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On spectrum and trace formula for one class of singular problems

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Abstract In this paper we obtain an asymptotic formula for eigenvalues of differential operator given on the whole axis and establish a formula for the regularized trace.

Keywords Spectrum · Eigenvalue · Asymptotics · Trace · Selfadjointness · Hilbert space

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Introduction

Let H be a separable Hilbert space with the scalar product $(\cdot,\cdot)$ and the norm $\|\cdot\|$. In $L_2((-\infty,+\infty),H)$ consider the problem $ly \equiv -y''(x)+|x|\ y\,(x)+Ay(x)+q\,(x)\,y\,(x) = \lambda y\,(x)$, where A is a self-adjoint operator, $A > E$, E is identity operator in H, $A^{-1} \in \sigma_\infty$, where σ_∞ is the set of all compact operators in H. Denote the eigenvalues of the operator A by $\gamma_1 \leq \gamma_2 \leq ...$ which are counted according to their multiplicities and let φ_1, φ_2..., denote eigenvalues of operator A respectively. We assume that for large k values $\gamma_k \sim a\cdot k^\alpha$, $a>0$, $\alpha>0$. Suppose that the operator-valued function $q\,(x)$ is weakly measurable, $\|q\,(x)\| < const$, $q^*\,(x) = q\,(x)\ \forall x \in (-\infty,+\infty)$ in H, also satisfies the following conditions:

1) $\sum_{k=1}^{\infty}\int_{-\infty}^{+\infty}|(q\,(x)\,\varphi_k,\,\varphi_k)|\,dx < \text{const}$;

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2) denoting $q_k(x) = (q(x)\varphi_k, \varphi_k)$ we assume that $\int_{-\infty}^{+\infty} \frac{q_k(x)}{x} dx < \infty$, and

$$\int_{-\infty}^{0} \frac{q_k(x)}{x} dx = 0, \ \int_{0}^{+\infty} \frac{q_k(x)}{x} dx = 0, \ \forall k = \overline{1, \infty},$$

3) $\int_{-\delta}^{\delta} |\frac{q_k(x)}{x^5}| dx < \infty$, where $\delta > 0$ is a sufficiently small number. Define the operators L_1 and L_2 in spaces $L_2((0,\infty),H)$ and $L_2((-\infty;0),H)$, respectively in the following way

$$\begin{aligned} D(L_1) = \{&y \in L_2((0,\infty),H)/L_1 y \equiv -y''(x) \\ &+ xy(x) + Ay(x) + q(x)y(x) \in L_2((0,\infty),H), \ y(0) = 0\}, \\ D(L_2) = \{&y \in L_2((-\infty,0),H)/L_2 y \equiv \\ &- y''(x) - xy(x) + Ay(x) + q(x)y(x) \in L_2((-\infty;0),H), y'(0) = 0\}. \end{aligned}$$

Denote the operators corresponding to case $q(x) \equiv 0$ by L_1^0 and L_2^0.

Evidently, L_1^0 and L_2^0 are self adjoint positive-definite operators with discrete spectrum. Define a self-adjoint operator $L_0 := L_1^0 \oplus L_2^0$. The spectra of this operator is a union of spectrums of operators L_1^0 and L_2^0. Define also the operator $L = L_1 \oplus L_2$.

In this paper, we investigate asymptotics of spectrum and trace of the operator L. Asymptotics of eigenvalues for differential operators is the subject of various works. Here we refer, for example, to works [1], [3-5], [8], [10], [12].

Trace formula for a differential operator equation with an unbounded operator coefficient is defined in [12]. Also we can refer to [2-6], [9], [11]. For abstract operators regularized traces are studied in [7, 13].

1 Asymptotic formula for eigenvalues of operator L

At the first step, investigate the spectrum of operators L_1 and L_2. Denote the eigenvalues of the operators L_1^0 and L_2^0 by λ_n^+ and λ_n^-, respectively.

