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Nonuniform exponential dichotomy for noninvertible evolution operators in Banach Spaces

Mihail Megan · Mihai Gabriel Babuţia

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Abstract The aim of this paper is to study two concepts of nonuniform exponential dichotomy and their correspondents for the case of uniform exponential dichotomy on the half-line in the general framework of evolution operators in Banach spaces. Characterizations of these concepts are obtained from the point of view of the families of projections. Some illustrative examples and counterexamples are given in order to prove that these dichotomy concepts are distinct. The first concept of exponential dichotomy is defined in the general case when the family of projections is only invariant for the evolution operator and hence the evolution operator is not invertible on the unstable manifold. Last but not least, an example which motivates the use of dichotomy with invariant families of projections for noninvertible evolution operators is presented.

Keywords Evolution operator · Nonuniform exponential dichotomy · Uniform exponential dichotomy · Invariant family of projections · Strongly invariant family of projections

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1 Introduction

In the qualitative theory of dynamical systems the study of linear systems is very important since a comprehensive analysis of nonlinear systems via perturbation techniques requires linear theory. This is due to the fact that in many cases the dichotomy properties of the solutions can be derived from the linearization along the solutions, the so-called variational equation.

Mihail Megan
Academy of Romanian Scientists
54 Independenţei St.
050094 Bucharest, Romania
E-mail: megan@math.uvt.ro

Mihai Gabriel Babuţia
Department of Mathematics
Faculty of Mathematics and Computer Science
West University of Timişoara
Timişoara, Romania
E-mail: mihai.babutia@gmail.com, mbabutia@math.uvt.ro

The notion of exponential dichotomy was introduced by PERRON in [21] and it plays a central role in a large part of the theory of dynamical systems. The exponential dichotomy property for linear dynamical systems has gained prominence since the appearance of two fundamental monographs of MASSERA and SCHÄFFER [14], DALECKII and KREIN [12]. These were followed by the important books of CHICONE and LATUSHKIN [11] and BARREIRA and VALLS [5]. The most important dichotomy concept used in the qualitative theory of autonomous ordinary differential equations is the uniform exponential dichotomy, fact that can be seen, for example, from the works of MEGAN [15], MEGAN and STOICA [19], VAN MINH, RÄBIGER, SCHNAUBELT [20] and THIEN-HUY [13]. In some situations, particularly in the nonautonomous setting, the concept of uniform exponential dichotomy is too restrictive and it is important to consider more general behaviors.

Two different perspectives can be identified when generalizing the uniform exponential dichotomy: on the one hand one can define dichotomies whose bounds depend on the initial time (and therefore are nonuniform) and on the other hand one can consider growth rates that are not necessarily exponential, in particular polynomial growth rates.

The first approach leads to concepts of nonuniform exponential dichotomies and can be found in the works of BARREIRA and VALLS ([6], [7], [9]), MEGAN, SASU and SASU ([17], [18]), PREDA and MEGAN ([23], [24]), SASU, BABUŢIA and SASU ([27]), POPA, MEGAN and CEAUŞU ([22]). We remark that the nonuniformities considered in this paper are of Barreira-Valls type ([5]-[9]).

The second approach is present in the works of BARREIRA and VALLS ([8]), BENTO and SILVA ([10]), RĂMNEANŢU, CEAUŞU and MEGAN ([25]), BABUŢIA, MEGAN, POPA ([4]).

One of the main reasons for weakening the assumption of uniform exponential behavior is that from the point of view of ergodic theory, almost all linear variational equations with nonzero Lyapunov exponents in a finite-dimensional space admit a nonuniform exponential dichotomy.

Several interesting results were also obtained in the papers [1], [2], [3], [26] and [28].

In the definitions of exponential dichotomies of an evolution operator U with respect to a family of projections P, this family of projections can be of two types: invariant for U (i.e. $U(t,s)P(s) = P(t)U(t,s)$) and strongly invariant for U (in addition $U(t,s)$ is an isomorphism from $Ker\ P(s)$ to $Ker\ P(t)$). We remark that in this paper we assume the existence of a family of projections P which is invariant for a given evolution operator U. At a first view the existence of such a family is a strong hypothesis. The impediment can be eliminated by using the notion of admissibility (see [9], [18], [27]). We also study the problem of uniqueness of P. These two types of families of projections are distinct even in the finite dimensional case (see Example 5.1).

Based on this fact, the purpose of this paper is to study the dichotomy concepts with respect to invariant families of projections (exponential dichotomy and uniform exponential dichotomy) and with respect to strongly invariant families of projections (strong exponential dichotomy and uniform strong exponential dichotomy). We give characterizations of these concepts and present connections (implications and counterexamples) between them.

Last but not least, we stress out that we consider evolution operators which are not not necessarily invertible (in particular evolution operators which are not generated by differential equations) and moreover, the time variable is considered on the half-line.

In addition, we present an example of dichotomic evolution operator U with invariant familiy of projections P for which P not strongly invariant for U (see Example 5.2) hence the study of dichotomy for noninvertible evolution operators is of interest.

2 Definitions, notations and preliminary results

Let X be a real or complex Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X. The norms on X and $\mathcal{B}(X)$ will be denoted by $\|\cdot\|$. Denote by I the identity operator on X.

If $A \in \mathcal{B}(X)$ then we shall denote by $Ker\ A$ and by $Range\ A$ respectively the kernel and range of A i.e.

$$Ker\ A = \{x \in X\ :\ Ax = 0\} \text{ and } Range\ A = \{Ax\ :\ x \in X\}.$$

We also denote by Δ the set of all pairs of real nonnegative numbers (t, s) with $t \geq s$ i.e. $\Delta = \{(t, s) \in \mathbb{R}_+^2 : t \geq s \geq 0\}$ and by T the set defined by $T = \{(t, s, t_0) \in \mathbb{R}_+^3 : t \geq s \geq t_0 \geq 0\}$.

Definition 2.1 *A map $U : \Delta \to \mathcal{B}(X)$ is called an evolution operator on X if*

(e_1) *$U(t, t) = I$, for every $t \geq 0$;*
(e_2) *$U(t, s)U(s, t_0) = U(t, t_0)$, for all $(t, s, t_0) \in T$.*

In addition,

(e_3) *if for all $(t, s) \in \Delta$ the linear operator $U(t, s)$ is bijective then we say that the evolution operator is reversible;*
(e_4) *if there are $M \geq 1$, $\varepsilon \geq 0$ and $\omega > 0$ such that $\|U(t, s)\| \leq Me^{\varepsilon s}e^{\omega(t-s)}$, for all $(t, s) \in \Delta$ then we say that the evolution operator U has exponential growth (e.g). In the particular case when $\varepsilon = 0$, we say that the evolution operator U has uniform exponential growth (u.e.g).*

Definition 2.2 *A map $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is called a family of projections on X if $P(t)^2 = P(t)$, for every $t \geq 0$. In addition, if there are $M \geq 1$ and $\gamma \geq 0$ such that $\|P(t)\| \leq Me^{\gamma t}$, for all $t \geq 0$, then we say that the family $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is exponentially bounded (e.b). In the particular case when $\gamma = 0$, P is called bounded.*

Remark 2.1 If $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is a family of projections on X then $Q : \mathbb{R}_+ \to \mathcal{B}(X)$ defined by $Q(t) = I - P(t)$ is also a family of projections on X, which is called the *complementary family of projections* of P.

