m+1$ and n' is not prime, then $\frac{3^{n'-1}+1}{2} = 2^n+1$ and hence, $3^{n'-1} = 2^{n+1}+1$. Thus Lemma 2.6 forces $n+1 = 3$, which is a contradiction.

1.6. If $K/H \cong {}^2D_{n'}(2)$, where $n' = 2^m+1 \geq 5$, then $2^{n'-1}+1 = 2^n+1$ and hence, $n'-1 = n$. Thus $K/H \cong {}^2D_{n+1}(2)$, as claimed.

1.7. If $K/H \cong {}^2D_r(3)$, where $5 \leq r \neq 2^m+1$, then $\frac{3^r+1}{4} = 2^n+1$ and hence, $3^r = 2^{n+2}+3$, which is a contradiction.

1.8. If $K/H \cong G_2(q)$, where $2 < q \equiv \epsilon \pmod 3$ and $\epsilon = \pm 1$, then $q^2-\epsilon q+1 = 2^n+1$. Thus $q(q-\epsilon) = 2^n$, which is impossible. The same reasoning rules out the case when $S \cong {}^3D_4(q)$ or $S \cong F_4(q)$, where q is odd.

1.9. If $K/H \cong {}^2F_4(2)'$, then $|S| = 2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$. Thus $2^n+1 = 13$, which is impossible.

1.10. If $K/H \cong {}^2A_3(2)$, then $|S| = 2^6 \cdot 3^4 \cdot 5$. Thus $2^n + 1 = 5$ and hence, $n = 2$, which is a contradiction.

1.11. Let $K/H \cong A_{r-1}(q)$, where $(r, q) \neq (3, 2)$, $(3, 4)$. Since $\frac{q^r-1}{(r,q-1)(q-1)} = p$, $p \in Z_r(q)$ and hence, Remark 2.1 shows that $r \mid p - 1 = 2^n$. Thus $r = 2$, which is a contradiction. The same reasoning rules out the case when $S \cong {}^2A_{r-1}(q)$.

1.12. Let $K/H \cong A_r(q)$, where $(q - 1) \mid (r + 1)$. Since $\frac{q^r-1}{(r,q-1)} = p$, $p \in Z_r(q)$ and hence, Remark 2.1 shows that $r \mid p - 1 = 2^n$. Thus $r = 2$, which is a contradiction. The same reasoning rules out the case when $(q + 1) \mid (r + 1)$, $(r, q) \neq (3, 3), (5, 2)$ and $S \cong {}^2A_r(q)$.

1.13. If $K/H \cong E_6(q)$, where $q = u^\alpha$, then $\frac{(q^6+q^3+1)}{(3,q-1)} = p$. Thus $p \in Z_9(q)$ and hence, Remark 2.1 shows that $9 \mid p - 1 = 2^n$, which is a contradiction. The same reasoning rules out the case when $S \cong {}^2E_6(q)$, where $q > 2$.

Case 2. Let $t(K/H) = 3$. Then $2^n + 1 \in \{OC_2(K/H), OC_3(K/H)\}$:

2.1. If $K/H \cong A_1(q)$, where $4 \mid q+1$, then $\frac{q-1}{2} = 2^n+1$ or $q = 2^n+1$. If $q = 2^n+1$, then $q + 1 = 2^n + 2$ and hence, $4 \nmid q + 1$, which is a contradiction. If $\frac{q-1}{2} = p$, then $q \equiv -1 \pmod 4$. Let $q = u^\alpha$, where u is a prime. Thus $p \in Z_\alpha(u)$ and hence, Remark 2.1 shows that $\alpha \mid p - 1 = 2^n$. So $\alpha = 2^t$ and hence, $q = u^\alpha \equiv 1 \pmod 4$, which is a contradiction.

2.2. If $K/H \cong A_1(q)$, where $4 \mid q - 1$, then $q = 2^n + 1$ or $\frac{q+1}{2} = 2^n + 1$.
If $q = 2^n+1$, then $q = p$ and hence, $|K/H| = p(p^2-1)/2 = 2^n p(2^{n-1}+1)$ and since $G/K \lesssim \mathrm{Out}(K/H) \cong \mathbb{Z}_2$, we deduce that $Z_n(2) \subseteq \pi(H)$, which is a contradiction with Step 3.
If $\frac{q+1}{2} = 2^n + 1$, then $q = 2^{n+1} + 1$. Thus $3 \mid q$ and hence, $3^\alpha = 2^{n+1} + 1$, which is a contradiction with Lemma 2.6.