From the spectral expansion of the self-adjoint operator A we get the following spectral problem for the coefficients $y_k(x) = (y(x), \varphi_k)$ in $L_2(0,\infty)$

$$-y_k''(x) + xy_k(x) + \gamma_k y_k(x) = \lambda y_k(x), \tag{1.1}$$

$$y_k(0) = 0. \tag{1.2}$$

The solution of equation (1.1) from $L_2(0,\infty)$ for $x + \gamma_k > \lambda$ is (see [14, p. 106]) $\psi^+(x,\lambda) = \sqrt{x + \gamma_k - \lambda}\, K_{\frac{1}{3}}(\frac{2}{3}(x + \gamma_k - \lambda)^{\frac{3}{2}})$, where $K_\nu(z) = \frac{\pi i}{2} e^{\frac{\pi}{2}\nu i} H_\nu^{(1)}(iz)$, $H_\nu^{(1)}(z)$ is a Henkel function [15, p.86] (it is solution of Bessel equation and is also called third type function). For $x + \gamma_k < \lambda$ one can write $\psi^+(x,\lambda)$ as a function of a real argument as

$$\psi^+(x,\lambda) = \frac{\pi}{\sqrt{3}} \sqrt{\lambda - x - \gamma_k} \left\{ J_{\frac{1}{3}} \left(\frac{2}{3} (\lambda - \gamma_k - x)^{\frac{3}{2}} \right) \right. \\ \left. + J_{-\frac{1}{3}} \left(\frac{2}{3} (\lambda - \gamma_k - x)^{\frac{3}{2}} \right) \right\}.$$

In virtue of (1.2) we get equation

$$\sqrt{\lambda-\gamma_k}\left\{J_{\frac{1}{3}}\left(\frac{2}{3}\left(\lambda-\gamma_k\right)^{\frac{3}{2}}\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}\left(\lambda-\gamma_k\right)^{\frac{3}{2}}\right)\right\}=0, \tag{1.3}$$

to determine the eigenvalues of L_1. Here J_ν is a Bessel function (see [15, p.51]). Taking $z=\sqrt{\lambda-\gamma_k}$, one can write (1.3) as

$$z\left\{J_{\frac{1}{3}}\left(\frac{2}{3}z^3\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}z^3\right)\right\}=0. \tag{1.4}$$

The following statement about roots of (1.4) is true.

Lemma 1.1 *Equation* (1.4) *has only real roots.*

Proof. Let $z=i\alpha,\ \alpha\in R,\ \alpha\neq 0$. Then, the self-adjoint operator associated with the problem $-u''\left(x\right)+xu\left(x\right)=z^2u\left(x\right),\ x\in\left(0,\infty\right),\ u\left(0\right)=0$ is positive. Indeed, if $Bu:=-u''(x)+xu(x), D(B)=\{u\in L_2(0,\infty)/Bu\in L_2(0,\infty),u(0)=0\}$, then

$$(Bu,u)=\int_0^\infty u'\left(x\right)^2dx+\int_0^\infty x\,u^2\left(x\right)dx\geq 0.$$

The eigenvalues of the operator B are the roots of equation (1.4). Have this equation imaginary root $z=i\alpha$, then $(Bu,u)=(z^2u,u)_{L_2(0,\infty)}=-\alpha^2(u(x),u(x))_{L_2(0,\infty)}<0$. So the left hand side of the last equation is positive, while the right right hand side is negative. Have this equation a complex root, then the self adjoint operator B would have unreal eigenvalues, which is contradiction.

The lemma is proved. □

We will now find asymptotics of the roots of equation (1.4). The following asymptotic relation at large $|z|$ [15, p.222]

$$J_\nu\left(z\right)=\sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\left(1+O\left(\frac{1}{z}\right)\right) \tag{1.5}$$

yields

$$z=\left(\frac{3\pi}{2}\right)^{\frac{1}{3}}m^{\frac{1}{3}}+\frac{1}{4}\frac{1}{m^{\frac{1}{3}}}+O\left(\frac{1}{m^{\frac{5}{3}}}\right). \tag{1.6}$$

Therefore, the eigenvalues of the operator L_1^0 are representable as $\lambda_{m,k}=\gamma_k+\alpha_m^2$, where

$$\alpha_m=c_1m^{\frac{1}{3}}+O\left(\frac{1}{m^{\frac{1}{3}}}\right). \tag{1.7}$$

For large $|z|$ consider a rectangular contour l_N with vertices at the points $\pm A_N\pm iB$, where $B=CA_N,\ A_N=\sqrt[3]{\frac{3\pi N}{2}+\frac{15\pi}{8}}$. In virtue of (1.7) for large N values $A_{N-1}<\alpha_N<A_N$.

Lemma 1.2 *Given sufficiently large N, then the number of the roots of equation* (1.4) *inside the contour l_N is $2N+1$.*

Proof. In virtue of asymptotics (1.5), by Rouche theorem for large $|z|$ the number of the zeros of function $F(z)=z\{J_{\frac{1}{3}}(\frac{2}{3}z^3)+J_{-\frac{1}{3}}(\frac{2}{3}z^3)\}$ inside the contour l_N equals the number of zeros $z\cos(\frac{2}{3}z^3-\frac{\pi}{4})$, i.e. $2N+1$. Denote the eigenvalue distribution function of the operator L_1^0 by $N_1(\lambda,L_1^0): N_1(\lambda,L_1^0)=\sum_{\lambda_{m,k}<\lambda}1$.

