

## Weak and strong convergence theorems for generalized nonexpansive mappings

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**Abstract** We consider a class of generalized nonexpansive mappings introduced by Karapinar and seen as a generalization of Suzuki (C)-condition. We prove some weak and strong convergence theorems for approximating fixed points of such mappings under suitable conditions in uniformly convex Banach spaces. Our results generalize those of KHAN and Suzuki to the case of this kind of mappings and, in turn, are related to a famous convergence theorem of Reich on nonexpansive mappings.

**Keywords** Generalized nonexpansive mappings · Fixed point · Convergence · Kadec–Klee property · Condition (I)

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### 1 Introduction

Let  $E$  be a Banach space and let  $K$  be a nonempty subset of  $E$ . A mapping  $T$  on  $K$  is called nonexpansive if  $\|Tx - Ty\| \leq \|x - y\|$ , for all  $x, y \in K$ . The set of fixed points of  $T$  is denoted by  $F(T)$ , i.e.,  $F(T) = \{x \in K : Tx = x\}$ . It is well known that if  $E$  is uniformly convex and  $K$  is a bounded, closed and convex subset of  $E$ , then  $F(T)$  is nonempty. Recall that a mapping  $T : K \rightarrow K$  is called quasi-nonexpansive [3] if  $\|Tx - p\| \leq \|x - p\|$ , for all  $x \in K$  and  $p \in F(T)$ . It is easy to see that every nonexpansive mappings with a fixed point is quasi-nonexpansive mapping.

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In 2008, SUZUKI [11] introduced the concept of generalized nonexpansive mappings (Condition (C)). This concept is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness.

**Condition (C).** For a mapping  $T$  defined from a subset  $K$  of a Banach space  $E$  into itself,  $T$  is said to satisfy the condition (C) if  $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$  implies  $\|Tx - Ty\| \leq \|x - y\|$ , for all  $x, y \in K$ .

SUZUKI [11] proved the following theorems for mappings satisfying this condition:

**Theorem 1.1 ([11])** *Let  $T$  be a mapping on a compact convex subset  $K$  of a Banach space  $E$ . Assume that  $T$  satisfies condition (C). Define a sequence  $\{x_n\}$  in  $K$  by  $x_1 \in K$  and  $x_{n+1} = \alpha Tx_n + (1 - \alpha)x_n$ , where  $\alpha$  is a real number belonging to  $[1/2, 1)$ . Then  $\{x_n\}$  converges strongly to a fixed point of  $T$ .*

**Theorem 1.2 ([11])** *Let  $T$  be a mapping on a weakly compact convex subset  $K$  of a Banach space  $E$  with the Opial property. Assume that  $T$  satisfies condition (C). Define a sequence  $\{x_n\}$  in  $K$  by  $x_1 \in K$  and  $x_{n+1} = \alpha Tx_n + (1 - \alpha)x_n$ , where  $\alpha$  is a real number belonging to  $[1/2, 1)$ . Then  $\{x_n\}$  converges weakly to a fixed point of  $T$ .*

It follows from Theorem 1.1 and Theorem 1.2 that for the mapping  $T$  satisfying the condition (C) defined on a convex subset  $C$  of a Banach space  $E$ , if either  $C$  is compact or  $C$  is a weakly compact and  $E$  has the Opial property, then  $T$  has a fixed point (see [11, Theorem 4]).

In 2013, KHAN and SUZUKI [8] gave the following weak convergence theorem for the mappings satisfying condition (C) defined on a bounded closed convex subset  $K$  of a uniformly convex Banach space  $E$ :

**Theorem 1.3 ([8])** *Let  $E$  be a uniformly convex Banach space whose dual has the Kadec–Klee property. Let  $T$  be a mapping on a bounded, closed and convex subset  $K$  of  $E$ . Assume that  $T$  satisfies condition (C). Define a sequence  $\{x_n\}$  in  $K$  by  $x_1 \in K$  and  $x_{n+1} = \alpha Tx_n + (1 - \alpha)x_n$ , where  $\alpha$  is a real number belonging to  $[1/2, 1)$ . Then  $\{x_n\}$  converges weakly to a fixed point of  $T$ .*

In 2013, KARAPINAR [7] introduced a new class of mappings as a modification of mappings satisfying the condition (C) of SUZUKI [11].

