An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat. (N.S.)
Tomul LXII, 2016, f. 2, vol. 3

A generalization of the $\mathcal{L}_2$-transform and its applications

Ayşenur Güncan · Faruk Uçar

Received: 15.VIII.2013 / Accepted: 25.VIII.2014

Abstract The purpose of this paper is to generalize the Laplace-type $\mathcal{L}_2$ transform to the $\mathcal{L}_{2n}$-transform by introducing the number n in the formula for $\mathcal{L}_2$. A similar generalization $\mathcal{P}_{2n}$ of the Widder potential-transform is also given. The main result is a collection of formulas linking e.g. the integral of a product of two $\mathcal{P}_{2n}$-transforms in terms of an integral of one of the functions against the the $\mathcal{P}_{2n}$-transform of the other. This equation is a kind of Parseval identity, as known from Fourier analysis.

Keywords Laplace transforms · Widder potential transforms · $\mathcal{L}_2$-transforms · $\mathcal{E}_{2,1}$-transforms · Parseval-Goldstein type theorems

Mathematics Subject Classification (2010) 44A10 · 44A20 · 44A15

1 Introduction

The following Laplace-type transform:

$$\mathcal{L}_2\big\{f(x);y\big\} = \int_0^\infty x\,\exp\big(-x^2\,y^2\big)\,f(x)\,dx \tag{1.1}$$

was introduced by Yürekli and Sadek [16]. Subsequently, various Parseval-Goldstein type identities for the $\mathcal{L}_2$-transform were given in [1, 2, 13, 14]. The $\mathcal{L}_2$-transform is related to the classical Laplace transform:

$$\mathcal{L}\big\{f(x);y\big\} = \int_0^\infty \exp\big(-x\,y\big)\,f(x)\,dx, \tag{1.2}$$

Ayşenur Güncan
Department of Mathematics
Süleyman Demirel University
TR-32100, Isparta, Turkey
E-mail: aysenurdayioglu@sdu.edu.tr

Faruk Uçar
Department of Mathematics
Marmara University
TR-34722, Kadıköy, İstanbul, Turkey
E-mail: fucar@marmara.edu.tr

by means of the following relationships:

$$\mathcal{L}_2\Big\{f(x);y\Big\} = \frac{1}{2}\,\mathcal{L}\Big\{f\big(x^{1/2}\big);y^2\Big\}, \tag{1.3}$$

and

$$\mathcal{L}_2\Big\{f(x);y\Big\} = 2\,\mathcal{L}\Big\{f\big(x^2\big);y^{1/2}\Big\}. \tag{1.4}$$

There are numerous analogous results in the literature on integral transforms (see, for instance, [9,12,15,17]).

In 1966, Widder [11] introduced so-called the Widder potential-transform as follows:

$$\mathcal{P}\left\{f(x);y\right\} = \int_0^\infty \frac{x\,f(x)}{x^2+y^2}\,dx. \tag{1.5}$$

Subsequently, Srivastava and Singh [8] gave the following Parseval-Goldstein type identity

$$\int_0^\infty y\,\mathcal{P}\left\{f(x);y\right\}\,g(y)\,dy = \int_0^\infty x\,\mathcal{P}\left\{g(y);x\right\}f(x)\,dx. \tag{1.6}$$

The Widder potential transform (1.5) is related to the Stieltjes transform as follows (see also [9,10]):

$$\mathcal{P}\left\{f(x);y\right\} = \frac{1}{2}\,\mathcal{S}\left\{f\big(x^{1/2}\big);y^2\right\}, \tag{1.7}$$

where the Stieltjes transform is defined by

$$\mathcal{S}\left\{f(x);y\right\} = \int_0^\infty \frac{f(x)}{x+y}\,dx. \tag{1.8}$$

The following Stieltjes-type transform:

$$\mathcal{G}\Big\{f(x);y\Big\} = \int_0^\infty \frac{f(x)}{\sqrt{x^2+y^2}}\,dx \tag{1.9}$$

was introduced by Glasser [6].