$N_1(\lambda,L_1^0)$ is the number of the positive integer pairs (m,k) for which $\alpha_{m,k}^2+\gamma_k\leq\lambda$ is valid. In virtue of asymptotics (1.7) and lemma 1.2 for sufficiently large m, $(c-\varepsilon)\,m^{\frac{2}{3}}\leq\alpha_{m,k}^2\leq(c+\varepsilon)\,m^{\frac{2}{3}}$. From asymptotics γ_k it follows that $(\alpha-\varepsilon)k^\alpha<\gamma_k<(a+\varepsilon)k^\alpha$ for sufficiently small $\varepsilon>0$. So, in virtue of lemmas 1.1, 1.2 $N_1''(\lambda)<N_1(\lambda,L_1^0)<N_1'(\lambda)$, where $N_1'(\lambda)$ is the number of positive integer pairs (m,k) satisfying

$$(c-\varepsilon)\,m^{\frac{2}{3}}+(a-\varepsilon)\,k^\alpha<\lambda, \tag{1.8}$$

and $N_2'(\lambda)$ is the one satisfying

$$(a+\varepsilon)\,k^\alpha+(c+\varepsilon)\,m^{\frac{2}{3}}<\lambda. \tag{1.9}$$

By using (1.8), (1.9), we have $N_1(\lambda,L_1^0)\sim c_1 n^{\frac{2+3\alpha}{2\alpha}}$ which yields that $\lambda_n^+\sim d_1 n^{\frac{2\alpha}{2+3\alpha}}$.
Then in similar way as in [12] we get for the eigenvalues of the operator L_1 denoted by μ_n^+ the asymptotics $\mu_n^+\sim d_1 n^{\frac{2\alpha}{2+3\alpha}}$. □

Thus, the following statement is true.

Theorem 1.3 *If $\gamma_k\sim ak^\alpha\,(a>0,\,\alpha>0)$, then*

$$\lambda_n^+\sim\mu_n^+\sim d_1 n^{\frac{2\alpha}{2+3\alpha}}. \tag{1.10}$$

2 Trace formula

Let A be a self-adjoint positive discrete operator, its fractional powers be also positive operators, defined by the spectral theorem. Denote its eigenvalues numbered in ascending order by $\{\lambda_n\}$, and denote by $\{\varphi_j\}$ a basis consisting of eigenvectors of A. Denote by B the perturbation operator and by $\{\mu_n\}$ the eigenvalues of $A+B$ arranged in ascending order of their real parts. In [13] the following theorem is proved.

Theorem 2.1 *Suppose there is a number $\delta\geq 0$ such that the operator BA^δ is bounded and $A^{-(1+\delta)}$ is a trace class operator. Then there exists a subsequence of natural numbers $\{n_m\}_{m=1}$ that*

$$\lim_{m\to\infty}\sum_{j=0}^{n_m}(\mu_j-\lambda_j-(B\varphi_j,\varphi_j))=0.$$

If we will take $L_1^0=A,\ B=q_+(x)\ (q_+(x)=q(x),\ x>0)$ then from (1.10) for $\delta>1-\frac{2\alpha}{2+\alpha}$ the operator $q_+{L_0^1}^{-(1+\delta)}$ is bounded and $\left(L_0^1\right)^{-(1+\delta)}$ is a trace class operator. Thus, in virtue of theorem 2.1

$$\lim_{m\to\infty}\sum_{j=0}^{n_m}\left(\mu_j^+-\lambda_j^+-\left(q_+\psi_j^+,\,\psi_j^+\right)\right)=0, \tag{2.1}$$

where ψ_j^+, $j=\overline{1,\infty}$ are orthonormal eigen-vectors of L_0^1.