**Definition 1.4 ([7])** Let  $T$  be a mapping on a subset  $K$  of a Banach space  $E$ . Then  $T$  is said to satisfy Reich-Suzuki-(C) condition (in short, (RSC)-condition) if  $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$  implies that  $\|Tx - Ty\| \leq \frac{1}{3}(\|x - y\| + \|y - Ty\| + \|x - Tx\|)$ , for all  $x, y \in K$ .

Also, Karapinar gave some properties of this kind of mappings and proved some weak and strong convergence theorems for the mappings satisfying the (RSC)-condition in Banach spaces. We will need the following which is Proposition 1 of KARAPINAR [7].

**Proposition 1.5** *If a mapping  $T$  satisfies (RSC)-condition and has a fixed point, then it is quasi-nonexpansive mapping.*

In this paper, we prove some weak and strong convergence theorems in uniformly convex Banach spaces. For weak convergence theorem, we assume that dual of uniformly convex Banach space has the Kadec-Klee property.

## 2 Preliminaries

Throughout this paper, we assume that all Banach spaces are real and denote by  $\mathbb{N}$  the set of all positive integers unless stated otherwise. In this section, we give some definitions, propositions and lemmas to use in the next section.

**Definition 2.1 ([2])** Let  $E$  be a Banach space.  $E$  is called uniformly convex if for each  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $\|x + y\| \leq 2 - \delta$ , for all  $x, y \in E$  with  $\|x\| = \|y\| = 1$  and  $\|x - y\| \geq \varepsilon$ .

Uniformly convex spaces are common examples of reflexive Banach spaces. The concept of uniform convexity was first introduced by CLARKSON [2] in 1936. In respect of these spaces, the following lemma is well known.

**Lemma 2.2 ([5])** Let  $E$  be a uniformly convex Banach space. Let  $\{x_n\}$  and  $\{y_n\}$  be sequences in  $E$  satisfying  $\lim_{n \rightarrow \infty} \|x_n\| = 1$ ,  $\lim_{n \rightarrow \infty} \|y_n\| = 1$  and  $\lim_{n \rightarrow \infty} \|x_n + y_n\| = 2$ . Then  $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ .

The following lemma was proved in [8] utilizing Lemma 2.2. We shall use it in the proof of our main theorem.

**Lemma 2.3 ([8])** Let  $E$  be a uniformly convex Banach space and let  $\{u_n\}, \{v_n\}$  and  $\{w_n\}$  be sequences in  $E$ . Let  $d$  and  $t$  be real numbers with  $d \in (0, \infty)$  and  $t \in (0, 1)$ . Assume that  $\lim_{n \rightarrow \infty} \|u_n - v_n\| = d$ ,  $\limsup_{n \rightarrow \infty} \|u_n - w_n\| \leq (1 - t)d$  and  $\limsup_{n \rightarrow \infty} \|v_n - w_n\| \leq td$ . Then  $\lim_{n \rightarrow \infty} \|tu_n + (1 - t)v_n - w_n\| = 0$ .

A Banach space  $E$  is said to have the Kadec–Klee property if, for every sequence  $\{x_n\}$  in  $E$  which converges weakly to a point  $x \in E$  with  $\|x_n\|$  converging to  $\|x\|$ ,  $\{x_n\}$  converges strongly to  $x$ . It is known that uniformly convex Banach spaces have Kadec–Klee property (see [1]).

**Lemma 2.4 ([4, 6])** Let  $E$  be a reflexive Banach space whose dual has the Kadec–Klee property. Let  $\{x_n\}$  be a bounded sequence in  $E$  and let  $y, z \in E$  be weak subsequential limits of  $\{x_n\}$ . Assume that for every  $t \in [0, 1]$ ,  $\lim_{n \rightarrow \infty} \|tx_n + (1 - t)p - q\|$  exists. Then  $y = z$ .