We observe that $Ker\ P(t) = Range\ Q(t)$, $Range\ P(t) = Ker\ Q(t)$ and $P(t)Q(t) = Q(t)P(t) = 0$, for all $t \geq 0$.

Lemma 2.3 *Let $P_1, P_2 : \mathbb{R}_+ \to \mathcal{B}(X)$ be two families of projections on X and let Q_1 and Q_2 be their complementary families of projections respectively. The following assertions are equivalent:*

(r_1) *$Range\ P_1(t) = Range\ P_2(t)$, for all $t \geq 0$;*
(r_2) *$P_1(t)P_2(t) = P_2(t)$ and $P_2(t)P_1(t) = P_1(t)$, for all $t \geq 0$;*
(r_3) *$Q_1(t)Q_2(t) = Q_1(t)$ and $Q_2(t)Q_1(t) = Q_2(t)$, for all $t \geq 0$.*

Proof. $(r_1) \Rightarrow (r_2)$. Let $t \geq 0$. From (r_1) it follows that for every $x \in X$ there are $x_1, x_2 \in X$ with $P_1(t)x_1 = P_2(t)x$ and $P_2(t)x_2 = P_1(t)x$. Then $P_1(t)P_2(t)x = P_1^2(t)x_1 = P_1(t)x_1 = P_2(t)x$ and $P_2(t)P_1(t)x = P_2^2(t)x_2 = P_2(t)x_2 = P_1(t)x$. $(r_2) \Rightarrow (r_3)$. For every $t \geq 0$ we have that $Q_1(t)Q_2(t) = I - P_2(t) - P_1(t) + P_2(t) = Q_1(t)$ and $Q_2(t)Q_1(t) = I - P_2(t) - P_2(t) + P_1(t) = Q_2(t)$. $(r_3) \Rightarrow (r_1)$. Let $t \geq 0$. If $Q_1(t)Q_2(t) = Q_1(t)$ then $P_1(t)P_2(t) = P_2(t)$ which implies that *Range* $P_2(t) \subset$ *Range* $P_2(t)$. Similarly, from $Q_2(t)Q_1(t) = Q_2(t)$ it results that $P_1(t)P_2(t) = P_2(t)$ and hence *Range* $P_2(t) \subset$ *Range* $P_1(t)$. □

Definition 2.4 *A family of projections $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is said to be invariant for the evolution operator $U : \Delta \to \mathcal{B}(X)$ if $U(t,s)P(s) = P(t)U(t,s)$, for all $(t,s) \in \Delta$.*

Remark 2.2 If the family of projections P is invariant for the evolution operator U then we also have that the complementary family of projections Q is invariant for the evolution operator U.

Characterizations of the invariance property of a family of projections for an evolution operator are given by

Proposition 2.5 *Let $P : \mathbb{R}_+ \to \mathcal{B}(X)$ be a family of projections on X and let $U : \Delta \to \mathcal{B}(X)$ be an evolution operator on X. The following statements are equivalent:*

(c_1) *P is invariant for U;*
(c_2) *$P(t)U(t,s)Q(s) = Q(t)U(t,s)P(s) = 0$, for all $(t,s) \in \Delta$;*
(c_3) *$U(t,s)\,(Ker\ P(s)) \subset Ker\ P(t)$ and $U(t,s)\,(Range\ P(s)) \subset Range\ P(t)$, for all $(t,s) \in \Delta$.*

Proof. $(c_1) \Rightarrow (c_2)$. If P is invariant for U then $P(t)U(t,s)Q(s) = P(t)Q(t)U(t,s) = 0$ and $Q(t)U(t,s)P(s) = Q(t)P(t)U(t,s) = 0$. $(c_2) \Rightarrow (c_3)$. By (c_2) it results that $U(t,s)\,(Ker\ P(s)) = U(t,s)\,(Range\ Q(s)) \subset Ker\ P(t)$ and $U(t,s)\,(Range\ P(s)) \subset Ker\ Q(t) = Range\ P(t)$, for all $(t,s) \in \Delta$.

$(c_3) \Rightarrow (c_1)$. We observe that for every $x \in X$ we have that $x - P(s)x \in Ker\ P(s)$ and $U(t,s)P(s)x \in Range\ P(t)$. Then from (c_3) it follows that $U(t,s)x - U(t,s)P(s)x \in Ker\ P(t)$ and hence

$$P(t)U(t,s)x = P(t)U(t,s)P(s)x. \tag{2.1}$$

Because $U(t,s)P(s)x \in Range\ P(t)$ it follows that

$$U(t,s)P(s)x = P(t)U(t,s)P(s)x. \tag{2.2}$$

The equalities from (2.1) and (2.2) prove that P is invariant for U. □

Definition 2.6 *Let $P : \mathbb{R}_+ \to \mathcal{B}(X)$ be a family of projections which is invariant for the evolution operator $U : \Delta \to \mathcal{B}(X)$. We say that P is strongly invariant for U if for every $(t,s) \in \Delta$ the linear operator $U(t,s)$ is an isomorphism from $Ker\ P(s)$ to $Ker\ P(t)$.*

Remark 2.3 From the preceding definition it follows that if P is strongly invariant for U then P is invariant for U. The converse implication is not valid, phenomenon illustrated by Example 5.1.

Proposition 2.7 *Let $P : \mathbb{R}_+ \to \mathcal{B}(X)$ be a family of projections which is invariant for the evolution operator $U : \Delta \to \mathcal{B}(X)$. If, for all $(t, s) \in \Delta$ the linear operator $U(t, s)$ is injective on $Ker\ P(s)$ then P is strongly invariant for U if and only if $Ker\ P(t) \subset Range\ U(t, s)$, for all $(t, s) \in \Delta$.*

Proof. Necessity. If P is strongly invariant for U and $y \in Ker\ P(t)$ then there is $x \in Ker\ P(s)$ with $y = U(t, s)x$ and hence $y \in Range\ U(t, s)$.

Sufficiency. We will prove that for every $y \in Ker\ P(t)$ there is $x \in Ker\ P(s)$ with $y = U(t, s)x$.

Let $y \in Ker\ P(t)$. Then $y \in Range\ Q(t)$ and hence $y = Q(t)y$. Moreover, from the hypothesis, there is $x_0 \in X$ such that $y = U(t, s)x_0$. Then we have that $y = Q(t)y = Q(t)U(t, s)x_0 = U(t, s)Q(s)x_0 = U(t, s)x$, where $x = Q(s)x_0 \in Ker\ P(s)$. □

Corollary 2.8 *If the family of projections $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is invariant for the reversible evolution operator $U : \Delta \to \mathcal{B}(X)$ then it is also strongly invariant for U.*

Proof. If P is invariant for the reversible evolution operator U and $y \in Ker\ P(t)$ then $y = U(t, s)U(t, s)^{-1}y \in Range\ U(t, s)$, for all $(t, s) \in \Delta$. By the preceding proposition it results that P is strongly invariant for U. □

Remark 2.4 If the family of projections P is strongly invariant for the evolution operator $U : \Delta \to \mathcal{B}(X)$ then there exists $V : \Delta \to \mathcal{B}(X)$ such that $V(t, s)$ is an isomorphism from $Ker\ P(t)$ to $Ker\ P(s)$ and

(v_1) $U(t, s)V(t, s)Q(t) = Q(t)$;
(v_2) $V(t, s)U(t, s)Q(s) = Q(s)$,

for all $(t, s) \in \Delta$. The map V is called the *skew-evolution operator* associated to the pair (U, P).