It can be calculated that orthonormal eigenvectors of L_0^1 in $L_2\left((0,\infty),\,H\right)$ are

$$\psi_{m,k}^+=\frac{\sqrt{3}\,\psi^+\left(x,\alpha_{m,k}^2\right)\varphi_k}{\pi\alpha_{m,k}^2\left\{J_{\frac{2}{3}}\left(\frac{2}{3}\alpha_{m,k}^3\right)-J_{-\frac{2}{3}}\left(\frac{2}{3}\alpha_{m,k}^3\right)\right\}}. \tag{2.2}$$

The following lemma is true

Lemma 2.2 *If the operator-valued function* $q\left(x\right)$ *satisfies condition* 1)*, then*

$$\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\int_0^{\infty}\frac{q_k^+\left(x\right)\psi^+\left(x,\,\alpha_{m,k}^2\right)^2dx}{\alpha_{m,k}^2\left\{J_{\frac{2}{3}}\left(\frac{2}{3}\alpha_{m,k}^3\right)-J_{-\frac{2}{3}}\left(\frac{2}{3}\alpha_{m,k}^3\right)\right\}}<\infty.$$

Proof. From recurrent formulas for the Bessel function

$$J_{-\frac{2}{3}}\left(\frac{2}{3}\beta^3\right)-J_{\frac{2}{3}}\left(\frac{2}{3}\beta^3\right)=\frac{1}{2\beta^2}\left[J_{\frac{1}{3}}\left(\frac{2}{3}\beta^3\right)\right.$$
$$\left.+\frac{1}{2\beta^3}\left\{J_{\frac{1}{3}}\left(\frac{2}{3}\beta^3\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}\beta^3\right)\right\}\right].$$

For large $|z|$ [15, p. 226]

$$J_{\frac{1}{3}}\left(z\right)+J_{-\frac{1}{3}}\left(z\right)=\frac{\sqrt{3}}{\pi i}K_{\frac{1}{3}}\left(iz\right)\sim e^{-iz}. \tag{2.3}$$

Hence, for large m values

$$J_{\frac{2}{3}}\left(\frac{2}{3}\alpha_{m,k}^3\right)-J_{-\frac{2}{3}}\left(\frac{2}{3}\alpha_{m,k}^3\right)\sim\frac{\sqrt{3}}{\pi i}\frac{e^{-i\frac{2}{3}\alpha_{m,k}^3}}{2\alpha_{m,k}^2}. \tag{2.4}$$

By using asymptotics (2.3) it can be shown that

$$\int_0^{\infty}\left|q_k^+\left(x\right)\psi^+\left(\alpha_{m,k}^2,x\right)\right|^2dx<c\int_0^{\infty}\left|q_k^+\left(x\right)\right|dx. \tag{2.5}$$

In virtue of condition 1), relations (2.4), (2.5) and $\alpha_{m,k}\sim c_m^{\frac{1}{3}}$ as $m\to\infty$ we obtain

$$\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\int_0^{\infty}\left|\frac{q_k^+\left(x\right)\psi^+\left(x,\,\alpha_{m,k}^2\right)^2}{\alpha_{m,k}^4\left\{J_{\frac{2}{3}}\left(\frac{2}{3}\alpha_{m,k}^3\right)-J_{-\frac{2}{3}}\left(\frac{2}{3}\alpha_{m,k}^3\right)\right\}^2}\right|bx$$
$$<\sum_{k=1}^{\infty}\int_0^{\infty}q_k^+\left(x\right)dx\sum_{m=1}^{\infty}O\left(\frac{1}{m^2}\right)<\infty.$$

The lemma is proved. □

We will now prove the following theorem.

Theorem 2.3 *If the operator valued function $q(x)$ satisfies conditions* 1) − 3)*, then the following formula holds:*

$$\lim_{m\to\infty}\sum_{n=1}^{n_m}\left(\mu_n^+ - \lambda_n^+\right) = 0.$$

Proof. By virtue of (2.1) and lemma 2.2, we have

$$\lim_{m\to\infty}\sum_{n=1}^{n_m}\left(\mu_n^+ - \lambda_n^+\right) = \lim_{m\to\infty}\sum_{n=1}^{n_m}\left(q_+(x)\psi_n^+, \psi_n^+\right)$$
$$= \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\int_0^{\infty}\frac{3}{\pi^2}\frac{q_k^+(x)\,\psi^+\left(x,\alpha_{m,k}^2\right)^2 dx}{\alpha_{m,k}^4\left(J_{\frac{2}{3}}\left(\frac{2}{3}\alpha_{m,k}^3\right) + J_{-\frac{2}{3}}\left(\frac{2}{3}\alpha_{m,k}^3\right)\right)^2}. \tag{2.6}$$

To calculate the inner sum in (2.6) investigate the asymptotic behavior of the partial sum

$$L_N(x) = \sum_{m=1}^{N}\frac{3}{\pi^2}\frac{\psi^+\left(x,\alpha_{m,k}^2\right)^2}{\alpha_{m,k}^4\left(J_{\frac{2}{3}}\left(\frac{2}{3}\alpha_{m,k}^3\right) + J_{-\frac{2}{3}}\left(\frac{2}{3}\alpha_{m,k}^3\right)\right)^2},$$

for each fixed k.

