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The singular value of $A+B$ and $\alpha A+\beta B$

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Abstract We present several inequalities regarding to the singular values of $A+B$ and $\alpha A+\beta B$, discussing when $\alpha, \beta \in \mathbb{C}$, A and B are Hermitian, and when one or both of them are definite.

Keywords Hermitian matrices · Positive (negative) definite matrices · Singular values · Majorizations

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1 Introduction

In [2] some properties of Hermitian matrices are investigated. Precisely, if A and B are Hermitian matrices, then some relations are established between singular values of $A+B$ and $A+iB$. Further, sometimes it is required that A or B are positive or negative semi-definite (see, also, [3]).

In this paper we study a similar problem: let A and B be Hermitian matrices, and let $\alpha, \beta \in \mathbb{C}$ be arbitrary. We consider the inequalities between the singular values of $A+B$ and the singular values of $\alpha A+\beta B$, under some assumptions.

One of the fields where this could be applied is the matrix pencil theory (see [4]), since, $\alpha A+\beta B$ is, in fact, a matrix pencil. Hence we find it interesting to consider this case separately from [2] and [3].

The vector space we observe is the n-dimensional complex vector space, that is, $\mathbb{C}^n$. The Euclidean norm in $\mathbb{C}^n$ is denoted by $\|\cdot\|$. All matrices are complex square and n-dimensional. The spectral norm of a complex matrix A is denoted by $\|A\|$. The matrix A^* is the complex conjugate matrix of A, and $|A|=(A^*A)^{1/2}$. The notation *p.d.* denotes positive definite, and *n.d.* denotes negative definite matrix (operator). Analogously, *p.s.d.* denotes positive semi definite, and *n.s.d.* denotes negative semi definite matrix.

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$\lambda_j(A)$ will represent an eigenvalue of the matrix A. If A is Hermitian, then we require $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$.

$s_j(A)$ will represent a singular value of the matrix A, i.e. $s_j(A) = \sqrt{\lambda_j(A^*A)}$. Hence, we always assume that singular values are ordered starting form the largest one: $\|A\| = s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A) \geq 0$.

Note that for arbitrary complex matrices A and B, the following stands:

$$(|A| - |B|) \text{ is } p.s.d. \Rightarrow (\forall k \in \{1, ..., n\}) s_k(A) \geq s_k(B) \tag{1.1}$$

$$\Rightarrow (\forall k \in \{1, ..., n\}) \prod_{j=1}^{k} s_j(A) \geq \prod_{j=1}^{k} s_j(B) \tag{1.2}$$

$$\Rightarrow (\forall k \in \{1, ..., n\}) \sum_{j=1}^{k} s_j(A) \geq \sum_{j=1}^{k} s_j(B). \tag{1.3}$$

The relation (1.3) is called the *weak majorization of the singular values of B by the singular values of A*, and it is denoted as $\{s_j(B)\} \prec_w \{s_j(A)\}$.

The relation (1.2) is called the *logarithmic weak majorization of the singular values of B by the singular values of A*, and it is denoted as $\{s_j(B)\} \prec_{log(w)} \{A\}$.

In this paper we prove some majorization results concerning the singular values of $A + B$ and $\alpha A + \beta B$, where $\alpha, \beta \in \mathbb{C}$, A, B are Hermitian, and in some cases A and B are *p.d.* (*p.s.d.*) or *n.d.* (*n.s.d.*).

2 Results

We formulate the main result of this paper.

Theorem 2.1 *Let A and B be Hermitian matrices. Let $\mu, \nu, \lambda, \eta \in \mathbb{R}$,*

$$\alpha := \mu + i\nu, \beta := \lambda + i\eta \tag{2.1}$$

denoted in a way that

$$|\alpha|, |\beta| \geq 1, \tag{2.2}$$

$$\mu\lambda + \nu\eta \geq 1. \tag{2.3}$$

Then:
(1) *If B and $A + B$ are* p.d. *and all singular values of $A + B$ are greater than $||B||$, then $\{s_j(A + B)\} \prec_w \{s_j(\alpha A + \beta B)\}$;*
(2) *If A and B are* n.d.*, then $\{s_j(A + B)\} \prec_{log(w)} \{s_j(\alpha A + \beta B)\}$;*
(3) *If A and B are* p.d.*, then $s_j(A + B) \leq s_j(\alpha A + \beta B)$ for all $j \in \{1, \ldots, n\}$.*