Remark 2.5 If the family of projections P is invariant for the reversible evolution operator U then the skew-evolution operator associated to the pair (U, P) is $V(t, s) = U(t, s)^{-1}$, for all $(t, s) \in \Delta$.

Proposition 2.9 *If the family of projections $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is strongly invariant for the evolution operator $U : \Delta \to \mathcal{B}(X)$ then the skew-evolution operator associated to the pair (U, P) has the following properties:*

(v_3) *$V(t, s)Q(t) = Q(s)V(t, s)Q(t)$;*
(v_4) *$Q(t) = V(t, t)Q(t) = Q(t)V(t, t)Q(t)$;*
(v_5) *$V(t, t_0)Q(t) = V(s, t_0)V(t, s)Q(t)$,*

for all $(t, s, t_0) \in T$

Proof. (v_3) We observe that, for all $(t, s, x) \in \Delta \times X$, we have that $Q(t)x \in Range\ Q(t) = Ker\ P(t)$ hence $V(t, s)Q(t)x \in Ker\ P(s) = Range\ Q(s)$ which implies $Q(s)V(t, s)Q(t)x = V(t, s)Q(t)x$.

(v_4) By (v_1) and (v_3) we have that for every $t \geq 0$, $Q(t) = V(t, t)Q(t) = Q(t)V(t, t)Q(t)$.

(v_5) If $(t, s, t_0) \in T$ then we have that

$$\begin{aligned}
V(t,t_0)Q(t) &\overset{(v_3)}{=} Q(t_0)V(t,t_0)Q(t) \\
&\overset{(v_2)}{=} V(s,t_0)U(s,t_0)Q(t_0)V(t,t_0)Q(t) \\
&= V(s,t_0)Q(s)U(s,t_0)Q(t_0)V(t,t_0)Q(t) \\
&\overset{(v_2)}{=} V(s,t_0)V(t,s)U(t,s)Q(s)U(s,t_0)Q(t_0) \\
&= V(s,t_0)V(t,s)U(t,t_0)Q(t_0)V(t,t_0)Q(t) \\
&\overset{(v_3)}{=} V(s,t_0)V(t,s)U(t,t_0)V(t,t_0)Q(t) \\
&\overset{(v_1)}{=} V(s,t_0)V(t,s)Q(t).
\end{aligned}$$

□

3 Exponential dichotomy with invariant families of projections

In this section we investigate two concepts of exponential dichotomy of an evolution operator $U : \Delta \to \mathcal{B}(X)$ with respect to a family of projections $P : \mathbb{R}_+ \to \mathcal{B}(X)$ invariant for U.

Definition 3.1 *Let $P : \mathbb{R}_+ \to \mathcal{B}(X)$ be a family of projections, invariant for the evolution operator $U : \Delta \to \mathcal{B}(X)$. We say that the evolution operator U admits an exponential dichotomy (or U is exponentially dichotomic) with respect to the family of projections P, and denote P-e.d, if there exist constants $N \geq 1$, $\alpha \geq 0$ and $\beta > 0$ such that*

(ed_1) $\|U(t,s)P(s)x\| \leq Ne^{\alpha s}e^{-\beta(t-s)}\|P(s)x\|$;
(ed_2) $\|Q(s)x\| \leq Ne^{\alpha t}e^{-\beta(t-s)}\|U(t,s)Q(s)x\|$,

for all $(t, s, x) \in \Delta \times X$, where Q is the complementary family of projections of P.

In the particular case in which $\alpha = 0$, the evolution operator is called uniformly exponentially dichotomic with respect to P and denote P-u.e.d.

If there exists a family of projections P such that U is P-e.d (P-u.e.d respectively) then U is called exponentially dichotomic and denote e.d (respectively uniformly exponentially dichotomic and denote u.e.d).

Remark 3.1 The above defined concept of uniform exponential dichotomy is studied for example in [14], [24], [15], [19], [20] and its correspondent in the nonuniform case was studied in [14,19,15,17,18,22,23,25] and [27].

Remark 3.2 If the evolution operator is u.e.d then it is e.d. The converse is not valid, fact illustrated by Example 5.3.

A simple characterization of the exponential dichotomy is given by

Proposition 3.2 *The evolution operator $U : \Delta \to \mathcal{B}(X)$ is exponentially dichotomic if and only if there are a family of projections $P : \mathbb{R}_+ \to \mathcal{B}(X)$ invariant for U and constants $N \geq 1$, $\alpha \geq 0$ and $\beta > 0$ such that*

(ed_1') $\|U(t,t_0)P(t_0)x_0\| \le Ne^{\alpha s}e^{-\beta(t-s)}\|U(s,t_0)P(t_0)x_0\|$;
(ed_2') $\|U(s,t_0)Q(t_0)x_0\| \le Ne^{\alpha t}e^{-\beta(t-s)}\|U(t,t_0)Q(t_0)x_0\|$,
for all $(t,s,t_0,x_0) \in T \times X$.

Proof. Necessity. If U is e.d then by Definition 3.1 we have that

$$\begin{aligned}\|U(t,t_0)P(t_0)x_0\| &= \|U(t,s)P(s)U(s,t_0)P(t_0)x_0\| \\ &\le Ne^{\alpha s}e^{-\beta(t-s)}\|U(s,t_0)P(t_0)x_0\|\end{aligned}$$

and

$$\begin{aligned}\|U(s,t_0)Q(t_0)x_0\| &= \|Q(s)U(s,t_0)Q(t_0)x_0\| \\ &\le Ne^{\alpha t}e^{-\beta(t-s)}\|U(t,t_0)Q(t_0)x_0\|,\end{aligned}$$

for all $(t,s,t_0,x_0) \in T \times X$.
Sufficiency. It is obvious, taking $s = t_0$ both in $(ed1')$ and $(ed2')$. □

As a particular case we have

Corollary 3.3 *The evolution operator* $U : \Delta \to \mathcal{B}(X)$ *is uniformly exponentially dichotomic if and only if there are a family of projections* $P : \mathbb{R}_+ \to \mathcal{B}(X)$ *invariant for* U *and constants* $N \ge 1$ *and* $\beta > 0$ *such that*

(ued_1') $\|U(t,t_0)P(t_0)x_0\| \le Ne^{-\beta(t-s)}\|U(s,t_0)P(t_0)x_0\|$;
(ued_2') $\|U(s,t_0)Q(t_0)x_0\| \le Ne^{-\beta(t-s)}\|U(t,t_0)Q(t_0)x_0\|$,
for all $(t,s,t_0,x_0) \in T \times X$.