Brown *et al.* [1] introduced the exponential integral transform

$$\mathcal{E}_1\Big\{f(x);y\Big\} = \int_0^\infty \exp(xy)E_1(xy)f(x)\,dx, \tag{1.10}$$

where $E_1(x)$ is the exponential integral function defined by

$$E_1(x) = -E_i(-x) = \int_x^\infty \frac{\exp(-u)}{u}du = \int_1^\infty \frac{\exp(-xt)}{t}dt.$$

In this article, we introduce several new integral transforms and establish potentially useful identities for these integral transforms. First of all, the $\mathcal{L}_{2n}$-transform is defined by

$$\mathcal{L}_{2n}\Big\{f(x);y\Big\} = \int_0^\infty x^{2n-1}\,\exp\big(-x^{2n}\,y^{2n}\big)\,f(x)\,dx, \tag{1.11}$$

for all $n \in \mathbb{N}$ which is related to the Laplace transform (1.2) as follows:

$$\mathcal{L}_{2n}\{f(x);y\} = \frac{1}{2n}\mathcal{L}\Big\{f\big(x^{1/2n}\big);y^{2n}\Big\}. \tag{1.12}$$

The $\mathcal{P}_{2n}$-transform is defined by

$$\mathcal{P}_{2n}\{f(x);y\} = \int_0^\infty \frac{x^{2n-1}\,f(x)}{x^{2n}+y^{2n}}\,dx, \tag{1.13}$$

for all $n \in \mathbb{N}$ which is related to the Widder potential transform (1.5) as follows:

$$\mathcal{P}_{2n}\{f(x);y\} = \frac{1}{n}\mathcal{P}\Big\{f\{x^{1/n}\};y^{n}\Big\}, \tag{1.14}$$

and the Stieltjes transform (1.8) as follows:

$$\mathcal{P}_{2n}\{f(x);y\} = \frac{1}{2n}\,\mathcal{S}\Big\{f\big(x^{1/2n}\big);y^{2n}\Big\}. \tag{1.15}$$

Remark 1.1 If we write $n = 2$ in (1.14), then we find following known identity due to Dernek, Srivastava and Yürekli [2]:

$$\mathcal{P}_{4}\{f(x);y\} = \frac{1}{2}\mathcal{P}\Big\{f\{x^{1/2}\};y^{2}\Big\}.$$

The $\mathcal{G}_n$-transform is defined by

$$\mathcal{G}_{n}\{f(x);y\} = \int_0^\infty \frac{x^{n-1}\,f(x)}{(x^{2n}+y^{2n})^{1/2}}\,dx, \tag{1.16}$$

for all $n \in \mathbb{N}$ which is related to the Glasser transform (1.9) as follows:

$$\mathcal{G}_{n}\{f(x);y\} = \frac{1}{n}\mathcal{G}\Big\{f\big(x^{1/n}\big);y^{n}\Big\}. \tag{1.17}$$

The $\mathcal{E}_{2n,1}$-transform is defined by

$$\mathcal{E}_{2n,1}\,\{f(x);y\} = \int_0^\infty x^{2n-1}\exp\big(x^{2n}\,y^{2n}\big)E_1\big(x^{2n}y^{2n}\big)f(x)\,dx, \tag{1.18}$$

for all $n \in \mathbb{N}$.

Remark 1.2 The $\mathcal{E}_{2n,1}$-transform is related to the $\mathcal{E}_{2,1}$-transform due to [1] as follows:

$$\mathcal{E}_{2n,1}\{f(x);y\} = \frac{1}{2n}\mathcal{E}_{2,1}\Big\{f\big(x^{1/n}\big);y^{2n}\Big\}. \tag{1.19}$$

In this paper, we consider several new integral transforms including the $\mathcal{L}_{2n}$- transform, the $\mathcal{P}_{2n}$-transform and the $\mathcal{E}_{2n,1}$-transform as generalizations of the classical Laplace transform, Widder potential transform and the Error transform, respectively. Many identities involving these transforms are given. By making use of these identities, a number of new Parseval-Goldstein type identities are obtained for these and many other well-known integral transforms. The identities proven in this paper are shown to give rise to useful corollaries for evaluating infinite integrals of special functions. Some example are also given as illustrations of the results presented here.

