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Some classes of alternating weighted binomial sums

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Abstract In this paper, we consider three classes of generalized alternating weighted binomial sums of the form

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i} f(n,i,k,t),$$

where $f(n,i,k,t)$ will be chosen as $U_{kti}V_{kn-k(t+2)i}$, $U_{kti}V_{kn-kti}$ and $U_{tki}V_{(k+1)tn-(k+2)ti}$. We use the Binet formula and the Newton binomial formula to prove the claimed results. Further we present some interesting examples of our results.

Keywords Binomial sums · Binary linear recurrences

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1 Introduction

Define second order linear recurrences $\{U_n\}$ and $\{V_n\}$ as for $n > 0$,

$$U_n = pU_{n-1} + U_{n-2} \text{ and } V_n = pV_{n-1} + V_{n-2},$$

where $U_0 = 0,\ U_1 = 1$, and, $V_0 = 2,\ V_1 = p$, respectively. If $p = 1$, then $U_n = F_n$ (nth Fibonacci number) and $V_n = L_n$ (nth Lucas number).

The Binet formulæ are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = \alpha^n + \beta^n,$$

where $\alpha, \beta = (p \pm \sqrt{\Delta})/2$ and $\Delta = p^2 + 4$.

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Let $A(x)$ and $B(x)$ be the exponential generating functions of sequences $\{a_n\}$ and $\{b_n\}$, that is

$$A(x)=\sum_{n\geq 0} a_n\frac{x^n}{n!} \text{ and } B(x)=\sum_{n\geq 0} b_n\frac{x^n}{n!}.$$

The convolution of the exponential generating functions is given as [2]

$$A(x)B(x)=\sum_{n\geq 0}(\sum_{k=0}^{n}\binom{n}{k}a_k b_{n-k})\frac{x^n}{n!}.$$

Many authors have computed various weighted binomial sums by various methods (for more details, see [1,3,6–11]). One of them is to use the convolution of the exponential generating functions. From the literature (see [3,10]), we recall the sums formulæ which can be written more compactly given by

$$\sum_{i=0}^{n}\binom{n}{i}F_{mi}L_{mn-mi}=2^n F_{mn}$$

and

$$\sum_{i=0}^{n}\binom{n}{i}F_{mi}F_{mn-mi}=(2^n L_{mn}-2L_m^n)/5.$$

Meanwhile, for some integers k,t and r such that $k\neq t$ or $k\neq r,$ the binomial sums

$$\sum_{i=0}^{n}\binom{n}{i}F_{ti}L_{kni-ri} \text{ and } \sum_{i=0}^{n}\binom{n}{i}(-1)^i F_{ti}L_{kni-ri}$$

can't be computed by the convolution of exponential generating functions.

Recently, the authors of [4] give general formulæ for the sums

$$\sum_{h=0}^{n}\binom{n}{h}h^m U_{ht}^{2m+\varepsilon}\ ,\quad \sum_{h=0}^{n}\binom{n}{h}h^m V_{ht}^{2m+\varepsilon},$$
$$\sum_{h=0}^{n}\binom{n}{h}(-1)^{n+h}h^m U_{ht}^{2m+\varepsilon}\ ,\quad \sum_{h=0}^{n}\binom{n}{h}(-1)^{n+h}h^m V_{ht}^{2m+\varepsilon},$$

where t is an positive integer and $\varepsilon\in\{0,1\}$.

Further the authors of [5] computed the weighted binomial sums

$$\sum_{k=0}^{n}\binom{n}{k}r_{mk}s_{m(tn+k)},$$

where r_n and s_n are the terms of $\{U_n\}$ and $\{V_n\}$ for some positive integers t and m. For example, for odd m,

$$\sum_{i=0}^{n}\binom{n}{i}U_{mi}V_{kmn+mi}=\Delta^{\lfloor\frac{n}{2}\rfloor}U_m^n\begin{cases}U_{(k+1)mn}, & \text{if } n \text{ is even,}\\ V_{(k+1)mn}, & \text{if } n \text{ is odd,}\end{cases}$$

and for even m,

$$\sum_{i=0}^{n}\binom{n}{i}V_{mi}V_{kmn+mi}=V_m^nV_{(k+1)mn}+2^nV_{kmn}.$$

In this paper we consider new three classes of generalized alternating binomial sums that couldn't be derived via the convolution of exponential generating functions :