**Proposition 2.5** Let  $T$  be a mapping on a subset  $K$  of a Banach space  $E$  and satisfy (RSC)-condition. Then

- (i)  $\|x - Ty\| \leq 7\|x - Tx\| + \|x - y\|$ ,
- (ii)  $\|y - Ty\| \leq 7\|x - Tx\| + 2\|x - y\|$ , hold for all  $x, y \in K$ .

*Proof.* (i) is proved in [7]. It follows from (i) that  $\|y - Ty\| \leq \|y - x\| + \|x - Ty\| \leq 7\|x - Tx\| + 2\|x - y\|$ . This completes the proof.  $\square$

## 3 Main results

In this section, we give a weak and a strong convergence theorem. First, we prove a couple of lemmas and a proposition which are useful for our weak convergence theorem. The following lemma is an extension of Lemma 8 of [8] to the case of mappings satisfying (RSC)-condition.

**Lemma 3.1** *Let  $T$  be a mapping on a bounded and convex subset  $K$  of a uniformly convex Banach space  $E$ . Assume that  $T$  satisfies (RSC)-condition. Then for any  $\varepsilon > 0$ , there exists  $\xi(\varepsilon) > 0$  such that for any  $t \in [0, 1]$  and for any  $u, v \in K$  with  $\|Tu - u\| < \xi(\varepsilon)$ ,  $\|Tv - v\| < \xi(\varepsilon)$ , we have  $\|T(tu + (1-t)v) - (tu + (1-t)v)\| < \varepsilon$ .*

*Proof.* Assume on contrary that there exist sequences  $\{u_n\}, \{v_n\} \subset K$ ,  $\{t_n\} \subset [0, 1]$  and  $\varepsilon > 0$  such that  $\|Tu_n - u_n\| < 1/n$ ,  $\|Tv_n - v_n\| < 1/n$  and  $\|T(t_n u_n + (1-t_n)v_n) - (t_n u_n + (1-t_n)v_n)\| \geq \varepsilon$ .

Set  $x_n = t_n u_n + (1-t_n)v_n$  and  $w_n = Tu_n$ . From Proposition 2.5 (ii), we get  $0 < \varepsilon \leq \liminf_{n \rightarrow \infty} \|Tx_n - x_n\| \leq \liminf_{n \rightarrow \infty} (7\|Tu_n - u_n\| + 2\|u_n - x_n\|) = 2 \liminf_{n \rightarrow \infty} \|u_n - x_n\|$ . Similarly, we can show that  $0 < \liminf_{n \rightarrow \infty} \|v_n - x_n\|$  and hence  $0 < \liminf_{n \rightarrow \infty} \|u_n - v_n\|$ . Since  $K$  is bounded and

$$0 < \liminf_{n \rightarrow \infty} \|v_n - x_n\| = \liminf_{n \rightarrow \infty} t_n \|u_n - v_n\| \leq \liminf_{n \rightarrow \infty} t_n \times \sup_{n \in \mathbb{N}} \|u_n - v_n\|,$$

we get  $0 < \liminf_{n \rightarrow \infty} t_n$ . Similarly, we can show that  $\limsup_{n \rightarrow \infty} t_n < 1$ . So, without loss of generality, we may assume that  $\|u_n - v_n\| \rightarrow d \in (0, \infty)$  and  $t_n \rightarrow t \in (0, 1)$  as  $n \rightarrow \infty$ . From  $\lim_{n \rightarrow \infty} \|Tu_n - u_n\| = 0$  and  $0 < \liminf_{n \rightarrow \infty} \|u_n - x_n\|$ , for sufficiently large  $n \in \mathbb{N}$ , we obtain

$$\frac{1}{2} \|Tu_n - u_n\| \leq \|u_n - x_n\|.$$

Since  $T$  satisfies (RSC)-condition, we have  $\|Tu_n - Tx_n\| \leq \frac{1}{3}(\|u_n - x_n\| + \|x_n - Tx_n\| + \|u_n - Tu_n\|)$ . Similarly, we can show that  $\|Tv_n - Tx_n\| \leq \frac{1}{3}(\|v_n - x_n\| + \|x_n - Tx_n\| + \|v_n - Tv_n\|)$ , for sufficiently large  $n \in \mathbb{N}$ . Now using Proposition 2.5 (ii) and the definition of (RSC)-condition, we get