2 The main theorem

In this section, we derive several identities involving the $\mathcal{L}_{2n}$-transform, $\mathcal{P}_{2n}$-transform, $\mathcal{G}_n$-transform and the $\mathcal{E}_{2n,1}$-transform. Using these identities, a number of new Parseval - Goldstein type identities are then obtained for these and many other well-known integral transforms. Our main theorem is shown to yield new identities for the integral transforms introduced above. In the following lemma, we give identities involving the $\mathcal{L}_{2n}$-transform (1.11) and the $\mathcal{P}_{2n}$-transform (1.13).

Lemma 2.1 *The following iteration identity*

$$\mathcal{L}_{2n}\Big\{\mathcal{L}_{2n}\{f(x);u\};y\Big\} = \frac{1}{2n}\mathcal{P}_{2n}\{f(x);y\} \tag{2.1}$$

holds true if the integrals involved converge absolutely.

Proof. In order to prove (2.1), we begin by using the definition of the $\mathcal{L}_{2n}$-transform (1.11). We thus observe that

$$\mathcal{L}_{2n}\Big\{\mathcal{L}_{2n}\{f(x);u\};y\Big\} = \int_0^\infty u^{2n-1}\exp\left(-u^{2n}y^{2n}\right)\cdot \cdot\left(\int_0^\infty x^{2n-1}\exp\big(-x^{2n}u^{2n}\big)f(x)dx\right)du. \tag{2.2}$$

Changing the order of integration, which is permissible by absolute convergence of the integrals involved, it follows from (2.2) that

$$\mathcal{L}_{2n}\Big\{\mathcal{L}_{2n}\{f(x);u\};y\Big\} = \int_0^\infty x^{2n-1}f(x)\cdot \Big(\int_0^\infty u^{2n-1}\exp\big(-(x^{2n}+y^{2n})u^{2n}\big)du\Big)dx. \tag{2.3}$$

Next, by evaluating the inner integral on the right-hand side of (2.3), we obtain

$$\mathcal{L}_{2n}\Big\{\mathcal{L}_{2n}\{f(x);u\};y\Big\} = \frac{1}{2n}\int_0^\infty \frac{x^{2n-1}f(x)}{x^{2n}+y^{2n}}\,dx. \tag{2.4}$$

Finally, by appliying the definition (1.13) on the right hand-side of (2.4), we deduce the identity (2.2) asserted by the Lemma. □

Remark 2.1 If we write $n = 2$ in (2.2), then we find following known identities due to Dernek, Srivastava and Yürekli [2]:

$$\mathcal{L}_{4}\Big\{\mathcal{L}_{4}\{f(x);u\};y\Big\} = \frac{1}{4}\mathcal{P}_{4}\{f(x);y\}$$

by Dernek, Srivastava and Yürekli [3, identity (23)].

Theorem 2.2 *The following Parseval-Goldstein type identities*

$$\int_0^\infty y^{2n-1}\mathcal{L}_{2n}\big\{f(x);y\big\}\mathcal{L}_{2n}\big\{g(u);y\big\}dy$$
$$= \frac{1}{2n}\int_0^\infty x^{2n-1}f(x)\mathcal{P}_{2n}\big\{g(u);x\big\}\,dx, \tag{2.5}$$

$$\int_0^\infty y^{2n-1}\mathcal{L}_{2n}\big\{f(x);y\big\}\mathcal{L}_{2n}\big\{g(u);y\big\}dy$$
$$= \frac{1}{2n}\int_0^\infty u^{2n-1}g(u)\mathcal{P}_{2n}\big\{f(x);u\big\}\,du, \tag{2.6}$$

$$\int_0^\infty x^{2n-1}f(x)\mathcal{P}_{2n}\big\{g(u);x\big\}dx = \int_0^\infty u^{2n-1}g(u)\mathcal{P}_{2n}\big\{f(x);u\big\}\,du \tag{2.7}$$

hold true if the integrals involved converge absolutely.