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i f(n,i,k,t),$$

where $f(n,i,k,t)$ will be chosen as

$$U_{kti}V_{kn-(t+2)ki},\ U_{kti}V_{kn-kti} \text{ and } U_{tki}V_{(k+1).tn-(k+2)ti}.$$

These binomial sums (except some special cases of k and t) have not been considered according to our best literature acknowledgement. To compute the claimed sums, our approach is to use Binet formula and the Newton binomial formula. In general, let $\{A_n\}$ and $\{B_n\}$ be two second order linear recurrences whose the Binet formulae are

$$A_n=c_1a_1^n+c_1a_2^n \text{ and } B_n=d_1b_1^n+d_2b_2^n.$$

Then we write for the sum

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}A_{ai}B_{m-bi}&=\sum_{i=0}^{n}\binom{n}{i}(c_1a_1^{ai}+c_1a_2^{ai})(d_1b_1^{m-bi}+d_2b_2^{m-bi})\\&=\sum_{1\le i,j\le 2}c_id_jb_j^m(a_i^ab_j^{-b}+1)^n.\end{aligned}$$

In computing our sums, we will choose required values of the scalers a_i,b_i,c_i and d_i for $1\le i\le 2$.

2 The main results

Fist we give an auxiliary lemma and then give our first result.

Lemma 2.1 *Let t be any integer.*
i) For odd k,

$$\begin{aligned}(-1)^t\alpha^{-k(2t+1)}-\alpha^k&=(-1)^{t+1}V_{k(t+1)}\beta^{kt},\\(-1)^t\beta^{-k(2t+1)}-\beta^k&=(-1)^{t+1}V_{k(t+1)}\alpha^{kt}.\end{aligned}$$

ii) For even k,

$$(\alpha^{-k(2t+1)}-\alpha^k)=-\sqrt{\Delta}U_{k(t+1)}\beta^{kt} \textit{ and } (\beta^{-k(2t+1)}-\beta^k)=\sqrt{\Delta}U_{k(t+1)}\alpha^{kt}.$$

Theorem 2.2 *For $n > 0$, any integer t and odd k,*

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i U_{kti} V_{kn-ki(t+2)} = (-1)^{tn} V_{k(t+1)}^n U_{ktn}$$

and for even k,

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i U_{kti} V_{kn-k(t+2)i} = \begin{cases} \Delta^{\frac{n-1}{2}} [2U_k^n - U_{k(t+1)}^n V_{ktn}], & \text{if } n \text{ is odd,} \\ \Delta^{\frac{n}{2}} U_{k(t+1)}^n U_{ktn}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. First assume that k is an odd integer. We write by recalling $\alpha\beta = -1$

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i U_{kti} V_{kn-k(t+2)i}$$
$$= \frac{1}{\alpha-\beta} \sum_{i=0}^{n} \binom{n}{i} (-1)^i [(\alpha^{kn-2ki} - \beta^{kn-2ki})$$
$$- (-1)^{ti} (\alpha^{kn-2ki(t+1)} - \beta^{kn-2ki(t+1)})]$$
$$= \frac{\alpha^{kn}}{\alpha-\beta} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \alpha^{-2ki} - \frac{\beta^{kn}}{\alpha-\beta} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \beta^{-2ki}$$
$$- \frac{1}{\alpha-\beta} \sum_{i=0}^{n} \binom{n}{i} (-1)^{i(t+1)} (\alpha^{kn-2ki(t+1)} - \beta^{kn-2ki(t+1)})$$
$$= \frac{1}{\alpha-\beta} [(\alpha^k - \alpha^{-k})^n - (\beta^k - \beta^{-k})^n]$$
$$- \frac{1}{\alpha-\beta} \sum_{i=0}^{n} \binom{n}{i} (-1)^{i(t+1)} (\alpha^{kn-2ki(t+1)} - \beta^{kn-2ki(t+1)}),$$

which, since $\alpha^k - \alpha^{-k} = \beta^k - \beta^{-k}$ for odd k, equals

$$- \frac{1}{\alpha-\beta} \sum_{i=0}^{n} \binom{n}{i} (-1)^{i(t+1)} [\alpha^{kn-2ki(t+1)} - \beta^{kn-2ki(t+1)}]$$
$$= -\frac{1}{\alpha-\beta} [\alpha^{kn} \sum_{i=0}^{n} \binom{n}{i} (-1)^{i(t+1)} \alpha^{-2ki(t+1)}$$
$$- \beta^{kn} \sum_{i=0}^{n} \binom{n}{i} (-1)^{i(t+1)} \beta^{-2ki(t+1)}]$$
$$= -\frac{1}{\alpha-\beta} [(\alpha^k - (-1)^t \alpha^{-k(2t+1)})^n - (\beta^k - (-1)^t \beta^{-k(2t+1)})^n],$$

