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A note on semilinear fractional equations governed by abstract differential operators

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Abstract Let $P(x)$ be an arbitrary non-zero complex polynomial, let $n \in \mathbb{N}$, and let $iA_j,\ 1 \leq j \leq n$ be commuting generators of bounded C_0-groups on a Banach space E. In this note, we shall prove some important results concerning the existence and uniqueness of mild solutions of the following semilinear fractional problem

$$\mathbf{D}_t^\alpha u(t) = \overline{P(A)}u(t) + f(t, u(t)),\ t \geq 0; \quad u(0) = x, \tag{1}$$

where $0 < \alpha < 1$, $\mathbf{D}_t^\alpha$ denotes the Caputo fractional derivative of order α (Bazhlekova) and the function $f(\cdot, \cdot)$ satisfies certain properties.

Keywords Semilinear fractional equations · Abstract differential operators · (g_α, C)-regularized resolvent families · Mild solutions

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1 Introduction and preliminaries

The main purpose of this note is to continue the researches of semilinear fractional differential equations raised in the papers [13] and [5]. It should be noted that, in our approach, the operator $\overline{P(A)}$ generates an exponentially bounded (g_α, C)-regularized resolvent family $(R_\alpha(t))_{t\geq 0}$ for a suitably chosen injective operator $C \in L(E)$ and that $(R_\alpha(t))_{t\geq 0}$ is not necessarily analytic, in contrast to [13] and [5] (cf. (1)). Compared with the above-mentioned papers, and with a great number of other papers from the existing literature that are not cited here, our results can be applied in the analysis of problem (1) with $\overline{P(A)}$ being (for example) the operator $\Delta_\alpha := e^{i(2-\alpha)\frac{\pi}{2}}\Delta$ acting on $E := L^p(\mathbb{R}^n)$ with its maximal distributional domain ($1 \leq p \leq \infty$, $0 < \alpha \leq 1$).

Let E be a complex Banach space, let $n \in \mathbb{N}$ and let $iA_j,\ 1 \leq j \leq n$ be commuting generators of bounded C_0-groups on E. By $L(E)$ we denote the space of all bounded

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continuous mappings from E into E; $g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$ $(t > 0,\ \alpha > 0)$, where $\Gamma(\cdot)$ denotes the Gamma function. Set $k := 1+\lfloor n/2 \rfloor$, $A := (A_1, \cdots, A_n)$ and $A^\eta := A_1^{\eta_1} \cdots A_n^{\eta_n}$ for any tuple $\eta = (\eta_1, \cdot \cdot \cdot, \eta_n) \in \mathbb{N}_0^n$. If $\xi = (\xi_1, \cdot \cdot \cdot, \xi_n) \in \mathbb{R}^n$ and $u \in \mathcal{F}L^1(\mathbb{R}^n) := \{\mathcal{F}f : f \in L^1(\mathbb{R}^n)\}$, put $|\xi| := (\sum_{j=1}^n \xi_j^2)^{1/2}$, $(\xi, A) := \sum_{j=1}^n \xi_j A_j$ and

$$u(A)x := \int_{\mathbb{R}^n} \mathcal{F}^{-1}u(\xi)e^{-i(\xi,A)}x\, d\xi,\ x \in E,$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse transform, respectively. Then $u(A) \in L(E)$, $u \in \mathcal{F}L^1(\mathbb{R}^n)$ and there exists a constant $M < \infty$ such that $\|u(A)\| \le M\|\mathcal{F}^{-1}u\|_{L^1(\mathbb{R}^n)}$, $u \in \mathcal{F}L^1(\mathbb{R}^n)$. For any non-zero complex polynomial $P(x) = \sum_{|\eta| \le N} a_\eta x^\eta$, $x \in \mathbb{R}^n$ $(N \in \mathbb{N})$, we define $P(A) := \sum_{|\eta| \le N} a_\eta A^\eta$ with maximal domain. Denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$. Set $E_0 := \{\phi(A)x : \phi \in \mathcal{S}(\mathbb{R}^n),\ x \in E\}$. Then it is well known that $P(A)$ is closable and that the following holds:

($\triangleright$) $\overline{E_0} = E$, $E_0 \subseteq \bigcap_{\eta \in \mathbb{N}_0^n} D(A^\eta)$, $\overline{P(A)_{|E_0}} = \overline{P(A)}$ and $\phi(A)P(A) \subseteq P(A)\phi(A) = (\phi P)(A)$, $\phi \in \mathcal{S}(\mathbb{R}^n)$.

