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On the projective change between two special Finsler spaces of (α, β)-metrics

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Abstract In the present paper, we find some relations to characterize Projective change between two special (α, β)-metrics such as Matsumoto metric and Kropina metric on a manifold with dimension $n > 2$, where α and $\tilde{\alpha}$ are two Riemannian metrics, β and $\tilde{\beta}$ are two non-zero 1-forms. Moreover, we consider this Projective change when Matsumoto metric has some special curvature properties.

Keywords Finsler metric · (α, β)-metric · Matsumoto metric · Kropina metric · Projective change · Douglas metric and S-curvature

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1 Introduction

It is an interesting topic to study Projective changes of Finsler metrics on a manifold in Finsler geometry. Two Finsler metrics F and $\tilde{F}$ on a manifold M are said to be Projectively related if any geodesic of the first is also a geodesic of the second and vice versa. The Projective relation is said to be trivial if the corresponding sprays are equal. The Projective changes between two Finsler spaces have been studied by many authors [1], [4], [10], [11], [12], [14], [17] and [19].

The (α, β)-metric is an important class of Finsler metrics, which can be expressed in the form $F = \alpha\phi(s), s = \frac{\beta}{\alpha}$, where α is a Riemannian metric, β is a 1-form and ϕ is a C^∞ positive function on the domain. In particular, when $\phi = \frac{1}{s}$, the Finsler metric $F = \frac{\alpha^2}{\beta}$ is called Kropina metric. Kropina metric was first introduced by L. Berwald in connection with a two-dimensional Finsler space with rectilinear extremal and was investigated by Kropina [5]. Kropina metrics are non-regular Finsler metrics. Kropina metrics are simplest non-trivial Finsler metrics with many interesting application in

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physics, electron optics with a magnetic field, dissipative mechanics and irreversible thermodynamics (see [15]). Also, they have interesting applications in relativistic field theory, control theory, evolution and developmental biology.

A crucial result related with the Projective change was given by Rapcsak's paper (see [14]). He proved a necessary and sufficient condition of Projective change. In 1984, Park and Lee [12] studied Projective changes between a Finsler space with (α,β)-metric and the associated Riemannian metric. Bacso and Matsumoto [1] in 1994, considered the Projective change between Finsler spaces with (α,β)-metric.

In 2008, Shen and Civi Yildirim [18] studied on a class of Projectively flat metrics with constant flag curvature.

In 2009, Cui and Shen [4] studied Projective change between Z. Shen square metric $\frac{(\alpha+\beta)^2}{\alpha}$ and a Randers metric and in 2011, Zohrehvand and Rezaii [20] studied Projectively related Matsumoto metric and a Randers metric.

Further in 2012, Mu and Cheng [11] studied on the Projective equivalence between (α,β)-metric and a Kropina metric.

The aim of present paper is to find Projective change between the Matsumoto metric $F=\frac{\alpha^2}{\alpha-\beta}$ and Kropina metric $\tilde{F}=\frac{\tilde{\alpha}^2}{\tilde{\beta}}$, where α and $\tilde{\alpha}$ are two Riemannian metrics, β and $\tilde{\beta}$ are two non-zero 1-forms. We can characterize such Projective change. Precisely, we have the following theorem:

Theorem 1.1 *Let $F=\frac{\alpha^2}{\alpha-\beta}$ and $\tilde{F}=\frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be two (α,β)-metrics on a manifold M with dimension $n>2$, where α and $\tilde{\alpha}$ are two Riemannian metrics, $\beta=b_iy^i$ and $\tilde{\beta}=\tilde{b}_iy^i$ are two non-zero collinear 1- forms. Then F is Projectively related to $\tilde{F}$ if and only if the following relations hold*

(a) $G^i_\alpha=\tilde{G}^i_{\tilde{\alpha}}+\frac{1}{2\tilde{b}^2}[\tilde{\alpha}^2\tilde{s}^i+\tilde{r}_{00}\tilde{b}^i]+\theta y^i$;