which, Lemma 2.1 (i) and the Binet formula, equals

$$-\frac{1}{\alpha-\beta} [(-1)^{tn} V_{k(t+1)}^n \beta^{ktn} - (-1)^{tn} V_{k(t+1)}^n \alpha^{ktn}] = (-1)^{tn} V_{k(t+1)}^n U_{ktn},$$

as claimed.

Now we consider the case k is even. We write

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i}(-1)^i U_{kti}V_{kn-ki(t+2)}\\&=\frac{1}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}(-1)^i[(\alpha^{k(n-2i)}-\beta^{k(n-2i)})-(\alpha^{kn-2ki(t+1)}-\beta^{kn-2ki(t+1)})],\end{aligned}$$

which, after some arrangements similar to the previous one, equals

$$\alpha^{kn}(1-\alpha^{-2k})^n-\beta^{kn}(1-\beta^{-2k})^n-\alpha^{kn}(1-\alpha^{-2k(t+1)})^n+\beta^{kn}(1-\beta^{-2k(t+1)})^n,$$

which, Lemma 2.1 (ii), equals

$$\begin{aligned}&\alpha^{kn}(1-\alpha^{-2k})^n-\beta^{kn}(1-\beta^{-2k})^n-\alpha^{kn}(1-\alpha^{-2k(t+1)})^n+\beta^{kn}(1-\beta^{-2k(t+1)})^n\\&=\frac{1}{\alpha-\beta}U_k^n\Delta^{\frac{n}{2}}[1-(-1)^n]+\frac{1}{\alpha-\beta}U_{k(t+1)}^n\Delta^{\frac{n}{2}}[(-1)^n\alpha^{ktn}-\beta^{ktn}]\\&=U_k^n\Delta^{\frac{n-1}{2}}[1-(-1)^n]+\Delta^{\frac{n-1}{2}}U_{k(t+1)}^n[(-1)^n\alpha^{ktn}-\beta^{ktn}],\end{aligned}$$

which gives the claimed result according to the case of n. □

For example, we have

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}(-1)^i U_{10i}V_{5(n-4i)}&=V_{15}^nU_{10n},\\\sum_{i=0}^{n}\binom{n}{i}(-1)^i U_{-3i}V_{3n-3i}&=-V_0^nU_{3n}\end{aligned}$$

and

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^iF_{2i}L_{2(n-3i)}=\begin{cases}5^{(n-1)/2}(2-3^nL_{2n}), & \text{if } n \text{ is odd,}\\5^{\frac{n}{2}}3^nF_{2n}, & \text{if } n \text{ is even.}\end{cases}$$

Lemma 2.3 *Let t be an integer.*

i) For odd k,

$$\begin{aligned}(-1)^t\alpha^{k(1-2t)}-\alpha^k&=(-1)^tU_{kt}\beta^{k(t-1)}\sqrt{\Delta},\\(-1)^t\beta^{k(1-2t)}-\beta^k&=(-1)^{t+1}U_{kt}\alpha^{k(t-1)}\sqrt{\Delta}.\end{aligned}$$

ii) For even k,

$$\alpha^{k(1-2t)}-\alpha^k=-U_{kt}\beta^{k(t-1)}\sqrt{\Delta},\quad \beta^{k(1-2t)}-\beta^k=U_{kt}\alpha^{k(t-1)}\sqrt{\Delta}.$$