We will use the minimax principle (see [1]):
If A is a Hermitian matrix, then

$$\lambda_j(A) = \max_{\substack{M < \mathbb{C}^n \\ dim M = j}} \min_{\substack{x \in M \\ ||x|| = 1}} \langle x, Ax \rangle. \tag{2.4}$$

Moreover, if A is an arbitrary linear operator on $\mathbb{C}^n$, then

$$s_j(A) = \max_{\substack{M<\mathbb{C}^n \\ dimM=j}} \min_{\substack{x\in M \\ ||x||=1}} ||Ax||. \tag{2.5}$$

Further, we have:

$$k \in \{1,\ldots,n\} \implies \sum_{j=1}^{k} s_j(A) = \max\left|\sum_{j=1}^{k}\langle y_j, Ax_j\rangle\right|, \tag{2.6}$$

where the maximum is taken over all k-tuples of orthonormal vectors $x_1, ..., x_k$ and $y_1, ..., y_k$.

Finally, we have:

$$k \in \{1,2,\ldots,n\} \implies \prod_{j=1}^{k} s_j(A) = \max|\det W^*AW|, \tag{2.7}$$

where the maximum is taken over all $n \times k$ matrices W with the property $W^*W = I$.

Before we get to the proof of Theorem 2.1, we will prove a useful lemma:

Lemma 2.2 *Let a, b be real numbers, and let $\alpha = \mu + i\nu$, $\beta = \lambda + i\eta$ be complex numbers, which satisfy the conditions* (2.1) *and* (2.2). *The following hold:*
(1) *If $ab > 0$ and $\lambda\mu + \nu\eta \geq 1$, then $|a + b| \leq |\alpha a + \beta b|$;*
(2) *If $ab < 0$ and $\lambda\mu + \nu\eta \leq 1$, then $|a + b| \leq |\alpha a + \beta b|$;*
(3) *If $ab = 0$ then $|a + b| \leq |\alpha a + \beta b|$.*

Proof. Trivially, if $ab = 0$ that means that at least one of those numbers is zero. Lets assume that $b = 0$. Hence, $|a+b| = |a| \leq |\alpha||a| = |\alpha a| = |\alpha a+\beta b|$. Thus, (3) is proved.

In general, note that

$$|a + b| \leq |\alpha a + \beta b| \tag{2.8}$$

is equivalent to

$$\begin{aligned}
(a + b)^2 &\leq |\alpha a + \beta b|^2 \\
&\Leftrightarrow a^2 + 2ab + b^2 \leq (\alpha a + \beta b)\overline{(\alpha a + \beta b)} \\
&\Leftrightarrow a^2 + 2ab + b^2 \leq (\alpha a + \beta b)(\overline{\alpha}a + \overline{\beta}b) \\
&\Leftrightarrow a^2 + 2ab + b^2 \leq |\alpha|^2a^2 + |\beta|^2b^2 + ab(\overline{\alpha}\beta + \overline{\beta}\alpha).
\end{aligned}$$

Given the conditions of the lemma, one can see that if we prove the following inequality

$$2ab \leq ab(\overline{\alpha}\beta + \overline{\beta}\alpha) \tag{2.9}$$

then both (1) and (2) are proved.

Lets assume that $ab > 0$. Then (2.9) is equivalent to $2 \leq \overline{\alpha}\beta + \alpha\overline{\beta} \Leftrightarrow 2 \leq (2\lambda\mu + 2\nu\eta + i(\nu\lambda + \mu\eta - \mu\eta - \nu\lambda)) \Leftrightarrow 1 \leq \lambda\mu + \nu\eta$, which is true due to the assumption of the lemma. Thus, (1) holds.

(2) can be proved analogously. □

Combining Lemma 2.2 with the minimax principle, we have the proof of Theorem 2.1.