(b) β is parallel with respect to α, i.e. $b_{i|j}=0$;

(c) $\tilde{s}_{ij}=\frac{1}{\tilde{b}^2}[\tilde{b}_i\tilde{s}_j-\tilde{b}_j\tilde{s}_i]$, where $b_{i|j}$ denote the coefficients of the covariant derivatives of β with respect to α and $\tilde{b}^i:=\tilde{a}^{ij}\tilde{b}_j$, $\tilde{b}:=\|\tilde{\beta}\|_{\tilde{\alpha}}$ and $\theta=\theta_iy^i$ is a 1-form on M.

Note that a Matsumoto metric $F=\frac{\alpha^2}{\alpha-\beta}$ is a Douglas metric if and only if β is parallel with respect to α (see [7]). It is known that Kropina metric $\tilde{F}=\frac{\tilde{\alpha}^2}{\tilde{\beta}}$ is a Douglas metric if and only if $\tilde{s}_{ij}=\frac{1}{\tilde{b}^2}[\tilde{b}_i\tilde{s}_j-\tilde{b}_j\tilde{s}_i]$ (see [8]). Therefore we have immediately from Theorem 1.1 that

Corollary 1.2 *Let $F=\frac{\alpha^2}{\alpha-\beta}$ and $\tilde{F}=\frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be two (α,β)-metrics on a manifold M with dimension $n>2$, where α and $\tilde{\alpha}$ are two Riemannian metrics, β and $\tilde{\beta}$ are two non-zero collinear 1- forms. Then F is Projectively related to $\tilde{F}$ if and only if they are Douglas metrics and the spray coefficients of α and $\tilde{\alpha}$ have the following relation:*

$$G^i_\alpha=\tilde{G}^i_{\tilde{\alpha}}+\frac{1}{2\tilde{b}^2}[\tilde{\alpha}^2\tilde{s}^i+\tilde{r}_{00}\tilde{b}^i]+\theta y^i, \tag{1.1}$$

where $\tilde{b}^i:=\tilde{a}^{ij}\tilde{b}_j$, $\tilde{b}:=\|\tilde{\beta}\|_{\tilde{\alpha}}$ and $\theta=\theta_iy^i$ is a 1-form on M.

1.1 Preliminaries

Let M be an n-dimensional C^∞-manifold, T_xM denotes the tangent space of M at x. The tangent bundle TM is the union of tangent spaces, $TM := \bigcup_{x\in M} T_xM$. We denote the elements of TM by (x,y), where $x=(x^i)$ is a point of M and $y\in T_xM$ called supporting element. We denote $TM_0 = TM\setminus\{0\}$.

Definition 1.3 *A Finsler metric on M is a function $F : TM \to [0,\infty)$ with the following properties:*

(i) F is C^∞ on TM_0;
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM;
(iii) the Hessian of F^2 with element $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definite on TM_0.

The pair $(M,F) = F^n$ is called a Finsler space. F is called the fundamental function and g_{ij} is called the fundamental tensor of the Finsler space F^n. For a given Finsler metric $F = F(x,y)$, the geodesic of F are characterized locally by a following system of second order differential equations (see [3])

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where $G^i = G^i(x,y)$ are called the spray coefficients of F, given by

$$G^i = \frac{1}{4}g^{il}\left\{\left[F^2\right]_{x^m y^l} y^m - \left[F^2\right]_{x^l}\right\}.$$

A Finsler metric is Projectively related to another metric if they have the same geodesics as point sets. In Riemannian geometry, two Riemannian metrics α and $\tilde\alpha$ are Projectively related if and only if their spray coefficients are given by (see [4])

$$G^i_\alpha = G^i_{\tilde\alpha} + \lambda_{x^k} y^k y^i, \tag{1.2}$$

where $\lambda = \lambda(x)$ is a scalar function on the based manifold and (x^i, y^j) denotes the local coordinates in the tangent bundle TM.