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|u_n - w_n\| &\leq \limsup_{n \rightarrow \infty} (\|u_n - Tu_n\| + \|Tu_n - Tx_n\|) \\ &\leq \limsup_{n \rightarrow \infty} \left( \|u_n - Tu_n\| + \frac{1}{3}(\|u_n - x_n\| + \|x_n - Tx_n\| + \|u_n - Tu_n\|) \right) \\ &\leq \limsup_{n \rightarrow \infty} \left( \|u_n - Tu_n\| + \frac{1}{3}(\|u_n - x_n\| + 8\|u_n - Tu_n\| + 2\|u_n - x_n\|) \right) = (1-t)d \end{aligned}$$

and

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|v_n - w_n\| &\leq \limsup_{n \rightarrow \infty} (\|v_n - Tv_n\| + \|Tv_n - Tx_n\|) \\ &\leq \limsup_{n \rightarrow \infty} \left( \|v_n - Tv_n\| + \frac{1}{3}(\|v_n - x_n\| + \|x_n - Tx_n\| + \|v_n - Tv_n\|) \right) \\ &\leq \limsup_{n \rightarrow \infty} \left( \|v_n - Tv_n\| + \frac{1}{3}(\|v_n - x_n\| + 8\|v_n - Tv_n\| + 2\|v_n - x_n\|) \right) = td. \end{aligned}$$

It then follows from Lemma 2.3 that  $0 < \varepsilon \leq \lim_{n \rightarrow \infty} \|x_n - w_n\| = 0$ , which is a contradiction. This completes the proof.  $\square$

**Proposition 3.2** *Let  $T$  be a mapping on a bounded and convex subset  $K$  of a uniformly convex Banach space  $E$ . Assume that  $T$  satisfies (RSC)-condition. Then  $I - T$  is demiclosed at zero. That is, if  $\{x_n\}$  in  $K$  converges weakly to  $x_0 \in K$  and  $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$  then  $Tx_0 = x_0$ .*

*Proof.* Take a function  $\xi : (0, \infty) \rightarrow (0, \infty)$  satisfying the conclusion of Lemma 3.1. Let  $\{x_n\}$  be a sequence which converges weakly to  $x_0 \in K$  and  $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$ . Choose  $\varepsilon > 0$  arbitrarily. Define a strictly decreasing sequence  $\{\varepsilon_n\}$  in  $(0, \infty)$  by  $\varepsilon_1 = \varepsilon$  and  $\varepsilon_{n+1} = \min\{\varepsilon_n, \xi(\varepsilon_n)\}/2$ . It is easy to see that  $\varepsilon_{n+1} < \xi(\varepsilon_n)$ . Choose a subsequence  $\{x_{f(n)}\}$  of  $\{x_n\}$  such that  $\|x_{f(n)} - Tx_{f(n)}\| < \xi(\varepsilon_n)$ . Note that  $x_0$  belongs to the closed convex hull of  $\{x_{f(n)} : n \in \mathbb{N}\}$  because it is a weak limit of  $\{x_{f(n)}\}$ . Hence, there exist  $y \in K$  and  $v \in \mathbb{N}$  such that  $\|y - x_0\| < \varepsilon$  and  $y$  belongs to the convex hull of  $\{x_{f(n)} : n = 1, 2, \dots, v\}$ . Using Lemma 3.1, we have  $\|Ty - y\| < \varepsilon$ . Now Proposition 2.5 plays a role to assure that  $\|Tx_0 - x_0\| \leq 7\|Ty - y\| + 2\|y - x_0\| < 9\varepsilon$ . Since  $\varepsilon > 0$  is arbitrary, we obtain  $Tx_0 = x_0$ .  $\square$