Proof. We only give the proof of (2.5). The proof of (2.6) is similar to the proof of (2.5). The proof of (2.7) follows easily from (2.5) and (2.6). By the definition of the $\mathcal{L}_{2n}$-transform (1.11) and changing the order of integration, which is permissible by absolute convergence of the integrals involved, we have

$$\int_0^\infty y^{2n-1}\mathcal{L}_{2n}\big\{f(x);y\big\}\mathcal{L}_{2n}\big\{g(u);y\big\}\,dy$$
$$= \int_0^\infty u^{2n-1}g(u)\Big(\int_0^\infty y^{2n-1}\mathcal{L}_{2n}\big\{f(x);y\big\}\exp\big(-y^{2n}\,u^{2n}\big)\,dy\Big)\,du.$$

Using the definition (1.11) of the $\mathcal{L}_{2n}$-transform once again, we find that

$$\int_0^\infty y^{2n-1}\mathcal{L}_{2n}\big\{f(x);y\big\}\mathcal{L}_{2n}\big\{g(u);y\big\}\,dy$$
$$= \int_0^\infty u^{2n-1}g(u)\;\mathcal{L}_{2n}\Big\{\mathcal{L}_{2n}\big\{f(x);y\big\};u\Big\}\,du$$

and by result of Lemma 2.1, we get

$$\int_0^\infty y^{2n-1}\mathcal{L}_{2n}\big\{f(x);y\big\}\mathcal{L}_{2n}\big\{g(u);y\big\}\,dy$$
$$= \frac{1}{2n}\int_0^\infty u^{2n-1}g(u)\;\mathcal{P}_{2n}\big\{f(x);u\big\}\,du.$$

□

Remark 2.2 For $n = 2$ from (2.5), (2.6) and (2.7) we get the theorem, which held before by Dernek, Srivastava and Yürekli ([2, p. 402]). Therefore, Theorem 2.2 is a generalization of the earlier Theorem.

Corollary 2.3

$$\mathcal{P}_{2n}\left\{\mathcal{L}_{2n}\left\{f(x);y\right\};z\right\}=\mathcal{L}_{2n}\left\{\mathcal{P}_{2n}\left\{f(x);u\right\};z\right\} \tag{2.8}$$

holds true if the integrals involved converge absolutely.

Proof. We set $g(u)=\exp(-u^{2n}z^{2n})$ in the assertion (2.6) of the Theorem 2.2. By making use of the identity (1.11) and evaluating the integral, we obtain

$$\mathcal{L}_{2n}\left\{g(u);y\right\}=\mathcal{L}_{2n}\left\{\exp(-u^{2n}z^{2n});y\right\}=\frac{1}{2n}\frac{1}{y^{2n}+z^{2n}}. \tag{2.9}$$

The assertion (2.8) follows from by substituting the result (2.9) into Parseval-Goldstein type identity (2.6). □

Lemma 2.4 *The following identity*

$$\mathcal{P}_{2n}\Big\{\mathcal{L}_{2n}\big\{f(x);u\big\};z\Big\}=\frac{1}{2n}\mathcal{E}_{2n,1}\big\{f(x);z\big\} \tag{2.10}$$

holds true if the integrals involved converge absolutely.

Proof. We set $g(u)=\exp\big\{-u^{2n}\,z^{2n}\big\}$ in the assertion (2.2) of the Theorem 2.2 as in the proof of the Corollary 2.3 and using the identitiy (1.15) and the formula ([5], p.217, Entry (11)) we have

$$\begin{aligned}\mathcal{P}_{2n}\big\{g(u);x\big\}&=\frac{1}{2n}\mathcal{S}\Big\{g\big(u^{1/2n}\big);x^{2n}\Big\}=\frac{1}{2n}\mathcal{S}\Big\{\exp\big(-uz^{2n}\big);x^{2n}\Big\}\\&=\exp\big(x^{2n}\,z^{2n}\big)E_1\big(x^{2n}\,z^{2n}\big).\end{aligned}$$

Putting these results in (2.5), we have

$$\begin{aligned}&\int_0^\infty\frac{y^{2n-1}}{y^{2n}+z^{2n}}\mathcal{L}_{2n}\big\{f(x);y\big\}\,dy\\&\quad=\int_0^\infty x^{2n-1}f(x)\mathcal{P}_{2n}\big\{\exp\big\{-u^{2n}\,z^{2n}\big\};x\big\}\,dx\\&\mathcal{P}_{2n}\Big\{\mathcal{L}_{2n}\big\{f(x);y\big\};z\Big\}\\&\quad=\frac{1}{2n}\int_0^\infty x^{2n-1}f(x)\exp\big\{x^{2n}\,z^{2n}\big\}E_1\big(x^{2n}\,z^{2n}\big)\,dx\end{aligned}$$

and the taking into account (1.18) the definition of the $\mathcal{E}_{2n,1}$-transform, we get the desired result. □

Lemma 2.5 *The following iteration identity*

$$\mathcal{L}_{2n}\Big\{\frac{1}{u^n}\mathcal{L}_{2n}\big\{f(x);u\big\};y\Big\}=\frac{\sqrt{\pi}}{2n}\mathcal{G}_n\big\{x^nf(x);y\big\} \tag{2.11}$$

holds true if the integrals involved converge absolutely.

