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Factorizations of reflexive-EP elements in rings

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Abstract We present characterizations of reflexive–EP elements in rings using different kinds of factorizations, generalizing well-known characterizations of (weighted) EP matrices, (weighted) EP Hilbert space operators, (weighted) EP C^*-algebra elements and (weighted) EP Banach algebra elements. Also, we show that the product of two or more reflexive–EP elements is reflexive–EP.

Keywords EP elements · Outer inverse · Ring

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1 Introduction

A square complex matrix (or a Hilbert space operator with closed range) A is said to be EP or range-Hermitian, if $R(A) = R(A^*)$ (EP for equal projections onto $R(A)$ and $R(A^*)$). EP matrices are a wide class of objects that include many matrices as their special cases, such as, Hermitian and normal matrices. Several authors in [1,3,7,9] have addressed themselves to the question of characterizing EP matrices, EP linear operators on Banach or Hilbert spaces and EP elements of C^*–algebras or Banach algebras. In rings with involution EP elements are those elements for which the group and the Moore–Penrose inverse exist and coincide [19,22]. The EP elements are important since they are characterized by commutativity with their Moore–Penrose inverse.

Tian and Wang [24] introduced the notion of weighted–EP matrices (matrices that commute with their weighted Moore-Penrose inverse). Similar objects in the contexts of C^*-algebra elements, Banach space operators and Banach algebra elements are investigated in [5,20].

One of the main lines of research concerning EP matrices and EP operators consists in characterizing them through factorizations. In [4,10,12,21], the authors have char-

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acterized (weighted) EP matrices, Hilbert and Banach space operators and Banach and C^*-algebra elements through several different factorizations.

Several articles studied when the product of two EP matrices, Hilbert or Banach space operators, or elements of a C^*-algebra or a Banach algebra is again EP, see [2, 3, 8, 13, 14, 16].

In [17], the reflexive–EP elements in rings were defined and characterized, that is elements which commute with their reflexive image-kernel (p, q)-inverse. In this way EP objects are extended from C^*-algebras or rings with involution to rings, a context where no involution is available. Also, the reflexive–EP elements in rings were characterized using the factorization $a = uv$.

The objective of this article is to characterize reflexive–EP elements of rings using two kind of factorizations. Thus, we extend results [10] to the mentioned context where no involution is available, using an idea that is similar to the one that led to introduce EP Banach space operators and EP Banach algebra elements. Furthermore, we prove that the product of reflexive–EP elements is reflexive–EP.

2 Preliminary definitions and results

Let $\mathcal{R}$ be a ring with the unit 1. The set of all idempotents of $\mathcal{R}$ will be denoted by $\mathcal{R}^\bullet$. For $a \in \mathcal{R}$, we define the following kernel ideals $a^\circ = \{x \in \mathcal{R} : ax = 0\}$, ${}^\circ a = \{x \in \mathcal{R} : xa = 0\}$, and image ideals $a\mathcal{R} = \{ax : x \in \mathcal{R}\}$ and $\mathcal{R}a = \{xa : x \in \mathcal{R}\}$.

For $u, v \in \mathcal{R}^\bullet$, observe that $u^\circ = (1-u)\mathcal{R}$ and ${}^\circ u = \mathcal{R}(1-u)$. Also, we have

$$u\mathcal{R} = v\mathcal{R} \;\Leftrightarrow\; {}^\circ u = {}^\circ v$$

and

$$\mathcal{R}u = \mathcal{R}v \;\Leftrightarrow\; u^\circ = v^\circ.$$

Let $a \in \mathcal{R}$ and let $p, q \in \mathcal{R}^\bullet$. An element $b \in \mathcal{R}$ is the image-kernel (p, q)-inverse of a if

$$bab = b, \quad ba\mathcal{R} = p\mathcal{R} \quad \text{and} \quad (1-ab)\mathcal{R} = q\mathcal{R}.$$

The image-kernel (p, q)-inverse b is unique if it exists [15], and it will be denoted by $a^\times$.

Notice that the condition $ba\mathcal{R} = p\mathcal{R}$ is equivalent to ${}^\circ(ba) = {}^\circ p$, and the condition $(1-ab)\mathcal{R} = q\mathcal{R}$ is equivalent to $\mathcal{R}ab = \mathcal{R}(1-q)$ which is equivalent to $(ab)^\circ = (1-q)^\circ$.