We will choose a function of complex variable which has poles at the points $\alpha_{m,k}$ and whose residues at this poles equal the terms of the considered sum.

By taking in equation (1.1) $\lambda = \alpha^2$ and $\lambda = \beta^2$, respectively, then multiplying the first of the obtained equations by $\psi\left(x,\beta^2\right)$ and the second one by $\psi\left(x,\alpha^2\right)$, subtracting the second equation from the first and integrating along interval (x,∞) as $\alpha \to \beta$ we get

$$\int_x^{\infty}\psi^+\left(x,\,\beta^2\right)^2 dx$$
$$= \frac{\pi^2}{3}\left(\beta^2 - x\right)^2\left[\left\{J_{\frac{1}{3}}\left(\frac{2}{3}\left(\beta^2 - x\right)^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3}\left(\beta^2 - x\right)^{\frac{3}{2}}\right)\right\}^2\right. \tag{2.7}$$
$$\left. + \left\{J_{\frac{2}{3}}\left(\frac{2}{3}\left(\beta^2 - x\right)^{\frac{3}{2}}\right) - J_{-\frac{2}{3}}\left(\frac{2}{3}\left(\beta^2 - x\right)^{\frac{3}{2}}\right)\right\}^2\right].$$

Denoting the right hand side of (2.7) by $F(x,\beta)$, we have

$$F'(\beta) = 2\beta\psi\left(x,\beta^2\right)^2. \tag{2.8}$$

Consider the function $g(z) = \frac{F(x,z)}{z\{J_{\frac{1}{3}}(\frac{2}{3}z^3) + J_{-\frac{1}{3}}(\frac{2}{3}z^3)\}^2}$.

It can be easily shown that all the zeros of $J_{\frac{1}{3}}\left(\frac{2}{3}z^3\right) + J_{-\frac{1}{3}}\left(\frac{2}{3}z^3\right)$ are simple. Thus $g(x)$ has poles of second order at the points $\alpha_{k,m}$. Denote $u^2(z) = J_{\frac{1}{3}}\left(\frac{2}{3}z^3\right) +$

$J_{-\frac{1}{3}}\left(\frac{3}{2}z^3\right)$. Write the expansion of $u^2(z)$ in powers of $(z-\alpha_{m,k})$:

$$zu^2(z) = \alpha_{m,k}\, u'(\alpha_{m,k})^2 (z-\alpha_{m,k})^2 + u'(\alpha_{m,k})\left(\alpha_{m,k}u''(\alpha_{m,k}) + u'(\alpha_{m,k})\right)(z-\alpha_{m,k})^3 + ... \tag{2.9}$$

By taking $\frac{3}{2}z^3 = v(z)$, we have

$$\alpha_{m,k}u''(\alpha_{m,k}) + u'(\alpha_{m,k}) = 2\alpha_{m,k}^2\left[2\alpha_{m,k}^2\left(J_{\frac{1}{3}}(v(z)) + J_{-\frac{1}{3}}(v(z))''\right)_{v=\frac{2}{3}\alpha_{m,k}^3} +3.\left(J_{\frac{1}{3}}(v(z)) + J_{-\frac{1}{3}}(v(z))\right)'_{v=\frac{2}{3}\alpha_{m,k}^3}\right].$$

Since both the functions $J_{\frac{1}{3}}(z)$ and $J_{\frac{-1}{3}}$ satisfy the Bessel equation $z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + \left(z^2 - \frac{1}{9}\right)y = 0$, then their sum also satisfies this equation. Thus, we get

$$\alpha_{m,k}u''(\alpha_{m,k}) + u'(\alpha_{m,k}) = 0, \tag{2.10}$$

in other words, the coefficient at $(z-\alpha_{m,k})$ vanishes.

So, using (2.8) - (2.10) we get

$$\underset{z=\alpha_{m,k}}{res}\, g(z) = \frac{\psi^+\left(x,\alpha_{m,k}^2\right)^2}{2\alpha_{m,k}^4\left(J_{\frac{2}{3}}\left(\frac{2}{3}\alpha_{m,k}^3\right) - J_{-\frac{2}{3}}\left(\frac{2}{3}\alpha_{m,k}^3\right)\right)^2}.$$