If E is a function space on which translations are uniformly bounded and strongly continuous, then the obvious choice for A_j is $-i\partial/\partial x_j$ and, in this case, $\overline{P(A)}$ is just the operator $\sum_{|\eta| \le N} a_\eta (-i)^{|\eta|} \partial^{|\eta|}/\partial x_1^{\eta_1} \cdots \partial x_n^{\eta_n} \equiv \sum_{|\eta| \le N} a_\eta D^\eta$ acting with its maximal distributional domain on E (cf. [10] and [3] for more details about functional calculus of commuting generators of bounded C_0-groups). Recall that $P(x)$ is called *r-coercive* $(0 < r \le N)$ if there exist $M,\ L > 0$ such that $|P(x)| \ge M|x|^r$, $|x| \ge L$.

Let $p \in [1, \infty]$. We refer the reader to [8] for the notions and basic properties of Fourier multipliers on $L^p(\mathbb{R}^n)$, the Mittag-Leffler and Wright functions, which will be sufficient for our further work (we will follow the terminology already used in [8], cf. also [6, Section 2.5]). Fairly complete information about vector-valued Laplace transform can be obtained by consulting monograph [1]. With the exception of the proof of Theorem 2.2, hereafter $M > 0$ denotes a generic constant whose value may change from line to line.

We refer the reader to [7] for the notion of an exponentially bounded (analytic) (a, k)-regularized C-resolvent family $(R_\alpha(t))_{t \ge 0}$ and its subgenerator (the integral generator); cf. [12] and [6] for further information concerning the abstract Volterra integro-differential equations. If $a(t) = g_\alpha(t)$ for some $0 < \alpha < 1$, and $k(t) = 1$, then $(R_\alpha(t))_{t \ge 0}$ is also said to be an exponentially bounded (g_α, C)-regularized resolvent family. Motivated by the analyses given in [13] and [5], we will introduce the notions of an (α, α, C)-resolvent family and its integral generator in the following definition, which will be crucial for our further study of fractional Cauchy problem (1). By B we denote a closed linear operator on E.

Definition 1.1 *Let $0 < \alpha < 1$, let $C \in L(E)$ be injective and let $C^{-1}BC = B$. A strongly continuous operator family $(P_\alpha(t))_{t>0} \subseteq L(E)$ is said to be an (α, α, C)-resolvent family generated by B iff there exist $M \ge 1$ and $\omega \ge 0$ such that the mapping $t \mapsto \|t^{1-\alpha}P_\alpha(t)\|$, $t \in (0, 1]$ is bounded, $\|P_\alpha(t)\| \le Me^{\omega t}$, $t \ge 1$ and*

$$(\lambda^\alpha - B)^{-1}Cx = \int_0^\infty e^{-\lambda t}P_\alpha(t)x\, dt, \quad \Re\lambda > \omega,\ x \in E.$$

It can be simply proved that $(P_\alpha(t))_{t>0}$ has the following properties (cf. [5, Lemma 3] for the case $C = I$): $P_\alpha(t)B \subseteq BP_\alpha(t)$, $t > 0$; $P_\alpha(t)x = g_\alpha(t)Cx + \int_0^t g_\alpha(t-s)P_\alpha(s)Bx\,ds$, $t > 0$, $x \in D(B)$; $P_\alpha(t)x = g_\alpha(t)Cx + B\int_0^t g_\alpha(t-s)P_\alpha(s)x\,ds$, $t > 0$, $x \in E$; and $\lim_{t\to 0+} t^{1-\alpha}P_\alpha(t)x = Cx/\Gamma(\alpha)$, $x \in \overline{D(B)}$. Notice also that the assertion of [5, Theorem 5] can be reformulated for the generators of C-regularized semigroups and that it is not clear how one can prove an analog of the above-mentioned theorem for the generators of (g_β, C)-regularized resolvent families, provided that $\beta \in (\alpha, 1)$.