An equivalent condition for the existence of the image-kernel (p, q)-inverse without explicit reference to ideals is given in following theorem.

Theorem 2.1 ([23], Theorem 2.1) *Let $p, q \in \mathcal{R}^\bullet$ and let $a \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *$a^\times$ exists,*
(ii) *there exists some $b \in \mathcal{R}$ such that*

$$b = pb, \quad bap = p, \quad bq = 0, \quad 1-q = (1-q)ab.$$

The element b in the part (ii) of Theorem 2.1 satisfies $b = a^{\times}$.

The image-kernel (p, q)-inverse of Kantún-Montiel [15] coincides with the (p, q, l)-outer generalized inverse of Cao and Xue [6]. The particular case of the image-kernel (p, q)-inverse is the (p, q)-outer generalized inverse, for more details see [11]. Inner image-kernel (p, q)-inverses in rings can be found in [18].

If the image-kernel (p, q)-inverse b of a satisfies the equations $a = aba$, then b is a reflexive image-kernel (p, q)-inverse of a and it is denote by $a^{(1,\times)}$. It follows that $a^{(1,\times)}$ is also unique in the case when it exists.

In [17], the reflexive–EP elements in rings were introduced in the following way.

Definition 2.2 *Let $a \in \mathcal{R}$ and $p, q \in \mathcal{R}^{\bullet}$ such that $a^{(1,\times)}$ exists. The element a is reflexive–EP if $aa^{(1,\times)} = a^{(1,\times)}a$.*

Thus, the reflexive–EP elements are elements which commute with their reflexive image-kernel (p, q)-inverse. In this way EP objects are extended from C^*-algebras or rings with involution to rings, a context where no involution is available.

Theorem 2.3 [17, Theorem 2.1] *Let $p, q \in \mathcal{R}^{\bullet}$ and let $a \in \mathcal{R}$ such that $a^{(1,\times)}$ exists. Then the following statements are equivalent:*

(i) $aa^{(1,\times)} = a^{(1,\times)}a$,
(ii) $a\mathcal{R} = p\mathcal{R}$ *and* $\mathcal{R}a = \mathcal{R}(1 - q)$,
(iii) $a\mathcal{R} \supset p\mathcal{R}$ *and* $\mathcal{R}a \supset \mathcal{R}(1 - q)$,
(iv) ${}^{\circ}a = {}^{\circ}p$ *and* $a^{\circ} = (1 - q)^{\circ}$,
(v) $a \in a^{(1,\times)}\mathcal{R}^{-1} \cap \mathcal{R}^{-1}a^{(1,\times)}$.

3 Factorization $a^{(1,\times)} = sa$

In this section, we will characterize reflexive–EP elements of a ring through factorizations of the form $a^{(1,\times)} = sa$. Note that in contexts where an involution exists, i.e., matrices, Hilbert space operators or C^*-algebras, this kind of factorization correspond to $a^* = sa$.

Theorem 3.1 *Let $p, q \in \mathcal{R}^{\bullet}$ and let $a \in \mathcal{R}$ such that $a^{(1,\times)}$ exists. Then the following statements are equivalent:*

(i) *a is reflexive–EP element;*
(ii) $\exists\, s, t \in \mathcal{R} : s^{\circ} = \{0\}$, $t\mathcal{R} = \mathcal{R}$ *and* $a^{(1,\times)} = sa = at$;
(iii) $\exists\, s_1, t_1 \in \mathcal{R} : a^{(1,\times)} = s_1a = at_1$;
(iv) $\exists\, u, v \in \mathcal{R} : a^{(1,\times)}a = au$ *and* $aa^{(1,\times)} = va$;
(v) $\exists\, u_1, v_1 \in \mathcal{R} : a^{(1,\times)}a = u_1a^{(1,\times)}$ *and* $aa^{(1,\times)} = a^{(1,\times)}v_1$;
(vi) $\exists\, x, y \in \mathcal{R}^{-1} : a^{(1,\times)}a = xaa^{(1,\times)} = aa^{(1,\times)}y$;
(vii) $\exists\, x_1, y_1 \in \mathcal{R} : x_1^{\circ} = \{0\}$; $y_1\mathcal{R} = \mathcal{R}$ *and* $a^{(1,\times)}a = x_1aa^{(1,\times)} = aa^{(1,\times)}y_1$;
(viii) $\exists\, x_2, y_2 \in \mathcal{R} : x_2^{\circ} = \{0\}$ *and* $a^{(1,\times)}a = x_2aa^{(1,\times)} = aa^{(1,\times)}y_2$;
(ix) $\exists\, z_1, z_2 \in \mathcal{R} : a^{(1,\times)}a = az_1a^{(1,\times)}$ *and* $aa^{(1,\times)} = a^{(1,\times)}z_2a$.