Theorem 2.4 *For $n \geq 0$ and for odd k,*

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i U_{kti} V_{k(n-ti)} = U_{kt}^n \begin{cases} (-1)^t V_{kn(t-1)} \Delta^{\frac{n-1}{2}}, & \text{if } n \text{ is odd,} \\ U_{kn(t-1)} \Delta^{\frac{n}{2}}, & \text{if } n \text{ is even,} \end{cases}$$

and for even k,

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i U_{kti} V_{kn-kti} = U_{kt}^n \begin{cases} -\Delta^{\frac{n-1}{2}} V_{k(t-1)n}, & \text{if } n \text{ is odd,} \\ \Delta^{\frac{n}{2}} U_{k(t-1)n}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. First assume that k is odd. We write

$$\begin{aligned}
&\sum_{i=0}^{n} \binom{n}{i} (-1)^i U_{kti} V_{kn-kti} \\
&= \frac{1}{\alpha-\beta} \sum_{i=0}^{n} \binom{n}{i} (-1)^i [(\alpha^{kn} - \beta^{kn}) - (-1)^{it} (\alpha^{kn-2ikt} - \beta^{kn-2ikt})] \\
&= -\frac{1}{\alpha-\beta} \sum_{i=0}^{n} \binom{n}{i} (-1)^{i(t+1)} (\alpha^{kn-2ikt} - \beta^{kn-2ikt}) \\
&= -\frac{\alpha^{kn}}{\alpha-\beta} (1 + (-1)^{t+1} \alpha^{-2kt})^n + \frac{\beta^{kn}}{\alpha-\beta} (1 + (-1)^{t+1} \beta^{-2kt})^n \\
&= \frac{1}{\alpha-\beta} (\beta^k + (-1)^{t+1} \beta^{k(1-2t)})^n - \frac{1}{\alpha-\beta} (\alpha^k + (-1)^{t+1} \alpha^{k(1-2t)})^n,
\end{aligned}$$

which, by Lemma 2.3 (ii), equals

$$\frac{1}{\alpha-\beta} U_{kt}^n \Delta^{\frac{n}{2}} [(-1)^{tn} \alpha^{kn(t-1)} - (-1)^n (-1)^{tn} \beta^{kn(t-1)}],$$

which gives the claim according to the case of n.
The case k is even could be obtained similarly. □

For example, for $k = 1$, $t = 2$ and the Fibonacci-Lucas case:

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i F_{2i} L_{n-2i} = \begin{cases} 5^{\frac{n-1}{2}} V_n, & \text{if } n \text{ is odd,} \\ 5^{\frac{n}{2}} U_n, & \text{if } n \text{ is even.} \end{cases}$$

Lemma 2.5 *i) For odd k and t,*

$$\alpha^{t(k-1)} - \alpha^{t(k+1)} = -V_t \alpha^{tk}, \qquad \beta^{t(k-1)} - \beta^{t(k+1)} = -V_t \beta^{tk},$$
$$\alpha^{-t(k+1)} + \alpha^{t(k+1)} = \beta^{-t(k+1)} + \beta^{t(k+1)} = V_{(k+1)t}.$$

ii) For odd k and even t,

$$\alpha^{t(k-1)} - \alpha^{t(k+1)} = -U_t \alpha^{tk} \sqrt{\Delta}, \qquad \beta^{t(k-1)} - \beta^{t(k+1)} = U_t \beta^{tk} \sqrt{\Delta},$$
$$\alpha^{-t(k+1)} - \alpha^{t(k+1)} = -U_{(k+1)t} \sqrt{\Delta}, \qquad \beta^{-t(k+1)} - \beta^{t(k+1)} = U_{(k+1)t} \sqrt{\Delta}.$$

iii) For even k and t,

$$\alpha^{t(k-1)} - \alpha^{(k+1)t} = -U_t\alpha^{kt}\sqrt{\Delta}, \qquad \beta^{t(k-1)} - \beta^{(k+1)t} = U_t\beta^{kt}\sqrt{\Delta},$$
$$\alpha^{-t(k+1)} - \alpha^{t(k+1)} = -U_{(k+1)t}\sqrt{\Delta}, \qquad \beta^{-t(k+1)} - \beta^{t(k+1)} = U_{(k+1)t}\sqrt{\Delta}.$$

iv) For even k and odd t,

$$\alpha^{t(k-1)} - \alpha^{(k+1)t} = -V_t\alpha^{kt}, \qquad \beta^{t(k-1)} - \beta^{(k+1)t} = -V_t\beta^{kt},$$
$$\alpha^{-t(k+1)} - \alpha^{t(k+1)} = -V_{(k+1)t}, \qquad \beta^{-t(k+1)} - \beta^{t(k+1)} = -V_{(k+1)t}.$$

Similar to the previous results, we give the following result without proof.