2.3. If $K/H \cong A_1(q)$, where $q > 2$ and q is even, then $p \in \{q - 1, q + 1\}$. If $q - 1 = 2^n + 1$, then $q = 2(2^{n-1} + 1)$, which is a contradiction. If $q + 1 = 2^n + 1$, then $q = 2^n$ and hence, $|K/H| = 2^n(2^n - 1)(2^n + 1)$. But $G/K \lesssim \mathrm{Out}(K/H) \cong \mathbb{Z}_n$, so $Z_{n-1}(2) \subseteq \pi(H)$, which is a contradiction with Step 3.

2.4. If $K/H \cong {}^2A_5(2)$ or $K/H \cong A_2(2)$, then $|K/H| = 2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$ or $|K/H| = 8 \cdot 3 \cdot 7$. Clearly, $2^n + 1 \neq 11$ and $2^n + 1 \neq 7$, which is a contradiction.

2.5. If $K/H \cong {}^2D_r(3)$, where $r = 2^t + 1 \geq 5$, then $\frac{3^r+1}{4} = 2^n + 1$ or $\frac{3^{r-1}+1}{2} = 2^n + 1$. If $\frac{3^r+1}{4} = 2^n + 1$, then $3^r = 2^{n+2} + 3$, which is a contradiction. If $\frac{3^{r-1}+1}{2} = 2^n + 1$, then $2^{n+1} + 1 = 3^{r-1}$, which is contradiction with Lemma 2.6.

2.6. If $K/H \cong G_2(q)$, where $q \equiv 0 \pmod 3$, then $q^2 - q + 1 = 2^n + 1$ or $q^2 + q + 1 = 2^n + 1$ and hence, $q(q \pm 1) = 2^n$, which is impossible.

2.7. If $K/H \cong {}^2G_2(q)$, where $q = 3^{2t+1} > 3$, then $q - \sqrt{3q} + 1 = 2^n + 1$ or $q + \sqrt{3q} + 1 = 2^n + 1$. Thus $3^{t+1}(3^t \pm 1) = 2^n$, which is impossible.

2.8. If $K/H \cong F_4(q)$, where q is even, then $q^4 + 1 = 2^n + 1$ or $q^4 - q^2 + 1 = 2^n + 1$. If $q^4 - q^2 + 1 = 2^n + 1$, then $q^2(q^2 - 1) = 2^n$, which is impossible. If $q^4 + 1 = 2^n + 1$, then $q^4 = 2^n$, so $(q^{12} - 1) = (2^{3n} - 1) \mid |K/H|$ and hence, $Z_{3n}(2) \subseteq \pi(G) = \pi(C_n(2))$, which is a contradiction again.

2.9. If $K/H \cong {}^2F_4(q)$, where $q = 2^{2t+1} > 2$, then $q^2 + \sqrt{2q^3} + q + \sqrt{2q} + 1 = 2^n + 1$ or $q^2 - \sqrt{2q^3} + q - \sqrt{2q} + 1 = 2^n + 1$. Thus $2^n + 1 = 2^{2(2t+1)} + \epsilon 2^{3t+2} + 2^{2t+1} + \epsilon 2^{t+1} + 1$, where $\epsilon = \pm 1$ and hence, $2^n = 2^{t+1}(2^{3t+1} + \epsilon 2^{2t+1} + 2^t + \epsilon)$, which is a contradiction.

2.10. If $K/H \cong E_7(2)$, then $2^n + 1 \in \{73, 127\}$, which is impossible.

2.11. If $K/H \cong E_7(3)$, then $2^n + 1 \in \{757, 1093\}$, which is impossible.

Case 3. Let $t(K/H) \in \{4, 5\}$. Then $2^n+1 \in \{OC_2(K/H), OC_3(K/H), OC_4(K/H), OC_5(K/H)\}$, as follows:

3.1. If $K/H \cong A_2(4)$ or ${}^2E_6(2)$, then $2^n + 1 = 7$ or $2^n + 1 = 19$, which is impossible.

3.2. If $K/H \cong {}^2B_2(q)$, where $q = 2^{2t+1}$ and $t \geq 1$, then $2^n+1 \in \{q-1, q \pm \sqrt{2q}+1\}$. If $q-1 = 2^n+1$, then $2^{2t+1} = 2^n+2$ and if $q \pm \sqrt{2q}+1 = 2^n+1$, then $2^{t+1}(2^t \pm 1) = 2^n$, which are impossible.