Take as a contour of integration a rectangular contour l_N with vertices at the points $\pm A_N$ and $\pm A_N + iB$, $A_N = \sqrt[3]{\frac{3\pi N}{2} + \frac{15N}{8}}$, which by-passes $\alpha_{m,k}$ along small semicircle above, $-\alpha_{m,k}$ and zero below the real axis. For large N

$$\alpha_{N,k} < A_N < \alpha_{N+1,k}.$$

On the right hand side of the contour in virtue of asymptotics $J_{\frac{1}{3}}(z) + J_{-\frac{1}{3}}(z) \sim e^{iz}$, $J_{\frac{2}{3}}(z) - J_{-\frac{2}{3}}(z) \sim \frac{e^{-iz}}{2z^2} + e^{-iz}$ as $z\to\infty$ and relation $\sqrt{z^2-x}^3 \sim z^3 - \frac{3}{2}xz$, by taking $B = A_N$ we have

$$\int_0^\infty q_k^+(x)\int_0^{A_N} A_N^3 e^{-\frac{3}{2}xv}dv dx = \int_0^\infty q_k^+(x)\left[A_N^3\frac{e^{-\frac{3}{2}xA_N}}{-\frac{3}{2}x} + \frac{2A_N^3}{3x}\right]dx \to \infty.$$

In virtue of condition 2) it follows that $\int_0^\infty \frac{q_k^+(x)}{x}A_N^3 dx = 0$ From condition 3)

$$\int_0^\infty \left|\frac{q_k^+(x)}{x}\right| A_N^3 e^{-\frac{3}{2}xA_N}dx$$

$$= \int_0^\infty \left|\frac{q_k^+(x)}{x}\right| A_N^3 \frac{1}{1 + \frac{3}{2}xA_N + \frac{\left(\frac{3}{2}xA_N\right)^2}{2} + \frac{\left(\frac{3}{2}xA_N\right)}{3!} + \frac{\left(\frac{3}{2}xA_N\right)^4}{4!} + ...}$$

$$< \int_0^\infty \frac{1}{A_N}\frac{|q_x^+(x)|}{x^5}dx \to 0$$

as $N \to \infty$.

On the side of the contour with the vertices at $\pm A_N + iB$,

$$\int_0^\infty q_k^+(x) \int_{-A_{N+iB}}^{A_{N+iB}} g(z)\,dzdx \sim \int_0^\infty q_k^+(x) \int_{-A_N}^{A_N} e^{-\frac{3}{2}xA_N} A_N^3 dv dx \to 0.$$

In a similar way one can show the convergence to zero of the integral along the left hand side of the contour. Since $g(z)$ is even, then the integral along a part of the contour located on the real axis also vanishes. Therefore,

$$\lim_{N\to\infty} \int_0^\infty q_k^+(x) \int_{l_N} g(z)\,dzdx = 0.$$

By Cauchy theorem about the contour integration

$$\sum_{m=1}^{\infty} \int_0^\infty \frac{\psi\left(\alpha_{m,k}^2, x\right)^2 q_k^+(x)\,dx}{\alpha_{m,k}^4 \left(J_{\frac{1}{3}}\left(\frac{2}{3}\alpha_{m,k}^3\right) + J_{-\frac{1}{3}}\left(\frac{2}{3}\alpha_{m,k}^3\right)\right)}$$
$$= \lim_{N\to\infty} \int_0^\infty q_k^+(x) \int_{l_N} g(z)\,dzdx = 0.$$

The theorem is proved. □

Now we will find the eigenvalue asymptotics and the trace of the operator L_2.

We have the following spectral problem

$$-y_k''(x) - xy_k(x) + \gamma_k y_k(x) = \lambda y_k(x), \quad x \in (-\infty, 0), \tag{2.11}$$

$$y_k'(0) = 0. \tag{2.12}$$

The solution of (2.11) from $L_2(-\infty, 0)$ is

$$\psi^-(x,\lambda) = \sqrt{x+\lambda-\gamma_k} H_{\frac{1}{3}}\left(\frac{2}{3}(x+\lambda-\gamma_k)^{\frac{3}{2}}\right).$$

As $x \to -\infty$ it has the asymptotics $e^{\frac{2i}{3}\sqrt{x+\lambda-\gamma_k}}$. So, when taking the branch $i\sqrt{-x-\lambda+\gamma_k}$ of $\sqrt{x+\lambda-\gamma_k}$ it is easy to see that $\psi^-(x,\lambda) \in L_2(-\infty, 0)$.