2 Formulation and proof of main results

We start this section by stating the following theorem.

Theorem 2.1 *Let $0 < \alpha < 1$, $\omega \geq 0$, $N \in \mathbb{N}$, let $P(x)$ be a non-zero complex polynomial of degree N, and let $\sup_{x\in\mathbb{R}^n} \Re(P(x)^{1/\alpha}) \leq \omega$.*

(i) Suppose $r \in (0, N]$, $P(x)$ is r-coercive, $a \in \mathbb{C} \setminus P(\mathbb{R}^n)$, $\gamma > \frac{nN}{2r\alpha}$ and $\gamma' > \frac{N}{\alpha r}(1-\alpha+\frac{n}{2})$ ($\gamma = n|\frac{1}{p}-\frac{1}{2}|\frac{N}{r\min(1,\alpha)}$ and $\gamma' = \frac{N}{\alpha r}(1-\alpha+n|\frac{1}{p}-\frac{1}{2}|)$, if $E = L^p(\mathbb{R}^n)$ for some $1 < p < \infty$). Set

$$R_{\alpha,\gamma}(t) := \Big(E_\alpha(t^\alpha P(x))(a-P(x))^{-\gamma}\Big)(A), \quad t \geq 0$$

and

$$P_{\alpha,\gamma'}(t) := t^{\alpha-1}\Big(E_{\alpha,\alpha}(t^\alpha P(x))(a-P(x))^{-\gamma'}\Big)(A), \quad t > 0.$$

Then $(R_{\alpha,\gamma}(t))_{t\geq 0}$ is an exponentially bounded $(g_\alpha,\, R_{\alpha,\gamma}(0))$-regularized resolvent family with the integral generator $\overline{P(A)}$, $(P_{\alpha,\gamma'}(t))_{t>0}$ is an exponentially bounded $(\alpha,\alpha, R_{\alpha,\gamma'}(0))$-resolvent family with the integral generator $\overline{P(A)}$, $(R_{\alpha,\gamma}(t))_{t\geq 0}$ and $(P_{\alpha,\gamma'}(t))_{t>0}$ are norm continuous provided $\gamma > \frac{nN}{2r\alpha}$, $\gamma' > \frac{N}{\alpha r}(1-\alpha+\frac{n}{2})$, and the following estimates hold, for every $t > 0$,

$$\|R_{\alpha,\gamma}(t)\| \leq M(1+t^{n/2})e^{\omega t} \ \Big(\|R_{\alpha,\gamma}(t)\| \leq M(1+t^{n|\frac{1}{p}-\frac{1}{2}|})e^{\omega t}\Big), \tag{2.1}$$

$$\begin{aligned}\|P_{\alpha,\gamma'}(t)\| \leq Mt^{\alpha-1}(1+t^{1-\alpha+\frac{n}{2}})e^{\omega t} \ \Big(&\|P_{\alpha,\gamma'}(t)\| \\ &\leq Mt^{\alpha-1}(1+t^{1-\alpha+n|\frac{1}{p}-\frac{1}{2}|})e^{\omega t}\Big).\end{aligned} \tag{2.2}$$

(ii) Suppose $\beta > \frac{nN}{2\alpha}$ and $\beta' > \frac{nN}{2\alpha} + N\frac{1-\alpha}{\alpha}$ ($\beta = n|\frac{1}{p}-\frac{1}{2}|\frac{N}{\alpha}$ and $\beta' = \frac{N}{\alpha}n|\frac{1}{p}-\frac{1}{2}| + N\frac{1-\alpha}{\alpha}$, if $E = L^p(\mathbb{R}^n)$ for some $1 < p < \infty$). Set

$$R_{\alpha,\beta}(t) := \Big(E_\alpha(t^\alpha P(x))(1+|x|^2)^{-\beta/2}\Big)(A),\ t \geq 0$$

and

$$P_{\alpha,\beta'}(t) := t^{\alpha-1}\Big(E_{\alpha,\alpha}(t^\alpha P(x))(1+|x|^2)^{-\beta'/2}\Big)(A),\ t > 0.$$