The particular case of reversible evolution operators is considered in

Proposition 3.4 *The reversible evolution operator* $U : \Delta \to \mathcal{B}(X)$ *is exponentially dichotomic if and only if there exist a family of projections* $P : \mathbb{R}_+ \to \mathcal{B}(X)$ *invariant for* U *and constants* $N \ge 1$, $\alpha \ge 0$ *and* $\beta > 0$ *such that*

(red_1) $\|U(t,s)P(s)x\| \le Ne^{\alpha s}e^{-\beta(t-s)}\|P(s)x\|$;
(red_2) $\|U(t,s)^{-1}Q(t)x\| \le Ne^{\alpha t}e^{-\beta(t-s)}\|Q(t)x\|$,
for all $(t,s,x) \in \Delta \times X$.

Proof. It is sufficient to prove the equivalence $(ed_2) \Leftrightarrow (red_2)$.
Assume that (ed_2) holds. Then, for all $(t,s,x) \in \Delta \times X$ we have

$$\begin{aligned}\|U(t,s)^{-1}Q(t)x\| &= \|Q(s)U(t,s)^{-1}Q(t)x\| \\ &\le Ne^{\alpha t}e^{-\beta(t-s)}\|U(t,s)Q(s)U(t,s)^{-1}Q(t)x\| \\ &= Ne^{\alpha t}e^{-\beta(t-s)}\|Q(t)x\|.\end{aligned}$$

Conversely, from (red_2) it results that

$$\begin{aligned}\|Q(s)x\| &= \|U(t,s)^{-1}Q(t)U(t,s)Q(s)x\| \\ &\le Ne^{\alpha t}e^{-\beta(t-s)}\|Q(t)U(t,s)Q(s)x\| \\ &= Ne^{\alpha t}e^{-\beta(t-s)}\|U(t,s)Q(s)x\|,\end{aligned}$$

for all $(t,s,x) \in \Delta \times X$. □

It is natural to ask whether the family of projections P from Definition 3.1 is unique. For this purpose, we consider two families of projections $P_1, P_2 : \mathbb{R}_+ \to \mathcal{B}(X)$ which are invariant for the evolution operator $U : \Delta \to \mathcal{B}(X)$. Now, we prove

Theorem 3.5 *Let $P_1, P_2 : \mathbb{R}_+ \to \mathcal{B}(X)$ be two exponentially bounded families of projections which are invariant for the evolution operator $U : \Delta \to \mathcal{B}(X)$ such that*

$$\textit{Range } P_1(t) = \textit{Range } P_2(t), \quad \textit{for all } t \geq 0.$$

If U admits an exponential dichotomy with respect to the family of projections P_1 then it also admits an exponential dichotomy with respect to the family of projections P_2.

Proof. Let $M \geq 1$ and $\gamma \geq 0$ be two constants such that $\|P_1(t)\| + \|P_2(t)\| \leq Me^{\gamma t}$, for all $t \geq 0$. Let $N \geq 1$, $\alpha \geq 0$ and $\beta > 0$ as in Definition 3.1. Define $N_1 = 4M^2N$ and $\alpha_1 = \alpha + 2\gamma$. If U is e.d with respect to P_1 then by Definition 3.1 and Lemma 2.3 we obtain that

$$\begin{aligned}\|U(t,s)P_2(s)x\| &= \|U(t,s)P_1(s)P_2(s)x\| \leq Ne^{\alpha s}e^{-\beta(t-s)}\|P_1(s)P_2(s)x\| \\ &= MNe^{(\alpha+\gamma)s}e^{-\beta(t-s)}\|P_2(s)x\| \leq N_1e^{\alpha_1 s}e^{-\beta(t-s)}\|P_2(s)x\|,\end{aligned}$$

for all $(t,s,x) \in \Delta \times X$.

Similarly, if we denote by $Q_1(t) = I - P_1(t)$ and $Q_2(t) = I - P_2(t)$, for all $t \geq 0$, then

$$\begin{aligned}\|Q_2(s)x\| &= \|Q_2(s)Q_1(s)x\| \leq \|Q_2(s)\| \cdot \|Q_1(s)x\| \\ &\leq Me^{\gamma s}Ne^{\alpha t}e^{-\beta(t-s)}\|U(t,s)Q_1(s)x\| \\ &\leq MNe^{(\alpha+\gamma)t}e^{-\beta(t-s)}\|U(t,s)Q_1(s)Q_2(s)x\| \\ &= MNe^{(\alpha+\gamma)t}e^{-\beta(t-s)}\|Q_1(t)U(t,s)Q_2(s)x\| \\ &\leq N_1e^{\alpha_1 t}e^{-\beta(t-s)}\|U(t,s)Q_2(s)x\|,\end{aligned}$$

for all $(t,s,x) \in \Delta \times X$. Finally, it follows that U admits an exponential dichotomy with the family of projections P_2. □

Concerning the range of the dichotomy projections, we give the following:

Proposition 3.6 *If the evolution operator U admits a P-exponential dichotomy, with the dichotomy constants N, α, β such that $\alpha \in [0, \beta]$ then*

$$\textit{Range } P(s) = \left\{ x \in X \ : \ \lim_{t\to\infty} U(t,s)x = 0 \right\}.$$

Proof. Let $s \geq 0$. If $x \in \textit{Range } P(s)$ then $P(s)x = x$ and from (ed_1) we obtain that $\|U(t,s)x\| = \|U(t,s)P(s)x\| \leq Ne^{\alpha s}e^{-\beta(t-s)}\|P(s)x\|$, which implies that $\lim_{t\to\infty} U(t,s)x = 0$.

Conversely let $x \in X$ such that $\lim_{t\to\infty} U(t,s)x = 0$. By (ed_2), (ed_1) we have that

$$\begin{aligned}\|Q(s)x\| &\leq Ne^{\alpha t}e^{-\beta(t-s)}\|U(t,s)Q(s)x\| \\ &\leq Ne^{(\alpha-\beta)t}e^{\beta s}\left(\|U(t,s)x\| + \|U(t,s)P(s)x\|\right) \\ &\leq Ne^{(\alpha-\beta)t}e^{\beta s}\left(\|U(t,s)x\| + Ne^{\alpha s}e^{-\beta(t-s)}\|P(s)x\|\right)\end{aligned}$$

and for $t \to \infty$ it results that $Q(s)x = 0$ i.e. $x = P(s)x \in \textit{Range } P(s)$. □

Corollary 3.7 *If the evolution operator U admits a P-uniform exponential dichotomy then*

$$Range\ P(s) = \left\{ x \in X \ : \ \lim_{t\to\infty} U(t,s)x = 0 \right\}.$$

Corollary 3.8 *If the evolution operator U admits a uniform exponential dichotomy with respect to the families of projections P_1 and P_2 then*

$$Range\ P_1(s) = Range\ P_2(s), \quad \text{for all } s \geq 0.$$

A necessary condition for exponential dichotomy is given by

Proposition 3.9 *If the evolution operator $U : \Delta \to \mathcal{B}(X)$ admits an exponential dichotomy then there exists a family of projections $P : \mathbb{R}_+ \to \mathcal{B}(X)$ invariant for U having the property that for every $(t,s) \in \Delta$ the restriction of $U(t,s)$ on $Ker\ P(s)$ is injective.*

Proof. Indeed, if U has an e.d with respect to the family of projections P (invariant for U) and $x \in Ker\ P(s)$ with $U(t,s)x = 0$ then by Definition 3.1 we obtain that $\|x\| = \|Q(s)x\| \leq Ne^{\alpha t}e^{-\beta(t-s)}\|U(t,s)Q(s)x\| = 0$, hence $x = 0$. □

Corollary 3.10 *If the evolution operator $U : \Delta \to \mathcal{B}(X)$ admits an exponential dichotomy with the family of projections $P : \mathbb{R}_+ \to \mathcal{B}(X)$ invariant for U then P is strongly invariant for U if and only if*

$$Ker\ P(t) \subset Range\ U(t,s), \quad \text{for all } (t,s) \in \Delta.$$

Proof. It results from Proposition 2.7 and Proposition 3.9. □

4 Exponential dichotomy with strongly invariant families of projections

In this section we consider the particular case of exponential dichotomy with families of projections that are strongly invariant for an evolution operator.