Proof. (i) $\Rightarrow$ (ii): By Theorem 2.3, the statement (i) implies $a \in a^{(1,\times)}\mathcal{R}^{-1} \cap \mathcal{R}^{-1}a^{(1,\times)}$. So, there exist $s, t \in \mathcal{R}^{-1}$ such that $a^{(1,\times)} = sa = at$ and (ii) holds.

(ii) $\Rightarrow$ (iii): This is trivial.

(iii) $\Rightarrow$ (i): Using the statements (iii), we get $a^{(1,\times)}\mathcal{R} \subset a\mathcal{R}$ and $\mathcal{R}a^{(1,\times)} \subset \mathcal{R}a$, which give

$$p\mathcal{R} = a^{(1,\times)}a\mathcal{R} = a^{(1,\times)}\mathcal{R} \subset a\mathcal{R}$$

and

$$\mathcal{R}(1-q) = \mathcal{R}aa^{(1,\times)} = \mathcal{R}a^{(1,\times)} \subset \mathcal{R}a.$$

Hence, by Theorem 2.3, we deduce that a is reflexive–EP element.

(i) $\Rightarrow$ (iv): It follows for $u = v = a^{(1,\times)}$.

(iv) $\Rightarrow$ (i): If the hypothesis (iv) holds, then $p\mathcal{R} = a^{(1,\times)}a\mathcal{R} \subset a\mathcal{R}$ and $\mathcal{R}(1-q) = \mathcal{R}aa^{(1,\times)} \subset \mathcal{R}a$. Applying Theorem 2.3, we conclude that the statement (i) is satisfied.

(i) $\Rightarrow$ (v): This implication is evident for $u_1 = v_1 = a$.

(v) $\Rightarrow$ (i): Suppose that there exist $u_1, v_1 \in \mathcal{R}$ such that $a^{(1,\times)}a = u_1a^{(1,\times)}$ and $aa^{(1,\times)} = a^{(1,\times)}v_1$. Then

$$a^{(1,\times)}a = u_1a^{(1,\times)}aa^{(1,\times)} = a^{(1,\times)}aaa^{(1,\times)}$$

and

$$aa^{(1,\times)} = a^{(1,\times)}aa^{(1,\times)}v_1 = a^{(1,\times)}aaa^{(1,\times)}$$

yield $a^{(1,\times)}a = aa^{(1,\times)}$. Therefore, the statement (i) holds.

(i) $\Rightarrow$ (vi): Obviously, for $x = y = 1$.

(vi) $\Rightarrow$ (vii) $\Rightarrow$ (viii): It is easy to verify these implications.

(viii) $\Rightarrow$ (i): The equality $a^{(1,\times)}a = x_2aa^{(1,\times)}$ gives $(aa^{(1,\times)})^\circ \subset (a^{(1,\times)}a)^\circ$. On the other hand, if $y \in (a^{(1,\times)}a)^\circ$, then $a^{(1,\times)}ay = 0$, that is, $x_2aa^{(1,\times)}y = 0$. Since $x_2^\circ = \{0\}$, we have $aa^{(1,\times)}y = 0$ and so $y \in (aa^{(1,\times)})^\circ$. Thus, $(a^{(1,\times)}a)^\circ \subset (aa^{(1,\times)})^\circ$. Now, we obtain $(aa^{(1,\times)})^\circ = (a^{(1,\times)}a)^\circ$ which implies $\mathcal{R}aa^{(1,\times)} = \mathcal{R}a^{(1,\times)}a$. So, $\mathcal{R}(1-q) = \mathcal{R}a$.