Proof. To prove (2.11), we start with using the definition of the $\mathcal{L}_{2n}$-transform,

$$\mathcal{L}_{2n}\Big\{\frac{1}{u^n}\mathcal{L}_{2n}\{f(x);u\};y\Big\}$$
$$=\mathcal{L}_{2n}\Big\{\frac{1}{u^n}\Big\{\int_0^\infty x^{2n-1}\exp\big(-x^{2n}u^{2n}\big)f(x)dx\Big\};y\Big\}$$
$$=\int_0^\infty u^{n-1}\exp\big(-u^{2n}y^{2n}\big)\Big\{\int_0^\infty x^{2n-1}\exp\big(-x^{2n}u^{2n}\big)f(x)dx\Big\}du.$$

Changing the order of integration, which is permissible by absolute convergence of the integrals involved, we have

$$\mathcal{L}_{2n}\Big\{\frac{1}{u^n}\mathcal{L}_{2n}\{f(x);u\};y\Big\}$$
$$=\int_0^\infty x^{2n-1}f(x)\Big(\int_0^\infty u^{n-1}\exp\big(-(x^{2n}+y^{2n})u^{2n}\big)du\Big)dx. \tag{2.12}$$

Evaluating the inner integral on the right-hand side of (2.12), we obtain

$$\mathcal{L}_{2n}\Big\{\frac{1}{u^n}\mathcal{L}_{2n}\{f(x);u\};y\Big\}=\frac{\sqrt{\pi}}{2n}\int_0^\infty \frac{x^{n-1}}{(x^{2n}+y^{2n})^{1/2}}x^n f(x)dx$$

and using the definition of $\mathcal{G}_n$-transform given by (1.16), we get the desired result. □

Remark 2.3 For $n=1$ from (2.11) we get the lemma, which held before by DERNEK *et al.* [4, Lemma (2.1)]

$$\mathcal{L}_2\Big\{\frac{1}{u}\mathcal{L}_2\{f(x);u\};y\Big\}=\frac{\sqrt{\pi}}{2}\mathcal{G}\{xf(x);y\}. \tag{2.13}$$

3 Illustrative examples

In this section of paper, we give some illustrative examples as applications of the identities (contained in Lemmas) and the Theorem. An interesting illustration for the identity (2.10) asserted by Lemma 2.4 is contained in the following examples.

Example 3.1 We show that $\int_0^\infty \frac{x^{n(1-v)-1}}{x^{2n}+y^{2n}}dx=\frac{\pi}{4n}\sec\left(\frac{v\pi}{2}\right)y^{-2n(v+1)}$, where $|\mathrm{Re}\,v|<1$.

Proof. We set $f(x)=x^{n(v-1)}$ $(-1<\mathrm{Re}\,v<1)$ in the identity (2.10). Making use of (1.12) and (1.19), then tables of Laplace transforms (see [7, p.15, (1)]) and BROWN *et al.* [1, p.1559, (2.8)], we obtain

$$\mathcal{L}_{2n}\{f(x);u\}=\frac{1}{2n}\Gamma\left(\frac{\nu+1}{2}\right)u^{-n(v+1)}, \tag{3.1}$$

and

$$\mathcal{E}_{2n,1}\{f(x);z\}=\frac{\pi}{4n}\sec\left(\frac{v\pi}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)z^{-2n(v+1)}. \tag{3.2}$$

Substituting the function f into (2.10), and using (3.1) and (3.2), we find

$$\mathcal{P}_{2n}\left\{u^{-n(v+1)};z\right\}=\frac{\pi}{4n}\sec\left(\frac{v\pi}{2}\right)z^{-2n(v+1)}.$$

□

Example 3.2 We show that

$$\int_0^\infty \frac{x^{n-1}}{x^{2n}+y^{2n}}e^{-a^2/4x^{2n}}dx=\frac{\pi}{4ny^{2n}}e^{a^2/4y^{4n}}Erfc\left(\frac{a}{2y^{2n}}\right),$$

where $|\mathrm{Re}\,v|<1$.