Remark 4.1 Before we state our results, it is important to mention that the study of exponential dichotomy (without assuming the strong invariance for the family of projections) covers a more general framework. Example 5.2 gives us an evolution operator that has a dichotomy property with respect to an invariant family of projections which is not strongly invariant and hence there are systems that cannot be studied in the light of dichotomy with strongly invariant projection families.

Let $P : \mathbb{R}_+ \to \mathcal{B}(X)$ be a family of projections which is strongly invariant for the evolution operator $U : \Delta \to \mathcal{B}(X)$ and let $V : \Delta \to \mathcal{B}(X)$ be the skew-evolution operator associated to the pair (U,P). Under these hypotheses we prove

Theorem 4.1 *The evolution operator U is exponentially dichotomic with respect to the family of projections P if and only if there are $N \geq 1$, $\alpha \geq 0$ and $\beta > 0$ such that the following conditions hold:*

(ed_1'') $\|U(t,s)P(s)x\| \leq Ne^{\alpha s}e^{-\beta(t-s)}\|P(s)x\|$;
(ed_2'') $\|V(t,s)Q(t)x\| \leq Ne^{\alpha t}e^{-\beta(t-s)}\|Q(t)x\|$,

for all $(t, s, x) \in \Delta \times X$.

Proof. We only have to prove the equivalence between the instability properties (i.e. $(ed_2) \Leftrightarrow (ed_2'')$). For the *necessity*, we observe that from (ed_2) and the equalities (v_1) and (v_3) we obtain that

$$\begin{aligned}\|V(t,s)Q(t)x\| &\overset{(v_3)}{=} \|Q(s)V(t,s)Q(t)x\| \\ &\leq Ne^{\alpha t}e^{-\beta(t-s)}\|U(t,s)Q(s)V(t,s)Q(t)x\| \\ &= Ne^{\alpha t}e^{-\beta(t-s)}\|Q(t)U(t,s)V(t,s)Q(t)x\| \\ &\overset{(v_1)}{=} Ne^{\alpha t}e^{-\beta(t-s)}\|Q(t)x\|,\end{aligned}$$

for all $(t, s, x) \in \Delta \times X$.

The *sufficiency* follows similarly, by using (v_2) and (ed_2'') in the following manner:

$$\begin{aligned}\|Q(s)x\| &\overset{(v_2)}{=} \|V(t,s)U(t,s)Q(s)x\| = \|V(t,s)Q(t)U(t,s)Q(s)x\| \\ &\leq Ne^{\alpha t}e^{-\beta(t-s)}\|U(t,s)Q(s)x\|,\end{aligned}$$

for all $(t, s, x) \in \Delta \times X$. $\square$

Corollary 4.2 *The evolution operator $U : \Delta \to \mathcal{B}(X)$ is exponentially dichotomic with respect to the family of projections $P : \mathbb{R}_+ \to \mathcal{B}(X)$, which is assumed to be strongly invariant for U, if and only if there exist $N \geq 1$, $\alpha \geq 0$ and $\beta > 0$ such that*

(i) $\|U(t,t_0)P(t_0)x_0\| \leq Ne^{\alpha s}e^{-\beta(t-s)}\|U(s,t_0)P(t_0)x_0\|$;
(ii) $\|V(t,t_0)Q(t)x_0\| \leq Ne^{\alpha t}e^{-\beta(t-s)}\|V(t,s)Q(t)x_0\|$,

for all $(t, s, t_0, x_0) \in T \times X$.

Proof. Necessity. If U is e.d with the respect to the family of projections P then

$$\begin{aligned}\|U(t,t_0)P(t_0)x_0\| &\overset{(e_2)}{=} \|U(t,s)P(s)U(s,t_0)P(t_0)x_0\| \\ &\leq Ne^{\alpha t}e^{-\beta(t-s)}\|U(s,t_0)P(t_0)x_0\|\end{aligned}$$

and

$$\begin{aligned}\|V(t,t_0)Q(t)x_0\| &\overset{(v_5)}{=} \|V(s,t_0)V(t,s)Q(t)x_0\| \\ &\overset{(v_3)}{=} \|V(s,t_0)Q(s)V(t,s)Q(t)x_0\| \\ &\leq Ne^{\alpha t}e^{-\beta(s-t_0)}\|Q(s)V(t,s)Q(t)x_0\| \\ &\overset{(v_3)}{=} Ne^{\alpha t}e^{-\beta(s-t_0)}\|V(t,s)Q(t)x_0\|,\end{aligned}$$

for all $(t, s, t_0, x_0) \in T \times X$.

Sufficiency. It is obvious from Theorem 4.1 taking $s = t_0$ in (i) and $t = s$ in (ii). $\square$

Corollary 4.3 *Let $P : \mathbb{R}_+ \to \mathcal{B}(X)$ be a family of projections strongly invariant for the evolution operator $U : \Delta \to \mathcal{B}(X)$. The following statements are equivalent:*

(i) U is uniformly exponentially dichotomic with respect to P;

(ii) there are $N \geq 1$ and $\beta > 0$ such that
(a) $\|U(t,s)P(s)x\| \leq Ne^{-\beta(t-s)}\|P(s)x\|$;
(b) $\|V(t,s)Q(t)x\| \leq Ne^{-\beta(t-s)}\|Q(t)x\|$,
for all $(t,s,x) \in \Delta \times X$;

(iii) there exist $N \geq 1$ and $\beta > 0$ such that
(a) $\|U(t,t_0)P(t_0)x_0\| \leq Ne^{-\beta(t-s)}\|U(s,t_0)P(t_0)x_0\|$;
(b) $\|V(t,t_0)Q(t)x_0\| \leq Ne^{-\beta(t-s)}\|V(t,s)Q(t)x_0\|$,
for all $(t,s,t_0,x_0) \in T \times X$.

Proof. It follows from Theorem 4.1 and Corollary 4.2 □

Now we introduce another concept of exponential dichotomy through

Definition 4.4 *Let $P : \mathbb{R}_+ \to \mathcal{B}(X)$ be a family of projections which is strongly invariant for the evolution operator $U : \Delta \to \mathcal{B}(X)$. We say that U is strongly exponentially dichotomic with respect to P (and denote P-s.e.d) if there are constants $N \geq 1$, $\alpha \geq 0$ and $\beta > 0$ such that*

(sed_1) $\|U(t,s)P(s)x\| \leq Ne^{\alpha s}e^{-\beta(t-s)}\|x\|$;
(sed_2) $\|V(t,s)Q(t)x\| \leq Ne^{\alpha t}e^{-\beta(t-s)}\|x\|$,

for all $(t,s,x) \in \Delta \times X$.