Then $(R_{\alpha,\beta}(t))_{t\geq 0}$ *is an exponentially bounded* $(g_\alpha, R_{\alpha,\beta}(0))$*-regularized resolvent family with the integral generator* $\overline{P(A)}$, $(P_{\alpha,\beta'}(t))_{t>0}$ *is an exponentially bounded* $(\alpha,\alpha, R_{\alpha,\beta'}(0))$*-resolvent family with the integral generator* $\overline{P(A)}$, $(R_{\alpha,\beta}(t))_{t\geq 0}$ *and* $(P_{\alpha,\beta'}(t))_{t>0}$ *are norm continuous provided* $\beta > \frac{nN}{2\alpha}$, $\beta' > \frac{nN}{2\alpha} + N\frac{1-\alpha}{\alpha}$, *and* (2.1)-(2.2) *holds.*

Proof. The proof of theorem is very similar to those of [8, Theorem 2.1, Theorem 2.2], so that we will only outline the main details needed for the proof of (i), in the case that E is a general space under consideration.

Put $C := R_{\alpha,\gamma'}(0)$ and denote by $E_{\alpha,\beta}(\cdot)$ the Mittag-Leffler function; $E_\alpha(\cdot) = E_{\alpha,1}(\cdot)$. Then $C \in L(E)$ is injective, $C^{-1}\overline{P(A)}C = \overline{P(A)}$ and the required properties of $(R_{\alpha,\gamma}(t))_{t\geq 0}$ have been proved in [8, Theorem 2.1]. Furthermore, for every $j \in \mathbb{N}$, there exist uniquely determined real numbers $c_{l,j,\alpha}$ $(1 \leq l \leq j)$ such that $E'_\alpha(z) = \alpha^{-1}E_{\alpha,\alpha}(z)$, $z \in \mathbb{C}$, $E^{(j)}_\alpha(z) = \sum_{l=1}^{j} c_{l,j,\alpha}E_{\alpha,\alpha j-(j-l)}(z)$, $z \in \mathbb{C}$, and that, for every multi-index $\eta \in \mathbb{N}_0^n$ with $|\eta| \leq k$, there exist complex polynomials $Q_j(x)$ of degree $\leq Nj - |\eta|$ $(1 \leq j \leq |\eta|)$ such that

$$D^\eta\Big(E_{\alpha,\alpha}(t^\alpha P(x))\Big) = \sum_{j=1}^{|\eta|} t^{\alpha j}E^{(j)}_{\alpha,\alpha}(t^\alpha P(x))Q_j(x) \tag{2.3}$$

$$= \sum_{j=1}^{|\eta|} t^{\alpha j}\sum_{l=1}^{j+1} \alpha c_{l,j+1,\alpha}E_{\alpha,\alpha j-j+l+\alpha-1}(t^\alpha P(x))Q_j(x),\ t \geq 0,\ x \in \mathbb{R}^n. \tag{2.4}$$

Using [8, Lemma 1.1] and the pre-assumption $\sup_{x\in\mathbb{R}^n}\Re(P(x)^{1/\alpha}) \leq \omega$, we get that

$$\begin{aligned}&|E_{\alpha,\alpha j-j+l+\alpha-1}(t^\alpha P(x))|\\ &\leq M\Bigg[1 + t^{1-(\alpha j-(j-l)+\alpha-1)}|P(x)|^{\frac{1-(\alpha j-(j-l)+\alpha-1)}{\alpha}}e^{\omega t}\Bigg],\end{aligned} \tag{2.5}$$