We have

$$\begin{aligned} \psi^-(x,\lambda) &= \sqrt{x+\lambda-\gamma_k} \frac{2}{\pi i e^{\frac{\pi i}{6}}} K_{\frac{1}{3}}\left(-i\frac{2}{3}(x+\lambda-\gamma_k)^{\frac{3}{2}}\right) \\ &= \frac{\pi i}{\sqrt{3}} \left\{ J_{\frac{1}{3}}\left(-\frac{2}{3}(x+\lambda-\gamma_k)^{\frac{3}{2}}\right) \right. \\ &\left. + J_{-\frac{1}{3}}\left(-\frac{2}{3}(x+\lambda-\gamma_k)\right)^{\frac{3}{2}} \right\} \sqrt{x+\lambda-\gamma_k}. \end{aligned} \tag{2.13}$$

Substituting (2.13) into (2.12), we get the equation

$$(\lambda-\gamma_k)\left\{ J_{\frac{2}{3}}\left(-\frac{2}{3}(\lambda-\gamma_k)^{\frac{3}{2}}\right) - J_{-\frac{2}{3}}\left(-\frac{2}{3}(\lambda-\gamma_k)^{\frac{3}{2}}\right)\right\} = 0.$$

Denoting $z=\sqrt{\lambda-\gamma_k}$ in virtue of oddness of $\left\{J_{\frac{2}{3}}\left(z^3\right)-J_{-\frac{2}{3}}\left(z^3\right)\right\}$ we have

$$z^2\left\{J_{\frac{2}{3}}\left(\frac{2}{3}z^3\right)-J_{-\frac{2}{3}}\left(\frac{2}{3}z^3\right)\right\}=0. \tag{2.14}$$

From asymptotics $J_\nu\left(z\right)=\sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\left(1+O\left(\frac{1}{2}\right)\right)$ we get the equivalent equation $\sin(\frac{2}{3}z^3-\frac{\pi}{4})(1+O(\frac{1}{z}))=0$ from which

$$z=\sqrt[3]{\frac{3\pi m}{2}+\frac{3\pi}{8}+O\left(\frac{1}{m}\right)}=\left(\frac{3\pi m}{2}\right)^{\frac{1}{3}}+O\left(\frac{1}{m^{\frac{1}{3}}}\right).$$

Therefore, the eigenvalues of L_2^0 are

$$\lambda_{m,k}^-=\gamma_k+\beta_m^2,\ \ \beta_m=cm^{\frac{1}{3}}+O\left(\frac{1}{m^{\frac{1}{3}}}\right). \tag{2.15}$$

Consider the contour c_N with the vertices at the points $\pm A_N\pm iB$, $A_N=\sqrt[3]{\frac{3\pi N}{2}+\frac{9\pi}{8}}$, $B=CA_N$.

By Rouché theorem it is easy to show that the number of the roots of equations (2.14) inside this contour is $2N$.

Denote the eigenvalues of L_2^- by μ_n^-. In virtue of the above statements and (2.15) one can in a similar way as in proof of Theorem 1.1 get the following asymptotic formula

$$\mu_n^-\sim\lambda_n^-\sim d_2 n^{\frac{2\alpha}{2+3\alpha}}. \tag{2.16}$$

Denote $q_-(x)=q(x)$, $x<0$.

Let $q\left(x\right)$ satisfy conditions 1)-3), and $q_- {L_2^0}^\delta$ is continued to the bounded operator for $\delta>1-\frac{2\alpha}{2+3\alpha}$. Then, in virtue of theorem 2.1 there is a subsequence $\{k_m\}$ of natural numbers such that

$$\lim_{m\to\infty}\sum_{n=1}^{k_m}\left(\mu_n^--\lambda_n^--\left(q_-\psi_n^-,\psi_n^-\right)\right)=0, \tag{2.17}$$

where $\{\psi_n^-\}$ is a orthonormal basis formed by the eigenvectors of the operator L_2^0.

We have

$$\psi_{m,k}^-=\frac{\sqrt{3}\psi^-\left(x,\beta_{m,k}^2\right)\varphi_k}{\pi\beta_{m,k}^2\left(J_{\frac{1}{3}}\left(\frac{2}{3}\beta_m^3\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}\beta_m^3\right)\right)}. \tag{2.18}$$

By using similar arguments as in the proof of lemma 2.1 one can prove the following lemma.