In the particular case in which $\alpha = 0$ the evolution operator U is called uniformly strongly exponentially dichotomic with respect to P (P-u.s.e.d).

If there exists a family of projections P such that U is P-s.e.d (P-u.s.e.d respectively) then U is called strongly exponentially dichotomic and denote s.e.d (respectively uniformly strongly exponentially dichotomic and denote u.s.e.d).

Remark 4.2 The above defined concept is used for example, in the uniform case, in [11] and in the nonuniform case in [16] and [5–7, 9].

Remark 4.3 The evolution operator U is s.e.d with respect to the the family of projections P if and only if there are $N \geq 1$, $\alpha \geq 0$ and $\beta > 0$ such that

(i) $\|U(t,s)P(s)\| \leq Ne^{\alpha s}e^{-\beta(t-s)}$;
(ii) $\|V(t,s)Q(t)\| \leq Ne^{\alpha t}e^{-\beta(t-s)}$

for all $(t,s) \in \Delta$.

For $\alpha = 0$ we obtain a characterization of the u.e.d property.

Remark 4.4 If U is s.e.d (respectively s.u.e.d) then it is also e.d (respectively u.e.d).

Indeed, it is sufficient to observe that if we substitute x with $P(s)x$ in (sed_1) respectively with $Q(t)x$ in (sed_2), one obtains (ed_1'') and (ed_2'') respectively.

A connection between the concepts of s.e.d and e.d is given by

Theorem 4.5 *Let $P : \mathbb{R}_+ \to \mathcal{B}(X)$ be a family of projections which is strongly invariant for an evolution operator $U : \Delta \to \mathcal{B}(X)$. Then U is strongly exponentially dichotomic with respect to P if and only if P is exponentially bounded and U is exponentially dichotomic with respect to P.*

Proof. Necessity. If U is s.e.d with the respect to the family of projections P then from (sed_1), for $t = s$, it results that P is exponentially bounded. By Remark 4.6 it results that s.e.d $\Rightarrow$ e.d.

Sufficiency. If $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is exponentially bounded then there exist $M \geq 1$ and $\gamma \geq 0$ such that $\|P(t)\| + \|Q(t)\| \leq Me^{\gamma t}$, for all $t \geq 0$. Then from the exponential dichotomy of U it results (via Theorem 4.1) that there are $N \geq 1$, $\alpha \geq 0$ and $\beta > 0$ such that

$$\|U(t,s)P(s)x\| \leq MNe^{(\alpha+\gamma)s}e^{-\beta(t-s)}\|x\| \leq N_1 e^{\alpha_1 s}e^{-\beta(t-s)}\|x\|$$

and

$$\|V(t,s)Q(t)x\| \leq MNe^{(\alpha+\gamma)t}e^{-\beta(t-s)}\|x\| \leq N_1 e^{\alpha_1 t}e^{-\beta(t-s)}\|x\|,$$

for all $(t, s, x) \in \Delta \times X$, where $N_1 = MN$ and $\alpha_1 = \alpha + \gamma$. By Definition 4.4 it results that U is s.e.d. □

As a particular case ($\alpha = \gamma = 0$) we have

Corollary 4.6 *The evolution operator $U : \Delta \to \mathcal{B}(X)$ is strongly uniformly exponentially dichotomic with respect to the family of projections P (which is assumed to be strongly invariant for U) if and only if P is bounded and U is uniformly exponentially dichotomic with respect to P.*

Corollary 4.7 *If the evolution operator $U : \Delta \to \mathcal{B}(X)$ has uniform exponential growth then it is strongly uniformly exponentially dichotomic with respect to $P : \mathbb{R}_+ \to \mathcal{B}(X)$ if and only if it is uniformly exponentially dichotomic with respect to P.*

Proof. The *necessity* follows straightforward from the preceding corollary and for the *sufficiency*, from the fact that U is u.e.d and has uniform exponential growth, by [20] it follows that the family of projections P is bounded hence, by the preceding corollary, the conclusion follows. □

5 Examples and counterexamples

In this section we present some examples which illustrate the concepts defined in the preceding sections and connections between them.

The following example presents a family of projections P which is invariant for an evolution operator U and it is is not strongly invariant for U.

Example 5.1 Let $X = \mathbb{R}^3$ with the norm $\|(x_1, x_2, x_3)\| = |x_1| + |x_2| + |x_3|$ and let $U : \Delta \to \mathcal{B}(X)$ be the evolution operator defined by

$$U(t,s)(x_1,x_2,x_3) = \begin{cases} \left(x_1 e^{s-t}, x_2 e^{t-s}, x_3 e^{t-s}\right), & \text{if } t \geq s > 0 \text{ or } t = s = 0 \\ \left(x_1 e^{-t}, 0, x_3 e^{t}\right), & \text{if } t > s = 0. \end{cases}$$

It is easy to check that $P : \mathbb{R}_+ \to \mathcal{B}(X)$ defined by

$$P(t)(x_1,x_2,x_3) = \begin{cases} (x_1, x_2, 0), & t = 0 \\ (x_1 + x_2 e^{-2t}, 0, 0), & t > 0, \end{cases}$$

is a bounded family of projections having the complementary family given by $Q : \mathbb{R}_+ \to \mathcal{B}(X)$,

$$Q(t)(x_1, x_2, x_3) = \begin{cases} (0, 0, x_3), & t = 0 \\ (-x_2 e^{-2t}, x_2, x_3), & t > 0. \end{cases}$$

One can see that $U(t,s)P(s) = P(t)U(t,s)$ for all $(t,s) \in \Delta$ and moreover

$$U(t,s)P(s)(x_1, x_2, x_3) = \begin{cases} (x_1, x_2, 0), & t = 0 \\ (x_1 e^{-t}, 0, 0), & t > s = 0 \\ (x_1 e^{s-t} + x_2 e^{-s-t}, 0, 0), & s > 0, \end{cases}$$

for all $(t,s,x) \in \Delta \times X$. From the above facts we obtain in particular that P is invariant for U.

P is not strongly invariant for U, because $U(1,0)$ is not an isomorphism from $Ker\ P(0)$ to $Ker\ P(1)$. Indeed, we observe that

$$y = \left(-\frac{1}{e^2}, 1, 0\right) \in Ker\ P(1) \text{ and } y \notin Range\ U(1,0)$$

which can easily be proved by a contradiction, for if we suppose that there is $x = (x_1, x_2, x_3) \in X$ with $U(1,0)x = y$ then we would obtain the contradiction

$$\left(\frac{x_1}{e}, 0, ex_3\right) = \left(-\frac{1}{e^2}, 1, 0\right).$$

The below example gives an evolution operator U on a Banach space X which is u.e.d with respect to a family of projections P such that P is invariant for U but P is not strongly invariant for U.