Proof. We set $f(x)=x^{-n}\cos(ax^n)$ in the identity (2.10). Making use of (1.12) and (1.19), then tables of Laplace transforms (see [7, p.82, (19)]) and Brown *et al.* [1, p.1559, (2.15)], we obtain

$$\mathcal{L}_{2n}\left\{f(x);u\right\}=\frac{\sqrt{\pi}}{2nu^n}e^{-a^2/4u^{2n}}, \tag{3.3}$$

$$\mathcal{E}_{2n,1}\left\{f(x);z\right\}=\frac{1}{4n}\frac{\pi^{3/2}}{z^{2n}}e^{a^2/4z^{4n}}Erfc\left(\frac{a}{2z^{2n}}\right), \tag{3.4}$$

where $Erfc(x)$ is complementary error function. Substituting the function f into (2.10), and using (3.3) and (3.4), we find that $\mathcal{P}_{2n}\{u^{-n}e^{-a^2/4u^{2n}};z\}=\frac{\pi}{4nz^{2n}}e^{a^2/4z^{4n}}Erfc(\frac{a}{2z^{2n}})$. □

Example 3.3 We show that

$$\int_0^\infty \frac{x^{-(2n\nu+1)}}{x^{2n}+y^{2n}}e^{-a^2/4x^{2n}}dx=\frac{1}{4n}\frac{a^\nu\Gamma(\nu+1)}{2^{\nu+1}y^{4n(\nu+1)}}e^{a^2/4n}\Gamma\left(-\nu,a^2/4y^{4n}\right),$$

where $\mathrm{Re}\,v>-1$ and $|\arg a|<\pi$.

Proof. We set $f(x)=x^{nv}J_v(ax^n)$ $(\mathrm{Re}\,v>-1,|\arg a|<\pi)$ in the identity (2.10). Making use of (1.12) and (1.19), then tables of Laplace transforms (see [7, p.262, (8)]) and Brown *et al.* [1, p.1560, (2.23)], we obtain

$$\mathcal{L}_{2n}\left\{f(x);u\right\}=\frac{a^\nu}{n2^{\nu+1}}u^{-2n(\nu+1)}e^{-a^2/4u^{2n}}, \tag{3.5}$$

$$\mathcal{E}_{2n,1}\left\{f(x);z\right\}=\frac{1}{2n}\frac{a^\nu\Gamma(\nu+1)}{2^{\nu+1}z^{4n(\nu+1)}}e^{a^2/4z^{4n}}\Gamma\left(-\nu,a^2/4z^{4n}\right), \tag{3.6}$$

where $\Gamma(a,x)$ is incomplete gamma function. Substituting the function f into (2.10), and using (3.5) and (3.6), we find that

$$\mathcal{P}_{2n}\left\{u^{-2n(\nu+1)}e^{-a^2/4u^{2n}};z\right\}=\frac{\Gamma(\nu+1)}{4nz^{4n(\nu+1)}}e^{a^2/4z^{4n}}\Gamma\left(-\nu,a^2/4z^{4n}\right).$$

□

An interesting illustration for the identity (2.11) asserted by Lemma 2.5 is contained in the following examples.

Example 3.4 We show that

$$\int_0^\infty \frac{x^{nv-1}}{(x^{2n}+y^{2n})^{1/2}}dx = \frac{\pi^{-1/2}}{2n}\Gamma\left(\frac{v+1}{2}\right)\Gamma\left(\frac{-v}{2}\right)y^{n\nu},$$

where $-1 < \operatorname{Re} v < 0$.