Let $(M,\tilde F)$ be a Finsler space with Finsler metric $\tilde F$. A Finsler metric F on M is Projective to $\tilde F$ if and only if there exists a scalar function $P(x,y)$ on $TM\setminus\{0\}$, homogeneous of degree one in y, such that

$$G^i = \tilde G^i + P(x,y)y^i, \tag{1.3}$$

where G^i and $\tilde G^i$ are the spray coefficients of F and $\tilde F$ respectively.

A Finsler metric is called a Projectively flat metric if it is Projectively related to a locally Minkowskian metric.

The concept of (α,β)-metrics were first introduced in 1972 by Matsumoto [9]. By definition, an (α,β)-metric is a Finsler metric expressed in the following form $F = \alpha\phi(s), s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form satisfying $\|\beta_x\| < b_0$, $\forall x\in M$. It is well known that $F = \alpha\phi(s)$ is a regular

(α,β)-metric if the function $\phi(s)$ is a positive C^∞ function with $|s| < b_0$ satisfying (see [3])

$$\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0, \quad |s| \leq b < b_0. \tag{1.4}$$

In this case the metric tensor induced by F is positive definite. Let $r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i})$, $s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i})$, where $b_{i|j}$ means the coefficients of the covariant derivative of β with respect to α. Clearly β is closed if and only if $s_{ij} = 0$. An (α,β)-metric is said to be trivial if $r_{ij} = s_{ij} = 0$. Furthermore, we denote $r^i_j := a^{ik} r_{kj}, s^i_j := a^{ik} s_{kj}$, $r_{00} := r_{ij} y^i y^j, r_{i0} := r_{ij} y^j$, $s_i := b_j s^j_i, s_0 := s_i y^i$, $r := r_{ij} b^i b^j, s_{i0} := s_{ij} y^j$. The relation between the spray coefficients G^i of F and geodesic coefficients G^i_α of α are given by (see [3], [16])

$$G^i = G^i_\alpha + \alpha Q s^i_0 + \{-2Q\alpha s_0 + r_{00}\}\left\{\Psi b^i + \Theta \alpha^{-1} y^i\right\}, \tag{1.5}$$

where

$$\Theta = \frac{\phi\phi^{'} - s(\phi\phi^{''} + \phi^{'}\phi^{'})}{2\phi\left((\phi - s\phi^{'}) + (b^2 - s^2)\phi^{''}\right)}, \quad Q = \frac{\phi^{'}}{\phi - s\phi^{'}},$$

and

$$\Psi = \frac{1}{2}\frac{\phi^{''}}{(\phi - s\phi^{'}) + (b^2 - s^2)\phi^{''}}.$$

The tensor $D := D^i_{jkl}\partial_i \otimes dx^j \otimes dx^k \otimes dx^l$ is called Douglas tensor where,

$$D^i_{jkl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m} y^i\right). \tag{1.6}$$

A Finsler metric is called Douglas metric if the Douglas tensor vanishes (see [4]). The Douglas tensor is Projective invariant (see [13]). Since the spray coefficients of a Riemannian metric are quadratic forms. It can be easly seen that for Riemannian metric Douglas tensor vanishes, which shows that Douglas tensor is a non-Riemannian quantity. The fundamental fact is that all Berwald metrics must be Douglas metrics.