**Lemma 3.3** *Let  $T$  be a mapping on a bounded and convex subset  $K$  of a uniformly convex Banach space  $E$  satisfying (RSC)-condition. Let  $\{x_n\}$  be a sequence in  $K$  defined by  $x_{n+1} = \alpha Tx_n + (1 - \alpha)x_n$ , where  $x_1 \in K$  is arbitrarily but fixed and  $\alpha$  is a real number belonging to  $[1/2, 1)$ . Let  $p, q \in F(T)$  and  $t \in [0, 1]$ . If  $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$ , then  $\lim_{n \rightarrow \infty} \|tx_n + (1 - t)p - q\|$  exists.*

*Proof.* Since  $T$  satisfies (RSC)-condition, it is quasi-nonexpansive by Proposition 1.5. Let  $S$  be a mapping from  $K$  onto itself defined by  $Sx = \alpha Tx + (1 - \alpha)x$ . It is not difficult to see that  $F(S) = F(T)$  and  $S$  is quasi-nonexpansive. Note that  $x_{n+1} = \alpha Tx_n + (1 - \alpha)x_n = Sx_n = S^n x_1$ . Thus for any  $q \in F(S)$ , we have  $\|x_{n+1} - q\| \leq \|Sx_n - q\| \leq \|x_n - q\|$ , because  $S$  is quasi-nonexpansive. Thus we have

$$\|x_{n+1} - q\| \leq \|x_n - q\|, \quad (3.1)$$

which means that the sequence  $\{\|x_n - q\|\}$  is nonincreasing and hence converges. Also it is obvious that the sequence  $\{\|p - q\|\}$  converges. Thus it suffices to consider  $t \in (0, 1)$ . Let  $\lim_{n \rightarrow \infty} \|x_n - p\| = d$ . If  $d = 0$ , there is nothing to prove. Take  $d > 0$ . From hypothesis, we have  $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$ . We also have  $\liminf_{m, n \rightarrow \infty} \|x_n - S^\ell(tx_m + (1 - t)p)\| \geq \liminf_{m, n \rightarrow \infty} (\|x_n - p\| - \|p - S^\ell(tx_m + (1 - t)p)\|) \geq \liminf_{m, n \rightarrow \infty} (\|x_n - p\| - \|p - (tx_m + (1 - t)p)\|) = (1 - t)d > 0$ , for all  $\ell \in \mathbb{N} \cup \{0\}$ , where  $S^0$  is the identity mapping on  $K$ . Hence, there exists  $\nu \in \mathbb{N}$  such that  $\frac{1}{2}\|x_n - Tx_n\| \leq \|x_n - S^\ell(tx_m + (1 - t)p)\|$ , for all  $\ell \geq 0$  and  $m, n \geq \nu$ . Using (RSC)-condition and Proposition 2.5 (ii), we obtain

$$\begin{aligned} \|Tx_n - T \circ S^\ell(tx_m + (1 - t)p)\| &\leq \frac{1}{3} \|x_n - S^\ell(tx_m + (1 - t)p)\| \\ &+ \frac{1}{3} \left\| \begin{array}{l} S^\ell(tx_m + (1 - t)p) \\ -T \circ S^\ell(tx_m + (1 - t)p) \end{array} \right\| + \frac{1}{3} \|x_n - Tx_n\|. \end{aligned}$$