Proof. We set $f(x) = x^{n(v-1)}$ $(-1 < \operatorname{Re} v < 0)$ in identity (2.11) of Lemma 2.5. Making use of (1.12), and then tables of Laplace transforms (see [7, p.15, (1)]), we obtain

$$\mathcal{L}_{2n}\left\{f(x);u\right\} = \frac{1}{2n}\Gamma\left(\frac{\nu+1}{2}\right)u^{-n(v+1)}. \tag{3.7}$$

Substituting the function f into (2.11), and using (3.7), we find that

$$\begin{aligned}\mathcal{G}_n\left\{x^{nv};y\right\} &= \frac{2n}{\sqrt{\pi}}\mathcal{L}_{2n}\left\{\frac{1}{u^n}\mathcal{L}_{2n}\left\{x^{n(v-1)};u\right\};y\right\}\\ &= \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{v+1}{2}\right)\mathcal{L}_{2n}\left\{(u^n)^{-v-2};y\right\}\\ &= \frac{1}{2n\sqrt{\pi}}\Gamma\left(\frac{v+1}{2}\right)\Gamma\left(\frac{-v}{2}\right)y^{nv}.\end{aligned}$$

□

Example 3.5 We show that

$$\int_0^\infty \frac{x^{2n-1}sin\left(ax^n\right)}{(x^{2n}+y^{2n})^{1/2}}dx = \frac{y^n}{n}K_{-1}\left(ay^n\right),$$

where K_ν is the Bessel function of the second kind of order ν.

Proof. We set $f(x) = \sin(ax^n)$ $(\operatorname{Re} v > 0)$ in identity (2.11) of Lemma 2.5. Making use of (1.12), and then tables of Laplace transforms (see [7, p.80, (6)]), we obtain

$$\mathcal{L}_{2n}\left\{f(x);u\right\} = \frac{a\sqrt{\pi}}{4nu^{3n}}e^{-a^2/4u^{2n}}. \tag{3.8}$$

Substituting the function f into (2.11), and using (3.8) and [7, p.31, (1)], we find that

$$\begin{aligned}\mathcal{G}_n\left\{x^n\sin\left(ax^n\right);y\right\} &= \frac{2n}{\sqrt{\pi}}\mathcal{L}_{2n}\left\{\frac{1}{u^n}\mathcal{L}_{2n}\left\{\sin\left(ax^n\right);u\right\};y\right\}\\ &= \frac{a}{2}\mathcal{L}_{2n}\left\{u^{-4n}e^{-a^2/4u^{2n}};y\right\} = \frac{y^n}{n}K_{-1}\left(ay^n\right).\end{aligned}$$

□

Example 3.6 We show that

$$\int_0^\infty \frac{x^{n-1}cos(ax^n)}{(x^{2n}+y^{2n})^{1/2}}dx = \frac{1}{n}K_0(ay^n),$$

where K_ν is the Bessel function of the second kind of order ν.

Proof. We set $f(x) = x^{-n}\cos(ax^n)$ $(\operatorname{Re} a > 0)$ in identity (2.11) of Lemma 2.5. Making use of (1.12), and then tables of Laplace transforms (see [7, p.82, (19)]), we obtain

$$\mathcal{L}_{2n}\{f(x);u\} = \frac{\sqrt{\pi}}{2nu^n}e^{-a^2/4u^{2n}}. \tag{3.9}$$

Substituting the function f into (2.11), and using (3.9) and [7, p.31, (1)], we find that

$$\begin{aligned}\mathcal{G}_n\{\cos(ax^n);y\} &= \frac{2n}{\sqrt{\pi}}\mathcal{L}_{2n}\left\{\frac{1}{u^n}\mathcal{L}_{2n}\left\{\frac{\cos(ax^n)}{x^n};u\right\};y\right\}\\ &= \mathcal{L}_{2n}\left\{u^{-2n}e^{-a^2/4u^{2n}};y\right\} = \frac{1}{2n}\mathcal{L}\left\{u^{-1}e^{-a^2/4u};y^{2n}\right\}\\ &= \frac{1}{n}K_0(ay^n).\end{aligned}$$

□

Example 3.7 We show that

$$\int_0^\infty \frac{x^{n(v+1)-1}}{(x^{2n}+y^{2n})^{1/2}}dx = \frac{1}{n}\sqrt{\frac{2\pi}{a}}(y^n)^{v+1/2}K_{-v-1/2}(ay^n),$$

where K_ν is the Bessel function of the second kind of order ν.