The Douglas tensor of a general (α,β)-metric is determined by

$$D^i_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m} y^i\right), \tag{1.7}$$

where

$$T^i = \alpha Q s^i_0 + \Psi\{-2Q\alpha s_0 + r_{00}\} b^i, \quad Q = \frac{\phi^{'}}{\phi - s\phi^{'}}, \tag{1.8}$$

and

$$\begin{aligned} T^m_{y^m} &= Q^{'} s_0 + \Psi^{'}\alpha^{-1}(b^2 - s^2)[r_{00} - 2Q\alpha s_0] \\ &\quad + 2\Psi[r_0 - Q^{'}(b^2 - s^2)s_0 - Qss_0], \end{aligned} \tag{1.9}$$

where $Q^{'} = \frac{\phi\phi^{''}}{(\phi - s\phi^{'})^2}$. Thus, if F and $\tilde{F}$ have the same Douglas tensor, i.e. $D^i_{jkl} = D^{\tilde{i}}_{jkl}$. From (1.6) and (1.7), we get

$$\frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left[T^i - \tilde{T}^i - \frac{1}{n+1}(T^m_{y^m} - T^{\tilde{m}}_{y^m})y^i\right] = 0. \tag{1.10}$$

Then there exists a class of scalar functions $H^i_{jk} := H^i_{jk}(x)$, such that

$$T^i - \tilde{T}^i - \frac{1}{n+1}(T^m_{y^m} - \tilde{T^m_{y^m}})y^i = H^i_{00}, \tag{1.11}$$

where $H^i_{00} := H^i_{jk}(x)y^jy^k$, T^i and $T^m_{y^m}$ are given by the relations (1.8) and (1.9) respectively.

2 Projective change of two (α,β) metrics

In this section, we consider on the Projectively related two special (α,β)-metrics, namely Matsumoto metric ($F = \frac{\alpha^2}{\alpha-\beta}$) and Kropina metric ($\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$).

For (α,β)-metric $F = \frac{\alpha^2}{\alpha-\beta}$, one can prove by (see [6]), that F is a regular Finsler metric if and only if $\|\beta_x\| < \frac{1}{2}$ for any $x \in M$. The geodesic coefficients of the Finsler metric $F = \frac{\alpha^2}{\alpha-\beta}$ are given by (1.5), where

$$Q = \frac{1}{1-2s}, \quad \Theta = \frac{1-4s}{2(1-3s+2b^2)}, \quad \Psi = \frac{1}{1-3s+2b^2}. \tag{2.1}$$

For Kropina-metric $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$, one can see that F is not a regular (α,β)-metric, but the relation $\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0$ is still true for $|s| > 0$.

In view of equation (1.5), geodesic coefficients of the Finsler metric $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ are given by

$$\tilde{G}^i = \tilde{G}^i_{\tilde{\alpha}} + \tilde{\alpha}\tilde{Q}\tilde{s}^i_0 + \{-2\tilde{Q}\tilde{\alpha}\tilde{s}_0 + \tilde{r}_{00}\}\{\tilde{\Psi}\tilde{b}^i + \tilde{\Theta}\tilde{\alpha}^{-1}y^i\}, \tag{2.2}$$

with

$$\tilde{Q} = -\frac{1}{2\tilde{s}}, \quad \tilde{\Theta} = -\frac{\tilde{s}}{\tilde{b}^2}, \quad \tilde{\Psi} = \frac{1}{2\tilde{b}^2}. \tag{2.3}$$

For simplicity, we assume in this paper that $\lambda := \frac{1}{(n+1)}$.

Lemma 2.1 **([8])** *Let* $F = \frac{\alpha^2}{\beta}$ *be a Kropina metric on an* n*-dimensional manifold* M*. Then*

(1) ($n \geq 3$) *Kropina metric* F *with* $b^2 \neq 0$ *is a Douglas metric if and only if*

$$s_{ik} = \frac{1}{b^2}\left(b_i s_k - b_k s_i\right). \tag{2.4}$$

(2) ($n = 2$) *Kropina metric* F *is a Douglas metric.*

Since the Douglas tensor is a Projective invariant, we have

Proposition 2.2 *Let* $F = \frac{\alpha^2}{\alpha-\beta}$ *be a Matsumoto metric and* $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ *be a Kropina metric on a manifold* M *with dimension* $n \geq 3$*, where* α *and* $\tilde{\alpha}$ *are two Riemannian metrics,* β *and* $\tilde{\beta}$ *are two non-zero* 1*-forms. Then* F *and* $\tilde{F}$ *have the same Douglas tensor if and only if they are all Douglas metrics.*

Proof. The sufficiency is obvious, we just need to prove the necessity. Suppose that F and $\tilde{F}$ have the same Douglas tensor on an n-dimensional manifold M when $n \geq 3$.