From the assumption $a^{(1,\times)}a = aa^{(1,\times)}y_2$, it follows $p\mathcal{R} = a^{(1,\times)}a\mathcal{R} \subset a\mathcal{R}$. Using Theorem 2.3, we show that the element a is reflexive–EP.

(i) $\Rightarrow$ (ix): If we state that $z_1 = z_2 = 1$, this implication follows.

(ix) $\Rightarrow$ (i): Observe that the equalities $a^{(1,\times)}a = az_1a^{(1,\times)}$ and $aa^{(1,\times)} = a^{(1,\times)}z_2a$ imply

$$p\mathcal{R} = a^{(1,\times)}a\mathcal{R} = az_1a^{(1,\times)}\mathcal{R} \subset a\mathcal{R}$$

and

$$\mathcal{R}(1-q) = \mathcal{R}aa^{(1,\times)} = \mathcal{R}a^{(1,\times)}z_2a \subset \mathcal{R}a.$$

Consequently, by Theorem 2.3, (i) holds. □

4 Factorization of the form $a = ucv$

In this section, the reflexive–EP elements of the form $a = ucv$ will be characterized.

Theorem 4.1 *Let $p, q \in \mathcal{R}^\bullet$ and let $a \in \mathcal{R}$ such that $a^{(1,\times)}$ exists. Then the following statements are equivalent:*

(i) *a is reflexive–EP element;*
(ii) *$\exists\, c, d, u, v \in \mathcal{R} : a = ucv$, $a^{(1,\times)} = udv$, $v\mathcal{R} = \mathcal{R} = \mathcal{R}u$, $c\mathcal{R} = d\mathcal{R}$ and $\mathcal{R}c = \mathcal{R}d$;*
(iii) *$\exists\, c, d, u, v \in \mathcal{R} : a = ucv$, $a^{(1,\times)} = udv$, $u^\circ = \{0\} = {}^\circ v$, $c^\circ = d^\circ$ and ${}^\circ c = {}^\circ d$;*

(iv) $\exists\ c,d,u,v \in \mathcal{R} : a^{(1,\times)}a = ucv,\ aa^{(1,\times)} = udv,\ v\mathcal{R} = \mathcal{R} = \mathcal{R}u,\ c\mathcal{R} = d\mathcal{R}$ *and* $\mathcal{R}c = \mathcal{R}d$;

(v) $\exists\ c,d,u,v \in \mathcal{R} : a^{(1,\times)}a = ucv,\ aa^{(1,\times)} = udv,\ u^\circ = \{0\} = {}^\circ v,\ c^\circ = d^\circ$ *and* ${}^\circ c = {}^\circ d$.

Proof. (i) $\Rightarrow$ (ii): Assume that $u = v = 1$, $c = a$ and $d = a^{(1,\times)}$. Now, we have $v\mathcal{R} = \mathcal{R} = \mathcal{R}u$,

$$c\mathcal{R} = a\mathcal{R} = aa^{(1,\times)}\mathcal{R} = a^{(1,\times)}a\mathcal{R} = a^{(1,\times)}\mathcal{R} = d\mathcal{R}$$

and

$$\mathcal{R}c = \mathcal{R}a = \mathcal{R}a^{(1,\times)}a = \mathcal{R}aa^{(1,\times)} = \mathcal{R}a^{(1,\times)} = \mathcal{R}d.$$

(ii) $\Rightarrow$ (i): If there exist $c,d,u,v \in \mathcal{R}$ such that $a = ucv$, $a^{(1,\times)} = udv$, $v\mathcal{R} = \mathcal{R} = \mathcal{R}u$, $c\mathcal{R} = d\mathcal{R}$ and $\mathcal{R}c = \mathcal{R}d$, then

$$a\mathcal{R} = ucv\mathcal{R} = uc\mathcal{R} = ud\mathcal{R} = udv\mathcal{R} = a^{(1,\times)}\mathcal{R} = a^{(1,\times)}a\mathcal{R} = p\mathcal{R}$$

and

$$\mathcal{R}a = \mathcal{R}ucv = \mathcal{R}cv = \mathcal{R}dv = \mathcal{R}udv = \mathcal{R}a^{(1,\times)} = \mathcal{R}aa^{(1,\times)} = \mathcal{R}(1-q).$$

By Theorem 2.3, we conclude that a is reflexive–EP element.