This gives

$$\begin{aligned}
& \left\| x_{n+1} - S^{\ell+1}(tx_m + (1-t)p) \right\| \leq \left\| Sx_n - S \circ S^\ell(tx_m + (1-t)p) \right\| \\
& \leq \left\| \begin{array}{l} \alpha Tx_n + (1-\alpha)x_n \\ -\alpha T \circ S^\ell(tx_m + (1-t)p) \\ -(1-\alpha)S^\ell(tx_m + (1-t)p) \end{array} \right\| \\
& = \left\| \begin{array}{l} \alpha(Tx_n - T \circ S^\ell(tx_m + (1-t)p)) \\ + (1-\alpha)(x_n - S^\ell(tx_m + (1-t)p)) \end{array} \right\| \\
& \leq \alpha \left\| Tx_n - T \circ S^\ell(tx_m + (1-t)p) \right\| \\
& \quad + (1-\alpha) \left\| x_n - S^\ell(tx_m + (1-t)p) \right\| \tag{3.2} \\
& \leq \alpha \left\| x_n - S^\ell(tx_m + (1-t)p) \right\| + \frac{8}{3} \|x_n - Tx_n\| \\
& \quad + (1-\alpha) \left\| x_n - S^\ell(tx_m + (1-t)p) \right\| \\
& = \left\| x_n - S^\ell(tx_m + (1-t)p) \right\| + \frac{8}{3} \|x_n - Tx_n\|,
\end{aligned}$$

for all  $\ell \geq 0$  and  $m, n \geq \nu$ . Let  $h : \mathbb{N} \rightarrow [0, \infty)$  be a function defined by  $h(n) = \|tx_n + (1-t)p - q\|$ . Take two subsequences  $\{f(n)\}$  and  $\{g(n)\}$  of  $\{n\}$  such that  $\nu < f(1), f(n) < g(n)$  for each  $n \in \mathbb{N}$  and

$$\lim_{n \rightarrow \infty} h(f(n)) = \liminf_{n \rightarrow \infty} h(n), \quad \lim_{n \rightarrow \infty} h(g(n)) = \limsup_{n \rightarrow \infty} h(n).$$

Set  $u_n = x_{g(n)}$ ,  $v_n = p$  and  $w_n = S^{g(n)-f(n)}(tx_{f(n)} + (1-t)p)$ . Then we get that  $\lim_{n \rightarrow \infty} \|u_n - v_n\| = d$ ,

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \|u_n - w_n\| &= \limsup_{n \rightarrow \infty} \left\| x_{g(n)} - S^{g(n)-f(n)}(tx_{f(n)} + (1-t)p) \right\| \\
&\leq \limsup_{n \rightarrow \infty} \left\| x_{f(n)} - (tx_{f(n)} + (1-t)p) \right\| \\
&\quad + \frac{8}{3} \limsup_{n \rightarrow \infty} \|x_n - Tx_n\| \quad (\text{by (3.2)}) \\
&= (1-t) \limsup_{n \rightarrow \infty} \|x_{f(n)} - p\| \\
&= (1-t)d
\end{aligned}$$

and  $\limsup_{n \rightarrow \infty} \|v_n - w_n\| \leq td$ . From Lemma 2.3, we get that

$$\lim_{n \rightarrow \infty} \|tx_{g(n)} + (1-t)p - S^{g(n)-f(n)}(tx_{f(n)} + (1-t)p)\| = 0.$$

Making use of quasi-nonexpansiveness of  $S$  together with the above equation, we get

$$\begin{aligned} \limsup_{n \rightarrow \infty} h(n) &= \lim_{n \rightarrow \infty} h(g(n)) \\ &\leq \limsup_{n \rightarrow \infty} \left( \left\| \begin{array}{l} tx_{g(n)} + (1-t)p \\ -S^{g(n)-f(n)}(tx_{f(n)} + (1-t)p) \end{array} \right\| \right. \\ &\quad \left. + \left\| S^{g(n)-f(n)}(tx_{f(n)} + (1-t)p) - q \right\| \right) \\ &= \limsup_{n \rightarrow \infty} \left\| S^{g(n)-f(n)}(tx_{f(n)} + (1-t)p) - q \right\| \\ &\leq \limsup_{n \rightarrow \infty} \left\| (tx_{f(n)} + (1-t)p) - q \right\| \\ &\leq \lim_{n \rightarrow \infty} h(f(n)) = \liminf_{n \rightarrow \infty} h(n). \end{aligned}$$

Thus  $\lim_{n \rightarrow \infty} h(n) = \lim_{n \rightarrow \infty} \|tx_n + (1-t)p - q\|$  exists.  $\square$

Now, we can prove the following weak convergence theorem.