Example 5.2 Consider $X = \mathbb{C} \oplus l^2(\mathbb{N}, \mathbb{C})$ the Banach space of all the pairs $x = (z, u)$, with respect to the norm:

$$\|x\| = |z| + \left(\sum_{k=0}^{\infty} |u_k|^2\right)^{1/2}, \quad \text{where } z \in \mathbb{C} \text{ and } u = (u_n)_{n \geq 0} \in l^2(\mathbb{N}, \mathbb{C}).$$

Denote by χ_A the characteristic function of the set A, and by $[t]$ the largest integer less or equal to $t \in \mathbb{R}_+$. Define $U : \Delta \to \mathcal{B}(X)$ by

$$\begin{aligned} U(t,s)x &= \left(e^{s-t} \cdot z, e^{t-s} \cdot \chi_{\{[t]-[s],[t]-[s]+1,\ldots\}}(\cdot) u(\cdot - [t] + [s])\right) \\ &= \begin{cases} \left(e^{s-t} \cdot z, e^{t-s} \cdot u(n - [t] + [s])\right), & \text{if } n \geq [t] - [s] \\ \left(e^{s-t} \cdot z, 0\right), & \text{if } n \in \{0, \ldots, [t] - [s] - 1\}, \end{cases} \end{aligned}$$

for $x = (z, u) \in X$, $(t,s) \in \Delta$ with $t \geq s + 1$ and $n \in \mathbb{N}$. In the case $t = s$ we can see that $U(t,s)x = (z, u(n))$.

Also, for every $t \geq 0$, define $P(t) : X \to X$ by $P(t)(z, u) = (z, 0)$, for all $(z, u) \in X$. First of all, we will show that U is an evolution operator on X. Consider $x = (z, u) \in X$

and (t,s), $(s,t_0) \in \Delta$. It is obvious that $U(t,t) = I$. Moreover, if $t \geq s+1 \geq t_0+1$, then we have that

$$U(t,s)U(s,t_0)x = (e^{t_0-t} \cdot z, e^{t-t_0} \cdot \\ \cdot \chi_{\{[t]-[s],[t]-[s]+1,\ldots\}}(\cdot) v(\cdot - [t] + [s])) \\ = \begin{cases} (e^{t_0-t} \cdot z, e^{t-t_0} \cdot v(n-[t]+[s])), \\ \quad \text{if } n \geq [t]-[s] \\ (t^{t_0-t} \cdot z, 0), \\ \quad \text{if } n \in \{0,\ldots,[t]-[s]-1\} \end{cases} \tag{5.1}$$

where

$$v(k) = \chi_{\{[s]-[t_0],[s]-[t_0]+1,\ldots\}}(k-[t]+[s]) u(k-[t]+[t_0]), \quad k \in \mathbb{N}. \tag{5.2}$$

The cases $t = s$ or $s = t_0$ are obvious, since $U(t,t) = I$, for all $t \geq 0$, hence the above relation also holds in each of the mentioned cases. Because

$$\chi_{\{[s]-[t_0],[s]-[t_0]+1,\ldots\}}(n-[t]+[s]) = \begin{cases} 1, & \text{if } n-[t]+[s] \geq [s]-[t_0] \\ 0, & \text{otherwise.} \end{cases} \\ = \begin{cases} 1, & \text{if } n \geq [t]-[t_0] \\ 0, & \text{otherwise.} \end{cases} \\ = \chi_{\{[t]-[t_0],[t]-[t_0]+1,\ldots\}}(n), \tag{5.3}$$

for all $n \in \mathbb{N}$ and from the fact that $[t]-[s] \leq [t]-[t_0]$, meaning that $\{[t]-[s],\ldots\} \supset \{[t]-[t_0],\ldots\}$, it follows that

$$\chi_{\{[t]-[s],\ldots\}}(n) \cdot \chi_{\{[s]-[t_0],\ldots\}}(n-[t]+[s]) = \chi_{\{[t]-[t_0],\ldots\}}(n). \tag{5.4}$$

From (5.1), (5.2), (5.3) and (5.4), we get that for all $n \in \mathbb{N}$,

$$\chi_{\{[t]-[s],[t]-[s]+1,\ldots\}}(n) v(n-[t]+[s]) = \chi_{\{[t]-[t_0],\ldots\}}(n) \cdot u(n-[t]+[t_0])$$

from where it follows that $U(t,s)U(s,t_0) = U(t,t_0)$ hence $U : \Delta \to \mathcal{B}(X)$ is an evolution operator on X.

A simple computation shows us that $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is invariant for $U : \Delta \to \mathcal{B}(X)$. In order to show that U is u.e.d, let $(t,s) \in \Delta$ and $x = (z,u) \in X$. We have that $\|U(t,s)P(s)x\| = e^{s-t}\|P(s)x\|$ and

$$\|U(t,s)Q(s)x\|^2 = \left\|e^{t-s}\chi_{\{[t]-[s],\ldots\}}(\cdot)u(\cdot-[t]+[s])\right\|_2^2 \\ = \sum_{k=0}^{\infty} e^{2(t-s)}\chi_{\{[t]-[s],\ldots\}}(k)\,|u(k-[t]+[s])|^2 \\ = e^{2(t-s)} \sum_{k=[t]-[s]}^{\infty} |u(k-[t]+[s])|^2 = e^{2(t-s)} \sum_{j=0}^{\infty} |u(j)|^2 \\ = e^{2(t-s)}\|u\|_2^2,$$

from where it follows that $\|U(t,s)Q(s)x\| = e^{t-s}\|u\|_2 = e^{t-s}\|Q(s)x\|$. It follows that $U : \Delta \to \mathcal{B}(X)$ is uniformly exponentially dichotomic.

Finally, consider $(1,0)\in\Delta$ and assume that $U(1,0)$ is an isomorphism from $Ker P(0)$ to $Ker\ P(1)$. Consider $u_0 \in l^2(\mathbb{N},\mathbb{C})$ defined by $u_0(n) = i \cdot \chi_{\{0\}}(n)$, for all $n \in \mathbb{N}$. Then $(0,u_0) \in Q(0)X$ and, by our hypothesis, there exists $u \in l^2(\mathbb{R}_+,\mathbb{C})$ such that $U(1,0)(0,u) = (0,u_0)$, which means that $e \cdot \chi_{\{1,2,\dots\}}(n) \cdot u(n) = \cdot u_0(n)$, for all $n \in \mathbb{N}$. setting $n = 0$ we obtain $e \cdot \chi_{\{1,2,\dots\}}(0) \cdot u(0) = \chi_{\{0\}}(0) \cdot i$, which is a contradiction. Hence P is not strongly invariant for U.

The following example presents pairs (U,P) for which the converse of tie implications $u.e.d \Rightarrow e.d$ and $u.s.e.d \Rightarrow s.e.d$ are not valid.