Proof. (1) Under the assumptions, there exist orthonormal eigenvectors $e_1, ..., e_n$ of $A + B$, arranged in such a way that the following holds:

$$\forall j = \overline{1, n} : s_j(A + B) = |\langle e_j, (A + B)e_j\rangle| = |\langle e_j, Ae_j\rangle + \langle e_j, Be_j\rangle|. \tag{2.10}$$

Note that $A + B$ and B are positive definite matrices, so it is safe to say that $\langle e_j, Ae_j\rangle$ and $\langle e_j, Be_j\rangle$ are real numbers. Denote $a := \langle e_j, Ae_j\rangle$ and $b := \langle e_j, Be_j\rangle$ for given j. Then $ab = \langle e_j, Ae_j\rangle\langle e_j, Be_j\rangle = (\langle e_j, (A + B)e_j\rangle - \langle e_j, Be_j\rangle)\langle e_j, Be_j\rangle = (s_j(A + B) - \langle e_j, Be_j\rangle))\langle e_j, Be_j\rangle$. Since, $\langle e_j, Be_j\rangle > 0$, applying the Cauchy-Schwarz inequality, we have $ab \geq (\langle e_j, Be_j\rangle)(s_j(A + B) - ||Be_j|| \cdot ||e_j||)$ Since $||e_j|| = 1$ and $||Be_j|| \leq ||B||||e_j|| = ||B||$, we get:

$$ab \geq (\langle e_j, Be_j\rangle)(s_j(A + B) - ||B||) > 0. \tag{2.11}$$

Since $ab > 0$, we can now apply Lemma 2.2 in (2.10), $|\langle e_j, Ae_j\rangle + \langle e_j, Be_j\rangle| \leq |\alpha\langle e_j, Ae_j\rangle + \beta\langle e_j, Be_j\rangle| = |\langle e_j, (\alpha A + \beta B)e_j\rangle|$.

Combining the last inequality with (2.6), it follows that

$$\sum_{j=1}^{k} s_j(A + B) \leq \sum_{j=1}^{k} |\langle e_j, (\alpha A + \beta B)e_j\rangle| \leq \sum_{j=1}^{k} s_j(\alpha A + \beta B).$$

(2) Let A and B be Hermitian *n.d.* matrices. Let $\{\lambda_1, ..., \lambda_n\}$ denote the spectrum of $(-B)^{-\frac{1}{2}}A(-B)^{-\frac{1}{2}}$. Now we have:

$$\begin{aligned} |\det(A + B)| &= |\det[(-B)^{\frac{1}{2}}((-B)^{-\frac{1}{2}}A(-B)^{-\frac{1}{2}} - I)(-B)^{\frac{1}{2}}]| \\ &= |\det(-B)\det((-B)^{-\frac{1}{2}}A(-B)^{-\frac{1}{2}} - I)| \\ &= |\det(-B)| \prod_{j=1}^{n} |\lambda_j - 1|. \end{aligned}$$

Note that λ_j are negative real numbers, for all j, due to A and B being *n.d.* Therefore, we can apply Lemma 2.2. Let $a := \lambda_j$ and $b := -1$. Since the condition $\lambda\mu + \nu\eta \geq 1$ satisfied, due to the assumption of the theorem, the following inequality holds:

$$\begin{aligned} |\det(-B)| \prod_{j=1}^{n} |\lambda_j - 1| &\leq |\det(-B)| \prod_{j=1}^{n} | - \beta + \alpha\lambda_j| \\ &= |\det(-B)||\det(-\beta I + \alpha(-B)^{-\frac{1}{2}}A(-B)^{-\frac{1}{2}})| \\ &= |\det(\alpha A + \beta B)|. \end{aligned}$$

From (2.7), we know that there exists $W \in \mathbb{C}^{n\times k}$, $W^*W = I$, such that

$$\prod_{j=1}^{k} s_j(A+B) = |\det(W^*(A+B)W)| \leq |\det(W^*(\alpha A+\beta B)W)|$$
$$\leq \prod_{j=1}^{k} s_j(W^*(\alpha A+\beta B)W),$$
$$\prod_{j=1}^{k} s_j(W^*(\alpha A+\beta B)W) \leq \prod_{j=1}^{k} s_j(\alpha A+\beta B)$$
$$\Rightarrow \prod_{j=1}^{k} s_j(A+B) \leq \prod_{j=1}^{k} s_j(\alpha A+\beta B).$$