(i) $\Rightarrow$ (iii): The hypothesis (i) and Theorem 2.3 imply ${}^\circ a = {}^\circ p$ and $a^\circ = (1-q)^\circ$. Choose $u = v = 1$, $c = a$ and $d = a^{(1,\times)}$. Then

$$c^\circ = a^\circ = (1-q)^\circ = (aa^{(1,\times)})^\circ = (a^{(1,\times)})^\circ = d^\circ$$

and

$${}^\circ c = {}^\circ a = {}^\circ p = {}^\circ(a^{(1,\times)}a) = {}^\circ(a^{(1,\times)}) = {}^\circ d.$$

(iii) $\Rightarrow$ (i): Suppose that there exist $c,d,u,v \in \mathcal{R}$ satisfying $a = ucv$, $a^{(1,\times)} = udv$, $u^\circ = \{0\} = {}^\circ v$, $c^\circ = d^\circ$ and ${}^\circ c = {}^\circ d$. In order to prove that $a^\circ = (a^{(1,\times)})^\circ$, let $x \in a^\circ$, i.e. $ucvx = 0$. Now, because $u^\circ = \{0\}$, $cvx = 0$. Hence, $vx \in c^\circ = d^\circ$, that is, $dvx = 0$ which gives $a^{(1,\times)}x = udvx = 0$. So, $a^\circ \subseteq (a^{(1,\times)})^\circ$. The reverse inclusion follows similarly. Therefore, $a^\circ = (a^{(1,\times)})^\circ = (aa^{(1,\times)})^\circ = (1-q)^\circ$.

From the assumptions $\{0\} = {}^\circ v$ and ${}^\circ c = {}^\circ d$, we analogy obtain ${}^\circ a = {}^\circ(a^{(1,\times)})$. Thus, ${}^\circ a = {}^\circ(a^{(1,\times)}) = {}^\circ(a^{(1,\times)}a) = {}^\circ p$ and so the element a is reflexive–EP, by Theorem 2.3.

(i) $\Rightarrow$ (iv) $\wedge$ (v): For $u = v = 1$, $c = a^{(1,\times)}a$ and $d = aa^{(1,\times)}$, we easy check this part.

(iv) $\Rightarrow$ (i): The statement (iv) gives

$$p\mathcal{R} = a^{(1,\times)}a\mathcal{R} = ucv\mathcal{R} = uc\mathcal{R} = ud\mathcal{R} = udv\mathcal{R} = aa^{(1,\times)}\mathcal{R} = a\mathcal{R}$$

and

$$\mathcal{R}a = \mathcal{R}a^{(1,\times)}a = \mathcal{R}ucv = \mathcal{R}cv = \mathcal{R}dv = \mathcal{R}udv = \mathcal{R}aa^{(1,\times)} = \mathcal{R}(1-q).$$

Using Theorem 2.3, we deduce that (i) is satisfied.

(v) $\Rightarrow$ (i): In the similar way as the implication (iii) $\Rightarrow$ (i), we prove this part. □

Now, we characterize reflexive–EP elements in terms of the existence of idempotents.

Theorem 4.2 *Let $p, q \in \mathcal{R}^\bullet$ and let $a \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *a is reflexive–EP element;*
(ii) $\exists\, r \in \mathcal{R}^\bullet : p\mathcal{R} = r\mathcal{R}$, $\mathcal{R}(1-q) = \mathcal{R}r$ *and* $a = rar \in (r\mathcal{R}r)^{-1}$*;*
(iii) $\exists\, c \in \mathcal{R}^{-1}$, $\exists\, r \in \mathcal{R}^\bullet : rar \in (r\mathcal{R}r)^{-1}$, $a = crarc^{-1}$, $c^{-1}p\mathcal{R} = r\mathcal{R}$ *and* $\mathcal{R}(1-q)c = \mathcal{R}r$.