**Theorem 3.4** *Let  $E$  be a uniformly convex Banach space whose dual has the Kadec-Klee property. Under the assumptions of Lemma 3.3 on  $T, K$  and  $\{x_n\}$ ,  $\{x_n\}$  converges weakly to a fixed point of  $T$ .*

*Proof.* Let  $W$  be the set of all weak subsequential limits of  $\{x_n\}$ . Since  $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$ , by Proposition 3.2 we have  $W \subset F(T)$ . That  $W$  is singleton now follows by using Lemma 2.4 and Lemma 3.3. Since  $E$  is reflexive (being uniformly convex), every subsequence of  $\{x_n\}$  has a subsequence converging weakly to the unique element of  $W$ . Therefore  $\{x_n\}$  itself converges weakly to the unique element of  $W$ .  $\square$

**Remark 3.1** The above theorem generalizes Theorem 11 of KHAN and SUZUKI [8].

The following is also a direct consequence of Theorem 3.4.

**Corollary 3.5** *Let  $E$  be a uniformly convex Banach space whose norm is Fréchet differentiable. Let  $K, T$ , and  $\{x_n\}$  be as in Lemma 3.3. Then,  $\{x_n\}$  converges weakly to a fixed point of  $T$ .*

Now we prove strong convergence theorem by using Condition (I) of SENTOR and DOTSON [10].

Recall that a mapping  $T : K \rightarrow K$  where  $K$  is a subset of  $E$ , is said to satisfy condition (I) (see [10]) if there exists a nondecreasing function  $f : [0, \infty) \rightarrow [0, \infty)$  with  $f(0) = 0$ ,  $f(r) > 0$ , for all  $r \in (0, \infty)$  such that  $\|x - Tx\| \geq f(d(x, F(T)))$ , for all  $x \in K$  where  $d(x, F(T)) = \inf \{\|x - p\| : p \in F(T)\}$ .

**Theorem 3.6** *Let  $E$  be a uniformly convex Banach space. Let  $K, T$ , and  $\{x_n\}$  be as in Lemma 3.3. If  $T$  satisfies the condition (I), then  $\{x_n\}$  converges strongly to a fixed point of  $T$ .*

*Proof.* By Lemma 3.3, we know that  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists for all  $p \in F(T)$ . From the inequality (3.1),  $d(x_{n+1}, F(T)) \leq d(x_n, F(T))$  and so  $\lim_{n \rightarrow \infty} d(x_n, F(T))$  exists. Assume that  $\lim_{n \rightarrow \infty} \|x_n - p\| = c$  for some  $c \geq 0$ . If  $c = 0$ , it is clear that  $\{x_n\}$  converges strongly to  $p$ . Suppose  $c > 0$ . From hypothesis and the condition

(I), we have  $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$  and  $f(d(x_n, F(T))) \leq \|Tx_n - x_n\|$ . This gives  $\lim_{n \rightarrow \infty} f(d(x_n, F(T))) = 0$ . Since  $f$  is nondecreasing function, we have  $\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0$ . Thus, we can take a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  and a sequence  $\{y_k\} \subset F(T)$  such that  $\|x_{n_k} - y_k\| < 2^{-k}$ . So, it follows from method of proof of TAN and XU [12] that  $\{y_k\}$  is a Cauchy sequence in  $F(T)$  and so it converges to a point  $y$ . Since  $F(T)$  is closed, therefore  $y \in F(T)$  and then  $\{x_{n_k}\}$  converges strongly to  $y$ . Since  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists, we have that  $x_n \rightarrow y \in F(T)$ . This completes the proof.  $\square$

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