Proof. We set $f(x) = x^{nv}J_v(ax^n)$ $(\operatorname{Re}\nu > -1, \operatorname{Re} a > 0, |\arg a| < \pi)$ in identity (2.11) of Lemma 2.5. Making use of (1.12), and then tables of Laplace transforms (see [7, p.262, (8)]), we obtain

$$\mathcal{L}_{2n}\{f(x);u\} = \frac{a^\nu}{2^{\nu+1}n}u^{-2n(\nu+3)}e^{-a^2/4u^{2n}}. \tag{3.10}$$

Substituting the function f into (2.11), and using (3.10) and [7, p.31, (1)], we find that

$$\begin{aligned}\mathcal{G}_n\left\{x^{n(v+1)}J_v(ax^n);y\right\} &= \frac{2n}{\sqrt{\pi}}\mathcal{L}_{2n}\left\{\frac{1}{u^n}\mathcal{L}_{2n}\{x^{nv}J_v(ax^n);u\};y\right\}\\ &= \frac{1}{n}\sqrt{\frac{2}{\pi a}}(y^n)^{v+1/2}K_{-v-1/2}(ay^n).\end{aligned}$$

□

We conclude by remarking that many other infinite integrals can be evaluated in this manner by applying Lemmas as well as the Theorem and its corollaries considered here.

References

1. Brown, D.; Dernek, N.; Yürekli, O. – *Identities for the $\mathcal{E}_{2,1}$-transform and their applications*, Appl. Math. Comput., 187 (2007), 1557–1566.
2. Dernek, N.; Srivastava, H.M.; Yürekli, O. – Parseval-Goldstein type identities involving the $\mathcal{L}_4$-transform and the $\mathcal{P}_4$-transform and their applications, Integral Transform. Spec. Funct., 18 (2007), 397–408.
3. Dernek, N.; Srivastava, H.M.; Yürekli, O. – *Some Parseval-Goldstein type identities involving the $\mathcal{F}_{S,2}$-transform, the $\mathcal{F}_{C,2}$-transform and the $\mathcal{P}_4$-transform and their applications*, Appl. Math. Comput., 202 (2008), 327–337.
4. Dernek, A.; Dernek, N.; Yürekli, O. – *New identities for the Glasser transform and their applications*, arXiv:0807.4041v1 [math.CA].
5. Erdélyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F.G. – *Tables of Integral Transforms*, Vol. II, Based, in part, on notes left by Harry Bateman. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954.
6. Glasser, M.L. – *Some Bessel function integrals*, Kyungpook Math. J., 13 (1973), 171–174.
7. Prudnikov, A.P.; Brychkov, Yu.A.; Marichev, O.I. – *Integrals and Series*, Vol. 4, Direct Laplace transforms, Gordon and Breach Science Publishers, New York, 1992.
8. Srivastava, H.M.; Singh, S.P. – *A note on the Widder transform related to the Poisson integral for a half-plane*, Internat. J. Math. Ed. Sci. Tech., 16 (1985), 675–677.
9. Srivastava, H.M.; Yürekli, O. – *A theorem on a Stieltjes-type integral transform and its applications*, Complex Variables Theory Appl., 28 (1995), 159–168.
10. Srivastava, R. – *Some inversion formulas for the Widder potential transform*, Integral Transform. Spec. Funct., 3 (1995), 235–242.
11. Widder, D.V. – *A transform related to the Poisson integral for a half-plane*, Duke Math. J., 33 (1966), 355–362.
12. Yürekli, O. – *A Parseval-type theorem applied to certain integral transforms*, IMA J. Appl. Math., 42 (1989), 241–249.
13. Yürekli, O. – *Theorems on $\mathcal{L}_2$-transforms and its applications*, Complex Variables Theory Appl., 38 (1999), 95–107.
14. Yürekli, O. – *New identities involving the Laplace and the $\mathcal{L}_2$-transforms and their applications*, Appl. Math. Comput., 99 (1999), 141–151.
15. Yürekli, O.; Graziadio, C. – *A theorem on the Laplace transform and its applications*, Internat. J. Math. Ed. Sci. Tech., 28 (1997), 616–621.
16. Yürekli, O.; Sadek, I. – *A Parseval-Goldstein type theorem on the Widder potential transform and its applications*, Internat. J. Math and Math. Sci., 14 (1991), 517–524.
17. Yürekli, O.; Sayginsoy, O. – A theorem on a Laplace-type integral transform and its applications, Internat. J. Math. Ed. Sci. Tech., 29 (1998), 561–567.