provided $t \geq 0$, $x \in \mathbb{R}^n$, $j,\ l \in \mathbb{N}$, $1 \leq j \leq k$ and $1 \leq l \leq j+1$. By (2.3), we obtain that, for every $t \geq 0$ and for every $x \in \mathbb{R}^n$ with $|t^\alpha P(x)| \leq 1$,

$$\Big|D^\eta\Big(E_{\alpha,\alpha}(t^\alpha P(x))\Big)\Big| \leq M\Big(t^\alpha + t^{\alpha|\eta|}\Big)(1+|x|)^{|\eta|(N-1)},\ |\eta| \leq k. \tag{2.6}$$

Suppose now $1 \leq l \leq j+1$, $1 \leq j \leq |\eta| \leq k$, $t \geq 0$, $x \in \mathbb{R}^n$ and $|t^\alpha P(x)| \geq 1$. We have

$$\begin{aligned}&|t^\alpha P(x)|^{\frac{1-(\alpha j-(j-l)+\alpha-1)}{\alpha}}(1+|x|)^{Nj-|\eta|}\\ &\leq Mt^{1-(\alpha j-(j-l)+\alpha-1)}(1+|x|)^{|\eta|(\frac{N}{\alpha}-1)+N\frac{1-\alpha}{\alpha}},\end{aligned} \tag{2.7}$$

provided $1 - (\alpha j - (j-l) + \alpha - 1) \geq 0$, and

$$|t^\alpha P(x)|^{\frac{1-(\alpha j-(j-l)+\alpha-1)}{\alpha}}(1+|x|)^{Nj-|\eta|} \leq M(1+|x|)^{|\eta|(N-1)}, \tag{2.8}$$

provided $1-(\alpha j-(j-l)+\alpha-1)\leq 0$. Then (2.4)-(2.8) enables one to carry out a computation similar to that appearing in the later part of the proof of [8, Theorem 2.1], showing that, for every $t\geq 0$ and $x\in\mathbb{R}^n$,

$$\Big|D^{\eta}\Big(E_{\alpha,\alpha}(t^{\alpha}P(x))\Big)\Big|\leq M(1+t^{1-\alpha})(1+t)^{|\eta|}e^{\omega t}(1+|x|)^{|\eta|(\frac{N}{\alpha}-1)+N\frac{1-\alpha}{\alpha}}. \tag{2.9}$$

Set $g_t(x):=E_{\alpha,\alpha}(t^{\alpha}P(x))(a-P(x))^{-\gamma'},\ t\geq 0,\ x\in\mathbb{R}^n$ and $G_{\alpha,\gamma'}(t):=(g_t(x))(A)$, $t\geq 0$. Using [10, (3.19)], the product rule and (2.9), we get that the absolute value of $D^{\eta}(E_{\alpha,\alpha}(t^{\alpha}P(x))(a-P(x))^{-\gamma'})$ does not exceed $M(1+t^{1-\alpha})(1+t)^{|\eta|}e^{\omega t}(1+|x|)^{|\eta|(\frac{N}{\alpha}-1)+N\frac{1-\alpha}{\alpha}-r\gamma'}$ $(t\geq 0,\ x\in\mathbb{R}^n,\ |\eta|\leq k)$. Due to [8, Lemma 1.3(i)] and this fact, $(G_{\alpha,\gamma'}(t))_{t\geq 0}$ is a strongly continuous operator family in $L(E)$, obeying $G_{\alpha,\gamma'}(t)=t^{1-\alpha}P_{\alpha,\gamma'}(t)$, $t>0$, and (2.2) holds. Making use of ($\triangleright$) and the identity $E_{\alpha}(t^{\alpha}P(x))=\int_0^t g_{1-\alpha}(t-s)s^{\alpha-1}E_{\alpha,\alpha}(s^{\alpha}P(x))\,ds$, $t>0$, $x\in\mathbb{R}^n$ (cf. [13, p. 212, l. 4]), it readily follows that $R_{\alpha,\gamma'}(t)x=(g_{1-\alpha}*P_{\alpha,\gamma'}(\cdot)x)(t)$, $t>0$, $x\in E$, so that the proof can be completed through the routine arguments already seen in that of [8, Theorem 2.1]. □