Then, $D^i_{jkl} = \tilde{D}^i_{jkl}$, which implies that (1.11) holds and hence we have

$$\frac{A^i\alpha^5 + B^i\alpha^4 + C^i\alpha^3 + D^i\alpha^2 + E^i\alpha + F^i}{I\alpha^4 + J\alpha^3 + K\alpha^2 + L\alpha} + \frac{\tilde{A}^i\tilde{\alpha}^2 + \tilde{B}^i}{2\tilde{b}^2\tilde{\beta}} = H^i_{00}, \tag{2.5}$$

where

$A^i := (1+2b^2)[(1+2b^2)s^i_0 - 2b^i s_0]$,
$B^i := (1+2b^2)[b^i r_{00} - 2\lambda y^i r_0] - 2\lambda y^i s_0(1-b^2) + 6b^i s_0\beta$,
$C^i := (4b^2+5)\beta[2\lambda y^i(r_0+s_0) - b^i r_{00}] - 3\lambda y^i b^2 r_{00} + 9\beta^2 s^i_0$,
$D^i := 6\beta[\lambda y^i(b^2 r_{00} - \beta(2r_0+3s_0) + b^i\beta r_{00})]$,
$E^i := 3\lambda y^i\beta^2 r_{00}$,
$F^i := -6\lambda y^i\beta^3 r_{00}$,
$I := (1+2b^2)^2$, $J := -4\beta(1+2b^2)(2+b^2)$,
$K := 3\beta^2[4(1+2b^2)+3]$, $L := -18\beta^3$,

and

$\tilde{A}^i := \tilde{b}^2\tilde{s}^i_0 - \tilde{b}^i\tilde{s}_0$,
$\tilde{B}^i := \tilde{\beta}[2\lambda y^i(\tilde{r}_0 + \tilde{s}_0) - \tilde{b}^i\tilde{r}_{00}]$.

Further, (2.5) is equivalent to

$$\begin{aligned}&(A^i\alpha^5 + B^i\alpha^4 + C^i\alpha^3 + D^i\alpha^2 + E^i\alpha + F^i)(2\tilde{b}^2\tilde{\beta})\\ &\quad + (\tilde{A}^i\tilde{\alpha}^2 + \tilde{B}^i)(I\alpha^4 + J\alpha^3 + K\alpha^2 + L\alpha)\\ &\quad = H^i_{00}(2\tilde{b}^2\tilde{\beta})(I\alpha^4 + J\alpha^3 + K\alpha^2 + L\alpha).\end{aligned} \tag{2.6}$$

Replacing y^i by $-y^i$ in (2.6) yields

$$\begin{aligned}&(-A^i\alpha^5 + B^i\alpha^4 - C^i\alpha^3 + D^i\alpha^2 - E^i\alpha + F^i)(-2\tilde{b}^2\tilde{\beta})\\ &\quad - (\tilde{A}^i\tilde{\alpha}^2 + \tilde{B}^i)(I\alpha^4 - J\alpha^3 + K\alpha^2 - L\alpha)\\ &\quad = H^i_{00}(-2\tilde{b}^2\tilde{\beta})(I\alpha^4 - J\alpha^3 + K\alpha^2 - L\alpha).\end{aligned} \tag{2.7}$$