Example 5.3 Let $X = l^\infty$ be the Banach space considered in Example 5.4. Let P_1, P_2, S be the bounded linear operators defined by

$$\begin{aligned} P_1(x_1,x_2,x_3,x_4,\dots) &= (x_1,0,x_3,0,\dots) \\ P_2(x_1,x_2,x_3,x_4,\dots) &= (0,x_2,0,x_4,\dots) \\ S(x_1,x_2,x_3,x_4,\dots) &= (x_2,x_3,x_4,x_4,\dots). \end{aligned}$$

Consider a nondecreasing function $p : \mathbb{R}_+ \to \mathbb{R}_+$ and for every $t \geq 0$ we define $P(t) : X \to X$ by $P(t) = P_1 + p(t)SP_2$, or equivalently

$$P(t)(x_1,x_2,x_3,x_4,\dots) = (x_1 + p(t)x_2, 0, x_3 + p(t)x_4, 0, \dots).$$

Then P is a family of projections with the complementary

$$Q(t)(x_1,x_2,x_3,x_4,\dots) = (-p(t)x_2, x_2, -p(t)x_4, x_4, \dots).$$

It is immediate to see that

$$\|P(t)\| = 1 + p(t) \text{ and } \|Q(s)x\| = \max\{1, p(s)\} \sup_{n\geq 1} |x_{2n}| \leq \|Q(t)x\|,$$

for all $(t,s,x) \in \Delta \times X$.

Let $\mathcal{U}$ be the set of all functions $u : \mathbb{R}_+ \to \mathbb{R}_+$ having the properties

(u_1) there are $\alpha > 0$ and $\beta > 0$ such that $u(s) - u(t) \leq \alpha s - \beta(t-s)$, for all $(t,s) \in \Delta$;
(u_2) $\limsup_{t\to\infty} (u(t) - u(t+1/2)) = \infty$.

The set $\mathcal{U}$ is nonempty and for this it is sufficient to see that the function $u_0 : \mathbb{R}_+ \to \mathbb{R}_+,\ u_0(t) = \frac{t}{1+\{t\}}$ (where $\{t\}$ denotes the fractional part of t) satisfies condition (u_1) with $\alpha = \beta = \frac{1}{2}$ and by choosing $t_n = n+1$ with $t_n \to \infty$, we have that

$$\lim_{n\to\infty} (u_0(t_n) - u_0(t_n + 1/2)) = \lim_{n\to\infty} \frac{n}{3} = \infty$$

from where it follows that $u_0 \in \mathcal{U}$.

For $u \in \mathcal{U}$ we define the map $U : \Delta \to \mathcal{B}(X), U(t,s) = e^{u(s)-u(t)}P(s) + e^{u(t)-u(s)}Q(t)$. It is a simple verification to see that U is an evolution operator with the property that the family of projections P is strongly invariant for U and the skew-evolution operator associated to the pair (U,P) is defined by $V(t,s)Q(t) = e^{u(s)-u(t)}Q(s)$, for all $(t,s) \in \Delta$. Then, from (u_1) we obtain that $\|U(t,s)P(s)x\| \leq e^{\alpha s}e^{-\beta(t-s)}\|P(s)x\|$ and $\|V(t,s)Q(t)\| \leq e^{\alpha t}e^{-\beta(t-s)}\|Q(t)x\|$, for all $(t,s,x) \in \Delta \times X$.

By Theorem 4.1 it results that U is exponentially dichotomic with respect to P.

If we assume that U is u.e.d then there are $N \geq 1$ and $\beta > 0$ such that

$$\|U(s+1/2,s)P(s)x\| = e^{u(s)-u(s+1/2)}\|P(s)x\| \leq e^{-\beta/2}\|P(s)x\|,$$

which implies that $u(s) - u(s+1/2) + \beta/2 \leq 0$, for every $s \geq 0$. Using the condition (u_2) we obtain a contradiction.

For the particular case where $p(t) = e^t - 1$ we obtain that $\|P(t)\| = e^t$ and hence P is exponentially bounded. Then, by Theorem 4.5 it follows that U is s.e.d with respect to P. Because U is not u.e.d with respect to P it results that U is not u.s.e.d with respect to P.

On the other hand, by taking p to be the null function, we have that U is s.e.d (and hence e.d). The same contradiction as above shows us that U is not u.e.d (and hence not u.s.e.d).

The following example presents a family of projections P which is strongly invariant for an evolution operator U, such that U is e.d and not s.e.d with respect to P. Thus the implications *e.d* $\Rightarrow$ *s.e.d* and *u.e.d* $\Rightarrow$ *u.s.e.d* are do not generally hold.

Example 5.4 Let $X = l^\infty$ be the Banach space of bounded real-valued sequences, endowed with the norm

$$\|x\| = \sup_{n\geq 1}|x_n| \text{ for } x = (x_1, x_2, \ldots, x_n, \ldots) \in X.$$

Let $P : \mathbb{R}_+ \to \mathcal{B}(X)$ be the map defined by $P(t)(x_1,x_2,x_3,\ldots) = (x_1 + (e^{t^2} - 1)x_2, 0, x_3 + (e^{t^2} - 1)x_4, 0, \ldots)$. It is immediate that P is a family of projections with the complementary $Q : \mathbb{R}_+ \to \mathcal{B}(X)$ given by $Q(t)(x_1,x_2,x_3,\ldots) = ((1-e^{t^2})x_2, x_2, (1-e^{t^2})x_4, x_4, \ldots)$. In addition $\|P(t)\| = e^{t^2}$ and $\|Q(s)x\| = \max\{1, e^{s^2} - 1\}\sup_{n\geq 1}|x_{2n}| \leq \|Q(t)x\|$, for all $(t,s) \in \Delta$ and for all $x = (x_1, x_2, \ldots) \in X$.

Furthermore we have that

$$P(t)P(s) = P(s), \quad Q(t)Q(s) = Q(t) \text{ and } Q(t)P(s) = 0 \tag{5.5}$$

for all $(t,s) \in \Delta$.

We observe that $U : \Delta \to \mathcal{B}(X)$ defined by $U(t,s) = e^{s-t}P(s) + e^{t-s}Q(t)$ is an evolution operator with the property that P is strongly invariant for U and the skew-evolution operator associated to the pair (U,P) is defined by $V(t,s)Q(t) = e^{s-t}Q(s)$, for $(t,s) \in \Delta$. Taking into account (5.5) we have that

$$\|U(t,s)P(s)x\| = e^{s-t}\|P(s)x\| = e^{-(t-s)}\|P(s)x\|$$

and

$$\|V(t,s)Q(t)x\| = e^{s-t}\|Q(s)x\| \leq e^{-(t-s)}\|Q(t)x\|,$$

for all $(t,s,x) \in \Delta \times X$.

By Corollary 4.3 it follows that U is u.e.d (and hence e.d) with respect to P.

Because P is not exponentially bounded, from Theorem 4.5 it follows that U is not s.e.d (and hence not s.u.e.d) with respect to P.

Remark 5.1 In the virtue of Definitions 3.1 and 4.4, Theorem 4.5, Corollary 4.6 and Examples 5.4 and 5.3, we obtain the connections between the dichotomy concepts studied in this paper.

These are illustrated by the following diagram:

$$\begin{array}{ccccc} u.s.e.d & \Longrightarrow & s.e.d & \nLeftarrow & u.e.d \\ & \nLeftarrow & & \nRightarrow & \\ \Downarrow\not\Uparrow & & \Downarrow\not\Uparrow & & \Uparrow\not\Downarrow \\ & \nLeftarrow & & \nRightarrow & \\ u.e.d & \Longrightarrow & e.d & \Longleftarrow & u.s.e.d \end{array}$$

Finally, we obtain that the studied dichotomy concepts are distinct.

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