Before proceeding further, we would like to point out that the assertion of Theorem 2.1 continues to hold, with appropriate technical modifications, in the case that $E=C_b(\mathbb{R}^n)$ or $E=L^{\infty}(\mathbb{R}^n)$. Suppose now that $T>0$, $x\in R(C)$ and the requirements of Theorem 2.1(i) or Theorem 2.1(ii) hold. Following the considerations given in the papers [13] and [5], it will be said that a continuous function $t\mapsto u(t)$, $t\in[0,T]$ is a mild solution of the semilinear fractional abstract Cauchy problem (1) on $[0,T]$ iff

$$u(t)=R_{\alpha}(t)C^{-1}x+\int_0^t P_{\alpha}(t-s)C^{-1}f(s,u(s))\,ds,\quad t\in[0,T].$$

Define the operator $Q_{\alpha}:C([0,T]:X)\to C([0,T]:X)$ by $(Q_{\alpha}u)(t):=R_{\alpha}(t)C^{-1}x+\int_0^t P_{\alpha}(t-s)C^{-1}f(s,u(s))\,ds$, $t\in[0,T]$. The most common technique to proving existence and uniqueness of mild solutions of semilinear fractional evolution equations is to apply some of the fixed point theorems; in our concrete situation, we must prove that the mapping $Q_{\alpha}(\cdot)$ has a unique fixed point. Complete analysis is beyond the scope of this paper; we shall only state and prove the following adaptation of [11, Theorem 1.2, p. 184] to illustrate our results obtained so far.

Theorem 2.2 *Let $T>0$, let $x\in R(C)$, and let the requirements of Theorem 2.1(i) or Theorem 2.1(ii) hold. Put $C:=R_{\alpha,\gamma'}(0)$, in the case of Theorem 2.1(i), and $C:=R_{\alpha,\beta'}(0)$, in the case of Theorem 2.1(ii). Suppose that the mapping $C^{-1}f:[0,T]\times E\to E$ is continuous in t on $[0,T]$ and uniformly Lipschitz continuous (with constant L) on E. Then the semilinear fractional Cauchy problem (1) has a unique mild solution $u\in C([0,T]:E)$. Moreover, the mapping $x\to u(\cdot)$ is Lipschitz continuous from $R(C)$ (endowed with the norm $\|\cdot\|_{R(C)}:=\|C^{-1}\cdot\|$, $R(C)$ becomes a Banach space) into $C([0,T]:E)$.*

Proof. Set $M:=\max_{t\in(0,T]}(t^{1-\alpha}\Gamma(\alpha)\|P_{\alpha}(t)\|)$. Arguing as in the proof of [11, Theorem 1.2, p. 184], we get that, for every $u,\ v\in C([0,T]:E)$,

$$\|(Q_{\alpha}^n u)(t)-(Q_{\alpha}^n v)(t)\|_{C([0,T]:E)}\leq\frac{(MLT^{\alpha})^n}{\Gamma(n\alpha+1)}\|u-v\|_{C([0,T]:E)},\ n\in\mathbb{N},\ t\in[0,T].$$

For a sufficiently large number $n \in \mathbb{N}$, one has $\frac{(MLT^{\alpha})^n}{\Gamma(n\alpha+1)} < 1$ so that a well known extension of the Banach contraction principle implies that the mapping $Q_{\alpha}(\cdot)$ has a unique fixed point, finishing the proof of existence and uniqueness of mild solutions of problem (1) on $[0, T]$. Keeping in mind a Gronwall-type inequality [4, Lemma 6.19, p. 111], the remaining part of proof follows similarly as in that of [11, Theorem 1.2, p. 184]. □

We close the paper with the observation that the method proposed here can be successfully applied in the analysis of certain classes of semilinear degenerate fractional equations associated with abstract differential operators. The reader may find more details in [9].

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