Theorem 2.6 *For odd k,*

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i U_{tki}V_{(k+1)tn-(k+2)ti}$$
$$= \begin{cases} V_t^n U_{ktn}, & \text{if } t \text{ is odd,} \\ \Delta^{\frac{n-1}{2}}(U_t^n V_{ktn} - 2U_{(k+1)t}^n), & \text{if } n \text{ is odd and } t \text{ is even,} \\ \Delta^{\frac{n}{2}} U_t^n U_{ktn}, & \text{if } n \text{ is even and } t \text{ is even,} \end{cases}$$

and for even k,

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i U_{tki}V_{(k+1)tn-(k+2)ti}$$
$$= \begin{cases} V_t^n U_{ktn}, & \text{if } t \text{ is odd,} \\ \Delta^{\frac{n-1}{2}}(U_t^n V_{ktn} - 2U_{(k+1)t}^n), & \text{if } n \text{ is odd and } t \text{ is even,} \\ \Delta^{\frac{n}{2}} U_t^n U_{ktn} & \text{if } n \text{ and } t \text{ are even.} \end{cases}$$

As consequences of the Lemma 2.1, we have the following results:

Theorem 2.7 *For even m,*

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i V_{2mi} = \begin{cases} \Delta^{\frac{n}{2}} V_{nm}, & \text{if } n \text{ is even,} \\ -\Delta^{\frac{n+1}{2}} U_m^n U_{nm}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. By Lemma 2.1 (ii), for even k, we have

$$\alpha^{-k(2t+1)} - \alpha^k = -\sqrt{\Delta}U_{k(t+1)}\beta^{kt} \text{ and } \beta^{-k(2t+1)} - \beta^k = \sqrt{\Delta}U_{k(t+1)}\alpha^{kt}$$

or

$$1 - \alpha^{-2k(t+1)} = \sqrt{\Delta}U_{k(t+1)}\alpha^{-k}\beta^{kt} \text{ and } 1 - \beta^{-2k(t+1)} = -\sqrt{\Delta}U_{k(t+1)}\beta^{-k}\alpha^{kt}$$

and write

$$(1 - \alpha^{-2k(t+1)})^n = \Delta^{\frac{n}{2}} U_{k(t+1)}^n \alpha^{-kn}\beta^{knt},$$
$$(1 - \beta^{-2k(t+1)})^n = (-1)^n \Delta^{\frac{n}{2}} U_{k(t+1)}^n \beta^{-kn}\alpha^{knt},$$

or

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i \alpha^{-2k(t+1)i} = \Delta^{\frac{n}{2}} U_{k(t+1)}^n \alpha^{-kn} \beta^{knt},$$

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i \beta^{-2k(t+1)i} = (-1)^n \Delta^{\frac{n}{2}} U_{k(t+1)}^n \beta^{-kn} \alpha^{knt}.$$

By adding these equalities side by side, we obtain

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i (\alpha^{-2k(t+1)i} + \beta^{-2k(t+1)i}) = \Delta^{\frac{n}{2}} U_{k(t+1)}^n (\alpha^{-kn} \beta^{knt} + (-1)^n \beta^{-kn} \alpha^{knt}),$$

or since $V_{-k} = (-1)^k V_k$,

$$\begin{aligned} \sum_{i=0}^{n} \binom{n}{i} (-1)^i V_{-2k(t+1)i} &= \sum_{i=0}^{n} \binom{n}{i} (-1)^i V_{2k(t+1)i} \\ &= \Delta^{\frac{n}{2}} U_{k(t+1)}^n [(-1)^n \beta^{-kn} \alpha^{knt} + \alpha^{-kn} \beta^{knt}] \\ &= U_{k(t+1)}^n \begin{cases} \Delta^{\frac{n}{2}} V_{kn(t+1)}, & \text{if } n \text{ is even,} \\ (-1)^{k+1} \Delta^{\frac{n+1}{2}} U_{kn(t+1)}, & \text{if } n \text{ is odd,} \end{cases} \end{aligned}$$

which, by taking m instead of $k(t+1)$ for even k, completes the proof. □

Similar to the proof method of Theorem just above, we have the following results without proof by using Lemma 2.1 (i).

Theorem 2.8 *For any integer m and $n \geq 0$,*

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^{im} V_{2mi} = (-1)^{mn} V_m^n V_{nm}$$

and

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^{im} U_{2mi} = (-1)^{nm} V_m^n U_{nm}.$$

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