3.3. If $K/H \cong E_8(q)$, then $2^n+1 \in \{\frac{q^{10}+q^5+1}{q^2-q+1} = q^8-q^7+q^5-q^4+q^3-q+1, \frac{q^{10}-q^5+1}{q^2-q+1} = q^8 + q^7 - q^5 - q^4 - q^3 + q + 1, \frac{q^{10}+1}{q^2+1} = q^8 - q^6 + q^4 - q^2 + 1, q^8 - q^4 + 1\}$. Thus $qt = 2^n$, where $t > 1$ is a natural number such that $(t, q) = 1$, which is a contradiction.

The above cases show that $K/H \in \{C_n(2), {}^2D_n(2), {}^2D_{n+1}(2)\}$ (up to isomorphism), as claimed. $\square$

Step 7. $K/H \cong S$.

Proof. We are going to prove this step in the following cases:

Case 1. First let $S = C_n(2)$. If $K/H \ncong S$, then Step 6 shows that $K/H \cong {}^2D_n(2)$ or ${}^2D_{n+1}(2)$. If $K/H \cong {}^2D_{n+1}(2)$, then $Z_{n+1}(2) \subseteq \pi(K/H)$ and hence, $Z_{n+1}(2) \subseteq \pi(G) = \pi(C_n(2))$, which is a contradiction. Thus $K/H \cong {}^2D_n(2)$ and hence, for $r \in Z_n(2)$, $|K/H|_r = |C_n(2)|_r/|2^n - 1|_r$. So Lemmas 2.12 and 3.6 force $|G|_r/|K/H|_r = |2^n - 1|_r$. But $G/K \lesssim \mathrm{Out}(K/H) \cong \mathbb{Z}_2$ and hence, $|H|_r = |2^n - 1|_r$, which is a contradiction with Step 3.

Case 2. Let $S = {}^2D_n(2)$. If $K/H \ncong S$, then Step 6 shows that $K/H \cong C_n(2)$ or ${}^2D_{n+1}(2)$. Thus Lemma 3.6 (ii) shows that $|G|_2 \mid 2^n|S|_2/n = 2^{n^2}/n$. But $|K/H|_2 \geq 2^{n^2}$ and hence, $|G|_2 \geq 2^{n^2}$, which is a contradiction.

Case 3. Let $S = {}^2D_{n+1}(2)$. Then applying the previous argument shows that $Z_{n+1}(2) \subseteq \pi(K/H) \subseteq \pi(G) = \pi(S)$. Thus Step 6 forces $K/H \cong S$.

The above cases show that $K/H \cong S$, as claimed. $\square$

Step 8. $G \cong S$.

Proof. By Steps 3 and 7, and Lemma 3.6, $|H| = 2^t \mid 2^n/n$. Thus since by Lemma 3.4, $S_p(G)$ acts fixed point freely on $H - \{1\}$, we deduce that $p \mid 2^t - 1$ and hence, Remark 2.1 shows that either $2n \mid t$ or $t = 0$. This forces $t = 0$ and hence, $H = 1$, $K \cong S$ and $G \leq \mathrm{Aut}(S)$. If $S \cong C_n(2)$, then Aut(S) $= S$ and hence, $G \cong S$, as desired. Now let $S \in \{{}^2D_n(2), {}^2D_{n+1}(2)\}$. Then

$$\mathrm{Aut}(S) = \begin{cases} GO_{2n}^-(2), & \text{if } S = {}^2D_n(2) \\ GO_{2(n+1)}^-(2), & \text{if } S = {}^2D_{n+1}(2). \end{cases}$$

Thus $m_2(\mathrm{Aut}(S)) > m_2(S)$ and hence, Lemma 3.3 forces $S \cong G$, as claimed. □

Corollary 3.7 *Let $n \geq 4$, $S \in \{C_n(2), {}^2D_n(2), {}^2D_{n+1}(2)\}$ and $2^n + 1 = p$ is prime. If G is a finite group with $\mathrm{nse}(S) = \mathrm{nse}(G)$ and $|G| = |S|$, then $G \cong S$.*

Proof. It follows immediately from the main theorem. □

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