Adding (2.6) and (2.7), yields

$$\begin{aligned}&(A^i\alpha^5 + C^i\alpha^3 + E^i\alpha)(2\tilde{b}^2\tilde{\beta}) + (J\alpha^3 + L\alpha)(\tilde{A}^i\tilde{\alpha}^2 + \tilde{B}^i)\\ &\quad = H^i_{00}(2\tilde{b}^2\tilde{\beta})(J\alpha^3 + L\alpha).\end{aligned} \tag{2.8}$$

Again, subtracting (2.7) from (2.6), yields

$$\begin{aligned}&(B^i\alpha^4 + D^i\alpha^2 + F^i)(2\tilde{b}^2\tilde{\beta}) + (I\alpha^4 + K\alpha^2)(\tilde{A}^i\tilde{\alpha}^2 + \tilde{B}^i)\\ &\quad = H^i_{00}(2\tilde{b}^2\tilde{\beta})(I\alpha^4 + K\alpha^2)\end{aligned} \tag{2.9}$$

From above equation, we can see that $\tilde{A}^i\tilde{\alpha}^2(I\alpha^4 + K\alpha^2)$ can be divided by $\tilde{\beta}$. Since $\beta = \mu\tilde{\beta}$, then $\tilde{A}^i\tilde{\alpha}^2 I\alpha^4$ can be divided by $\tilde{\beta}$. Because $\tilde{\beta}$ is prime with respect to α and $\tilde{\alpha}$, therefore $\tilde{A}^i := \tilde{b}^2\tilde{s}^i_0 - \tilde{b}^i\tilde{s}_0$ can be divided by $\tilde{\beta}$. Hence, there is a scalar function $\varphi^i(x)$ such that

$$\tilde{b}^2\tilde{s}^i_0 - \tilde{b}^i\tilde{s}_0 = \tilde{\beta}\varphi^i. \tag{2.10}$$

Contracting (2.10) by $\tilde{y}_i := \tilde{a}_{ij}y^j$, we get that $\varphi^i(x) = -\tilde{s}^i$. Then we have

$$\tilde{s}_{ij} = \frac{1}{\tilde{b}^2}[\tilde{b}_i\tilde{s}_j - \tilde{b}_j\tilde{s}_i], \tag{2.11}$$

provided, $b^2 \neq 0$. Thus, by Lemma (2.1), $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ is a Douglas metric. Since F and $\tilde{F}$ have the same Douglas tensor, both of them are Douglas metrics. When $n = 2$, $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ is a Douglas metric by Lemma (2.1). Thus, F and $\tilde{F}$ having the same Douglas tensor means that they are all Douglas metrics. This completes the proof of proposition (2.2). □

Now, we are in the position to prove Theorem (1.1).

3 Proof of Theorem 1.1

First we prove the necessity. Since Douglas tensor is an invariant under Projective change between two Finsler metrics. If F is Projectively related to $\tilde{F}$, then they have the same Douglas tensor. By proposition (2.2), we know that F and $\tilde{F}$ are both Douglas metrics. It is well known that $F = \frac{\alpha^2}{\alpha-\beta}$ is a Douglas metric if and only if

$$b_{i|j} = 0, \tag{3.1}$$

where $b_{i|j}$ denote the coefficients of the covariant derivatives of β with respect to α.

Plugging (3.1) and (2.1) into (1.5) yields

$$G^i = G^i_\alpha. \tag{3.2}$$

It has been proved in [8] that $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ is a Douglas metric if and only if

$$\tilde{s}_{ij} = \frac{1}{\tilde{b}^2}[\tilde{b}_i\tilde{s}_j - \tilde{b}_j\tilde{s}_i]. \tag{3.3}$$

On the other hand, plugging (3.3) and (2.3) into (2.2) yields

$$\tilde{G}^i = \tilde{G}^i_{\tilde{\alpha}} - \frac{1}{2\tilde{b}^2}\left[-\tilde{\alpha}^2\tilde{s}^i + (2\tilde{s}_0y^i - \tilde{r}_{00}\tilde{b}^i) + \frac{2\tilde{r}_{00}\tilde{\beta}y^i}{\tilde{\alpha}^2}\right]. \tag{3.4}$$