Proof. (i) $\Rightarrow$ (ii): Denote by $r = a^{(1,\times)}a = aa^{(1,\times)}$. Now, we have $r^2 = r$, $r\mathcal{R} = a^{(1,\times)}a\mathcal{R} = p\mathcal{R}$ and $\mathcal{R}(1-q) = \mathcal{R}aa^{(1,\times)} = \mathcal{R}r$. Since

$$rar = aa^{(1,\times)}ar = ar = aa^{(1,\times)}a = a$$

and

$$ra^{(1,\times)}r = a^{(1,\times)}aa^{(1,\times)}r = a^{(1,\times)}r = a^{(1,\times)}aa^{(1,\times)} = a^{(1,\times)},$$

then

$$rarra^{(1,\times)}r = aa^{(1,\times)} = r = a^{(1,\times)}a = ra^{(1,\times)}rrar.$$

Hence, $rar \in (r\mathcal{R}r)^{-1}$ and $(rar)^{-1} = ra^{(1,\times)}r$.

(ii) $\Rightarrow$ (iii): It follows for $c = 1$.

(iii) $\Rightarrow$ (i): Because $rar \in (r\mathcal{R}r)^{-1}$, there exists $b \in \mathcal{R}$ such that $rarbr = r = rbrar$. Let $x = crbrc^{-1}$. The assumptions $a = crarc^{-1}$ implies

$$ax = crarbrc^{-1} = crc^{-1} = crbrarc^{-1} = xa. \tag{4.1}$$

From $c^{-1}p\mathcal{R} = r\mathcal{R}$ and $\mathcal{R}(1-q)c = \mathcal{R}r$, we get $c^{-1}pc\mathcal{R} = r\mathcal{R}$ and $\mathcal{R}c^{-1}(1-q)c = \mathcal{R}r$ which give $r = c^{-1}pcr$, $c^{-1}pc = rc^{-1}pc$, $rc^{-1}(1-q)c = r$ and $c^{-1}(1-q)cr = c^{-1}(1-q)c$. So, $cr = pcr$, $p = crc^{-1}p$, $rc^{-1}q = 0$ and $(1-q)cr = (1-q)c$ which imply

$$px = pcrbrc^{-1} = crbrc^{-1} = x,$$

$$xap = crc^{-1}p = p,$$

$$xq = crbrc^{-1}q = 0,$$

$$(1-q)ax = (1-q)crc^{-1} = (1-q)cc^{-1} = 1-q.$$

By these equalities and Theorem 2.1, we deduce that $x = a^\times$. Since the element x satisfies $axa = crarc^{-1} = a$ and (4.1), then $x = a^{(1,\times)}$ and a is reflexive–EP element. □

In the preceding theorem we can express $a^{(1,\times)}$ in terms of the idempotent r and ordinary inverse in $\mathcal{R}$:

$$a^{(1,\times)} = rbr = (a+1-r)^{-1}r$$

or

$$a^{(1,\times)} = crbrc^{-1} = c(a+1-r)^{-1}rc^{-1},$$

using the relation between the ordinary and $r\mathcal{R}r$ inverses; it is known that rar is invertible in $r\mathcal{R}r$ if and only if $rar+1-r$ is invertible in $\mathcal{R}$.

If $\mathcal{R}$ is a ring with involution and $a \in \mathcal{R}$ is a Moore-Penrose invertible element, then $b = a^\dagger$ is the reflexive image-kernel (p,q)-inverse of a, where $p = a^\dagger a$ and $q = 1 - aa^\dagger$. Notice that the results of the previous sections recover some results related to the corresponding factorizations presented in [4].

5 Product of reflexive–EP elements

In the following theorem, it will be proved that the product of two reflexive–EP elements is reflexive–EP.