(3) Let A and B be Hermitian *p.d.* It follows that the singular values of $A+B$ are also its eigenvalues, because $A+B$ is also Hermitian *p.d.* Let $j \in \{1,...,n\}$ be fixed, $e_1,...,e_j$ be the eigenvectors of $A+B$ that correspond to the eigenvalues (singular values) $\lambda_1,...,\lambda_j$ of $A+B$, in that order, and let M be the span over $\{e_1,\dots,e_j\}$. Now we have:

$$s_j(A+B) = \min_{\substack{x\in M\\ ||x||=1}} \langle x,(A+B)x\rangle.$$

Note that $\langle x,(A+B)x\rangle = \langle x,Ax\rangle + \langle x,Bx\rangle$, for an arbitrary normed $x \in M$, for an arbitrary $j-$dimensional subspace M of $\mathbb{C}^n$. Since A and B are *p.d.*, it follows that $\langle x,Ax\rangle > 0$ and $\langle x,Bx\rangle > 0$. Thus we have $\langle x,Ax\rangle + \langle x,Bx\rangle = |\langle x,Ax\rangle + \langle x,Bx\rangle| \leq |\alpha\langle x,Ax\rangle + \beta\langle x,Bx\rangle| = |\langle x,(\alpha A+\beta B)x\rangle|$. Using the condition of the theorem: $\mu\lambda+\nu\eta \geq 1$, we can apply Lemma 2.2. Since the majorization holds in all $M < \mathbb{C}^n, x \in M, ||x|| = 1$, we have:

$$\begin{aligned} s_j(A+B) &\leq |\langle x,(\alpha A+\beta B)x\rangle| \leq ||(\alpha A+\beta B)x|| \cdot ||x|| \\ &= ||(\alpha A+\beta B)x|| \\ &\Rightarrow \\ s_j(A+B) &\leq \min_{\substack{x\in M\\ ||x||=1}} ||(\alpha A+\beta B)x|| \leq \max_{\substack{M\langle \mathbb{C}^n\\ dim M=j}} \min_{\substack{x\in M\\ ||x||=1}} ||(\alpha A+\beta B)x|| \\ &= s_j(\alpha A+\beta B). \end{aligned}$$

□

First of all, note that we only applied part (1) of Lemma 2.2, that is, we considered the case where $ab > 0$. This can be generalized: the part (3) of the lemma claims that if $ab = 0$ then $|a+b| \leq |\alpha a+\beta b|$, having $|\alpha|,|\beta| \geq 1$, of course. This enables the following:

Corollary 2.3 *Let A and B be Hermitian matrices, $\mu,\nu,\lambda,\eta \in \mathbb{R}$, and $\alpha := \mu + i\nu, \beta := \lambda + i\eta$ denoted in a way that $|\alpha|,|\beta| \geq 1$, and $\lambda\mu+\nu\eta \geq 1$. Then:*
(1) *If B and $A+B$ are* p.s.d. *and none of the singular values of $A+B$ are smaller than $||B||$, then $\{s_j(A+B)\} \prec_w \{s_j(\alpha A+\beta B)\}$;*

(2) *If A is* n.s.d. *and B is* n.d.*, then* $\{s_j(A+B)\} \prec_{log(w)} \{s_j(\alpha A+\beta B)\}$*;*
(3) *If A and B are* p.s.d.*, then* $s_j\{A+B\} \leq s_j\{\alpha A+\beta B\}$.