Since F is Projectively related to $\tilde{F}$ again, there is a scalar function $P := P(x,y)$ on $TM\setminus\{0\}$ such that

$$G^i = \tilde{G}^i + Py^i. \tag{3.5}$$

From (3.2), (3.4) and (3.5), we have

$$\left[P - \frac{1}{\tilde{b}^2}\left(\tilde{s}_0 + \frac{\tilde{r}_{00}\tilde{\beta}}{\tilde{\alpha}^2}\right)\right]y^i = G^i_\alpha - \tilde{G}^i_{\tilde{\alpha}} - \frac{1}{2\tilde{b}^2}\left[\tilde{\alpha}^2\tilde{s}^i + \tilde{r}_{00}\tilde{b}^i\right]. \tag{3.6}$$

The Right-hand side of the above equation is a quadratic form in y. Then there exists a one form $\theta = \theta_i(x)y^i$ on M such that,

$$P - \frac{1}{\tilde{b}^2}\left(\tilde{s}_0 + \frac{\tilde{r}_{00}\tilde{\beta}}{\tilde{\alpha}^2}\right) = \theta. \tag{3.7}$$

Then we have

$$G^i_\alpha = \tilde{G}^i_{\tilde{\alpha}} + \frac{1}{2\tilde{b}^2}[\tilde{\alpha}^2\tilde{s}^i + \tilde{r}_{00}\tilde{b}^i] + \theta y^i. \tag{3.8}$$

From (3.1), (3.3) and (3.8), we complete the proof of the necessity.

Conversely, plugging (3.1) into (1.5) with (2.1) yields (3.2). Plugging (3.3) into (1.5) with (2.3) yields (3.4). From (3.2), (3.4) and (3.8) we have

$$G^i = \tilde{G}^i + \left[\theta + \frac{1}{\tilde{b}^2}\left(\tilde{s}_0 + \frac{\tilde{r}_{00}\tilde{\beta}}{\tilde{\alpha}^2}\right)\right] y^i. \tag{3.9}$$

i.e. F is Projectively related to $\tilde{F}$. We complete the proof of theorem (1.1).

4 Isotropic Berwald curvature

It is well known that the Berwald curvature tensor of a Finsler metric F is defined by $\mathbf{B} := B^i_{jkl}dx^j \otimes \partial_i \otimes dx^k \otimes dx^l$, where $B^i_{jkl} = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l} = [G^i]_{y^j y^k y^l}$ and G^i are the spray coefficients of F.

A Finsler metric F is of isotropic Berwald curvature if $B^i_{jkl} = c(F_{y^j y^k}\delta^i_l + F_{y^j y^l}\delta^i_k + F_{y^k y^l}\delta^i_j + F_{y^j y^k y^l}y^i)$, where $c = c(x)$ is a scalar function on M (see [3]).

The mean Berwald curvature tensor is defined by $\mathbf{E} := E_{ij}dx^i \otimes dx^j$, where $E_{ij} := \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j}(\frac{\partial G^m}{\partial y^m}) = \frac{1}{2}B^m_{mij}$. A Finsler metric is said to be of isotropic mean Berwald curvature if $E_{ij} = \frac{n+1}{2}c(x)F_{y^i y^j}$, where $c = c(x)$ is a scalar function on M (see [17]). Clearly, the Finsler metric of isotropic Berwald curvature must be of isotropic mean Berwald curvature.

A Finsler metric F is said to have isotropic S-curvature if $\mathbf{S} = (n+1)c(x)F$, for some scalar function $c(x)$ on M (see [3]).