Lemma 2.4 *If operator-valued function $q\left(x\right)$ satisfies condition* 1) *then*

$$\frac{3}{\pi^2}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\left|\int_\infty^0\frac{q_k^-\left(x\right)\psi^-\left(x,\beta_{m,k}^2\right)^2dt}{\beta_{m,k}^4\left(J_{\frac{1}{3}}\left(\frac{2}{3}\beta_{m,k}^3\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}\beta_{m,k}^3\right)\right)^2}\right|<\infty.$$

Theorem 2.5 *If the operator-valued function* $q(x)$ *satisfies conditions* $1)-3)$, *then*

$$\lim_{m\to\infty}\sum_{n=1}^{k_m}\left(\mu_n^- - \lambda_n^-\right)=0.$$

Proof. In virtue of relation (2.18) and lemma 2.4 we have

$$\lim_{m\to\infty}\sum_{n=1}^{k_m}\left(\mu_n^- - \lambda_n^-\right) = \frac{3}{\pi^2}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\int_{-\infty}^{0}\frac{q_k^-(x)\,\psi^-\left(\beta_{m,k}^2,x\right)^2dx}{\beta_{m,k}^4\left(J_{\frac13}\left(\frac23\beta_{m,k}^3\right)+J_{-\frac13}\left(\frac23\beta_{m,k}^3\right)\right)^2}. \tag{2.19}$$

Denote

$$M_N(x)=\sum_{m=1}^{N}\frac{3}{\pi^2}\frac{\psi^-\left(\beta_{m,k}^2,x\right)^2}{\beta_{m,k}^4\left(J_{\frac13}\left(\frac23\beta_{m,k}^3\right)+J_{-\frac13}\left(\frac23\beta_{m,k}^3\right)\right)}. \tag{2.20}$$

It can be shown that

$$\int_{-\infty}^{x}\psi^-(x,z^z)^2\,dz = \frac{\pi^2}{3}(z^2+x)\left[\left\{J_{\frac13}(-\frac23(z^2+x)^{\frac23})+J_{-\frac13}(-\frac23(z^2+x)^{\frac23})\right\}^2 + \left\{J_{\frac23}(-\frac23(z^2+x)^{\frac23})-J_{-\frac23}(-\frac23(z^2+x)^{\frac23})\right\}^2\right]. \tag{2.21}$$

Let $z^2+x=\sigma(x,z)$.

Denoting the right hand side of (2.21) by f, we have $f'_x=f'_\sigma,\ f'_z=f'_\sigma\cdot 2z=f'_x\cdot 2z$. From (2.21) it follows that $f'_x=\psi^-\left(x,t^2\right)^2$, so $f'_z=2z\,\psi^-\left(x,z^2\right)^2$.

For evaluating (2.19), we will choose the function

$$g(z)=\frac{\pi^2}{3}\frac{(z^2+x)\left[\left\{J_{\frac13}\left(-\frac23\left(x+z^2\right)^{\frac32}\right)+J_{-\frac13}\left(-\frac23\left(x+z^2\right)^{\frac32}\right)\right\}^2\right.}{2z\left(J_{\frac23}\left(\frac23 z^3\right)-J_{-\frac13}\left(\frac23 z^3\right)\right)^2} + \frac{\left.\left\{J_{\frac23}\left(-\frac23\left(z^2+x\right)^{\frac32}\right)-J_{-\frac23}\left(-\frac23\left(z^2+x\right)^{\frac32}\right)\right\}^2\right]}{2z\left(J_{\frac23}\left(\frac23 z^3\right)-J_{-\frac13}\left(\frac23 z^3\right)\right)^2}$$

which has poles of second order at the points $\beta_{m,k}$ and the residues at the poles equal the terms of sum (2.20). Further calculations are conducted in a way similar to one used in the proof of Theorem 2.3. Denote the eigenvalue distribution function of the operator L_0 by $N(\lambda, L_0)$. Then $N(\lambda, L_0) = \sum_{\lambda_n^+<\lambda} 1 + \sum_{\lambda_n^-<\lambda} 1 \sim N(\lambda, L_1^0) + N(\lambda, L_2^0) \sim cn^{\frac{2+3\alpha}{2\alpha}}$, $c = c_1 + c_2$, from which we obtain that the eigenvalues of the operator L denoted by $\mu_n(\{\mu_n\} = \{\mu_n^+\} \cup \{\mu_n^-\})$ have the asymptotics $\mu_n \sim dn^{\frac{2\alpha}{2+3\alpha}}$. □

Call $\lim\limits_{m\to\infty} \sum_{n=1}^{n_m}(\mu_m^+ - \lambda_n^+) + \sum_{n=1}^{n_m}(\mu_m^- - \lambda_n^-)$ a regularized trace of the operator L and denote it by $\sum_{n=1}^{\infty}{}'(\mu_n - \lambda_n)$.

From Theorems 2.3, 2.5 we get the following statement.

Theorem 2.6 *Under conditions* $1) - 3)$ $\sum_{n=1}^{\infty}(\mu_n - \lambda_n) = 0$.

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