Proof. (1) Note that (2.11) will be altered: since B is *p.s.d.*, i.e. $\langle x, Bx\rangle \geq 0, \forall x \in \mathbb{C}^n$, and for all $j = \overline{1,n}$ we have $s_j(A+B) \geq ||B||$, the following stands $ab \geq \langle e_j, Be_j\rangle(s_j(A+B) - ||B||) \geq 0 \Rightarrow ab > 0 \underline{\vee} ab = 0$ both of which, due to Lemma 2.2 and Theorem 2.1, lead to $\{s_j(A+B)\} \prec_w \{s_j(\alpha A+\beta B)\}$.
(2) If A is *n.s.d.*, we have: $\langle x, Ax\rangle \leq 0$, for an arbitrary vector x. Therefore $\langle x, ((-B)^{-\frac{1}{2}}A(-B)^{-\frac{1}{2}}x)\rangle = \langle (-B)^{-\frac{1}{2}}x, A((-B)^{-\frac{1}{2}}x)\rangle \leq 0$ which means that $(-B)^{-\frac{1}{2}}A(-B)^{-\frac{1}{2}}$ is a *n.s.d.*. It follows that $ab = (-1)\cdot(\lambda_j((-B)^{-\frac{1}{2}}A(-B)^{-\frac{1}{2}})) \geq 0$, for every $j \in \{1, ..., n\}$, so we have $ab \geq 0$. In the same way as in (1) of the corollary, this leads to the logarithmic weak majorization.
(3) Obviously, if A and B are *p.s.d.*, then $\langle x, Ax\rangle\langle x, Bx\rangle \geq 0$, for all $x \in \mathbb{C}^n$, so we trivially have $ab \geq 0$, from which the result follows. □

Note that in (2) the condition of B being *n.d.* cannot be weakened: it must be invertible due to the assumption of the existence of $(-B)^{-1}$. If one would suppose that A is positive semi-definite, then one should notice that $((-B)^{-\frac{1}{2}}A(-B)^{-\frac{1}{2}})$ is a positive operator.

Corollary 2.4 *Let A and B be Hermitian matrices, and $\mu, \nu, \lambda, \eta \in \mathbb{R}$, $\alpha := \mu + i\nu, \beta := \lambda + i\eta$ denoted in a way that $|\alpha|, |\beta| \geq 1$, and $\lambda\mu + \nu\eta \leq 1$. Then:*
(1) *If B is negative semi-definite, $A+B$ is positive semi-definite, and none of the singular values of $A+B$ are smaller than $||B||$, then $\{s_j(A+B)\} \prec_w \{s_j(\alpha A+\beta B)\}$;*
(2) *If A is positive semi-definite and B is* n.d.*, then $\{s_j(A+B)\} \prec_{log(w)} \{s_j(\alpha A+\beta B)\}$;*
(3) *If A is positive semi-definite, B is negative semi-definite and $A+B$ is positive semi-definite, then $s_j\{A+B\} \leq s_j\{\alpha A+\beta B\}$.*

Some of the conditions Corollary 2.3 cannot be weakened.
We illustrate our results by several examples. We used the mathematical software MATHEMATICA for computations.

Example 2.1 Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},\ B = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix},$$

be Hermitian square matrices, B is not positive semi definite (it contains a negative number in its spectrum), $\alpha = \beta = i - 10^{-12}$ and $C = A+B, D = \alpha A+\beta B$. It derives that $\{s_1(C), s_2(C)\} = \{\sqrt{11+2\sqrt{10}}, \sqrt{11-2\sqrt{10}}\}$ and $\{s_1(D), s_2(D)\} = \{4.16228, 2.16228\}$. Then the following holds $(s_1(C)+s_2(C)) - (s_1(D)+s_2(D)) \approx 2.16206 > 0$, which means that (1) in Corollary 2.3 is not true.

Example 2.2 Let

$$A = \begin{pmatrix} 10 & 0 \\ 0 & -20 \end{pmatrix},\ B = \begin{pmatrix} -1000 & 0 \\ 0 & -1000 \end{pmatrix},$$

be Hermitian square matrices, B is negative semi definite and A is not definite, $\alpha = 5i, \beta = \frac{i}{5}$ and $C = A+B, D = \alpha A+\beta B$. It derives that $\{s_1(C), s_2(C)\} = \{1020, 990\}$ and $\{s_1(D), s_2(D)\} = \{300, 150\}$, which means that (2) in Corollary 2.3 is not true.

Example 2.3 Let

$$A=\begin{pmatrix}10 & 0\\ 0 & -20\end{pmatrix},\ B=\begin{pmatrix}1000 & 0\\ 0 & 1000\end{pmatrix},$$

be Hermitian square matrices, B is positive semi definite and A is not definite, $\alpha = 5i, \beta = \frac{i}{5}$ and $C = A+B, D = \alpha A+\beta B$. It derives that $\{s_1(C), s_2(C)\} = \{1010, 980\}$ and $\{s_1(D), s_2(D)\} = \{250, 100\}$, which means that (3) in Corollary 2.3 is not true.

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