Cheng and Lu prove the following

Lemma 4.1 **([2])** *For the Matsumoto metric $F = \frac{\alpha^2}{\alpha-\beta}$ on an n-dimensional manifold M, the following are equivalent:*

(a) *F is of isotropic S-curvature, that is, $\boldsymbol{S} = (n+1)c(x)F$;*
(b) *F is of isotropic mean Berwald curvature, $\boldsymbol{E} = \frac{n+1}{2}c(x)F^{-1}\boldsymbol{h}$;*
(c) *F has vanished S-curvature, that is, $\boldsymbol{S} = 0$;*
(d) *F is a weak Berwald metric, that is, $\boldsymbol{E} = 0$;*
(e) *β is a Killing 1-form of constant length with respect to α, that is, $r_{00} = 0$ and $s_0 = 0$, where $c = c(x)$ is a scalar function.*

Theorem 4.2 *Let $F = \frac{\alpha^2}{\alpha-\beta}$ be Projectively equivalent to $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ and $\tilde{F}$ have isotropic Berwald curvature. Then F has isotropic Berwald curvature if and only if F has isotropic S-curvature.*

Proof. Suppose F has isotropic Berwald curvature, then F has isotropic mean Berwald curvature. Thus, by Lemma (4.1), F is of isotropic S-curvature. This proves necessary condition.

After this, we show the sufficiency. Since F and $\tilde{F}$ are Projectively equivalent, (1.3) holds. Suppose that F has isotropic S-curvature $\mathbf{S}=(n+1)c(x)F$.

By Lemma (4.1), we have F is of isotropic mean Berwald curvature, that is, $E_{ij}=\frac{n+1}{2}cF_{y^iy^j}$.

Given that $\tilde{F}$ has isotropic Berwald curvature, then

$$\tilde{B}^i_{jkl}=\tilde{c}(\tilde{F}_{y^jy^k}\delta^i_l+\tilde{F}_{y^jy^l}\delta^i_k+\tilde{F}_{y^ky^l}\delta^i_j+\tilde{F}_{y^jy^ky^l}y^i),$$

where $\tilde{c}=\tilde{c}(x)$ is a scalar function on M. Hence, by the definition of the mean Berwald tensor, it follows from (1.3) that $cF_{y^iy^j}=\tilde{c}\tilde{F}_{y^iy^j}+P_{y^iy^j}$, which gives that $cF_{y^iy^jy^k}=\tilde{c}\tilde{F}_{y^iy^jy^k}+P_{y^iy^jy^k}$. Now we have

$$\begin{aligned}B^i_{jkl}=\frac{\partial^3G^i}{\partial y^j\partial y^k\partial y^l}&=\tilde{B}^i_{jkl}+(P_{y^jy^k}\delta^i_l+P_{y^jy^l}\delta^i_k+P_{y^ky^l}\delta^i_j+P_{y^jy^ky^l}y^i)\\&=\tilde{c}(\tilde{F}_{y^jy^k}\delta^i_l+\tilde{F}_{y^jy^l}\delta^i_k+\tilde{F}_{y^ky^l}\delta^i_j+\tilde{F}_{y^jy^ky^l}y^i)\\&\quad+(P_{y^jy^k}\delta^i_l+P_{y^jy^l}\delta^i_k+P_{y^ky^l}\delta^i_j+P_{y^jy^ky^l}y^i)\\&=c(F_{y^jy^k}\delta^i_l+F_{y^jy^l}\delta^i_k+F_{y^ky^l}\delta^i_j+F_{y^jy^ky^l}y^i),\end{aligned}$$

this implies that F has isotropic Berwald curvature. We complete the proof. □

By the above methods, we could obtain the theorem.

Theorem 4.3 *Let $F=\frac{\alpha^2}{\alpha-\beta}$ be Projectively equivalent to $\tilde{F}=\frac{\tilde{\alpha}^2}{\tilde{\beta}}$ and F have isotropic Berwald curvature. Then $\tilde{F}$ has isotropic Berwald curvature if and only if $\tilde{F}$ has isotropic S-curvature.*

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