Theorem 5.1 *Let $p, q \in \mathcal{R}^\bullet$ and let $a, b \in \mathcal{R}$ be reflexive–EP elements. Then the following statements hold:*

(i) *ab is reflexive–EP element;*
(ii) *$ab\mathcal{R} = b\mathcal{R}$ and $\mathcal{R}ab = \mathcal{R}a$;*
(iii) *${}^\circ(ab) = {}^\circ b$ and $(ab)^\circ = a^\circ$;*
(iv) *$(ab)^{(1,\times)}$ exists and $(ab)^{(1,\times)} = b^{(1,\times)}a^{(1,\times)}$.*

Proof. (i) Since a and b are reflexive–EP elements, by Theorem 2.3, $a\mathcal{R} = p\mathcal{R} = b\mathcal{R}$ and $\mathcal{R}a = \mathcal{R}(1-q) = \mathcal{R}b$ which yield $aa^{(1,\times)}\mathcal{R} = bb^{(1,\times)}\mathcal{R}$ and $\mathcal{R}a^{(1,\times)}a = \mathcal{R}b^{(1,\times)}b$. There exist $x, y \in \mathcal{R}$ such that $aa^{(1,\times)} = bb^{(1,\times)}x$ and $b^{(1,\times)}b = ya^{(1,\times)}a$. So, from

$$aa^{(1,\times)} = bb^{(1,\times)}bb^{(1,\times)}x = bb^{(1,\times)}aa^{(1,\times)}$$

and

$$b^{(1,\times)}b = ya^{(1,\times)}aa^{(1,\times)}a = b^{(1,\times)}ba^{(1,\times)}a$$

we get $aa^{(1,\times)} = b^{(1,\times)}b$ implying $pb^{(1,\times)}a^{(1,\times)} = b^{(1,\times)}a^{(1,\times)}$, $b^{(1,\times)}a^{(1,\times)}q = 0$,

$$b^{(1,\times)}a^{(1,\times)}abp = b^{(1,\times)}bb^{(1,\times)}bp = b^{(1,\times)}bp = p,$$

$$(1-q)abb^{(1,\times)}a^{(1,\times)} = (1-q)aa^{(1,\times)}aa^{(1,\times)} = (1-q)aa^{(1,\times)} = 1-q$$

and

$$abb^{(1,\times)}a^{(1,\times)}ab = aa^{(1,\times)}aa^{(1,\times)}ab = ab.$$

Thus, we conclude that $(ab)^{(1,\times)}$ exists and $(ab)^{(1,\times)} = b^{(1,\times)}a^{(1,\times)}$. The equalities

$$ab\mathcal{R} = ap\mathcal{R} = aa^{(1,\times)}a\mathcal{R} = a\mathcal{R} = p\mathcal{R},$$

$$\mathcal{R}ab = \mathcal{R}(1-q)b = \mathcal{R}bb^{(1,\times)}b = \mathcal{R}(1-q)$$

and Theorem 2.3 give that ab is reflexive–EP element.

The statements (ii) and (iv) follow by the previous part.

(iii) The equality $ab\mathcal{R}{=}b\mathcal{R}$ is equivalent to $ab(ab)^{(1,\times)}\mathcal{R}{=}bb^{(1,\times)}\mathcal{R}$. Because $ab(ab)^{(1,\times)}, bb^{(1,\times)}{\in}\mathcal{R}^\bullet$, the last equality is equivalent to ${}^\circ[ab(ab)^{(1,\times)}] = {}^\circ[bb^{(1,\times)}]$, i.e. ${}^\circ(ab) = {}^\circ b$. In the similar way, we verify that $\mathcal{R}ab = \mathcal{R}a$ is equivalent to $(ab)^\circ = a^\circ$. □

Using Theorem 5.1, we can show the next result.

Corollary 5.2 *Let $p, q \in \mathcal{R}^\bullet$, $n \in N$ and let $a_1, a_2, \dots, a_n \in \mathcal{R}$ be reflexive–EP elements. Then the following statements hold:*

(i) *$a_1a_2\dots a_n$ is reflexive–EP element;*
(ii) *$a_1a_2\dots a_n\mathcal{R} = a_n\mathcal{R}$ and $\mathcal{R}a_1a_2\dots a_n = \mathcal{R}a_1$;*
(iii) *${}^\circ(a_1a_2\dots a_n) = {}^\circ a_n$ and $(a_1a_2\dots a_n)^\circ = a_1^\circ$;*
(iv) *$(a_1a_2\dots a_n)^{(1,\times)}$ exists and $(a_1a_2\dots a_n)^{(1,\times)} = a_n^{(1,\times)}\dots a_1^{(1,\times)}$.*

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