An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat. (N.S.)
Tomul LXII, 2016, f. 2, vol. 3

Symmetric lightlike hypersurfaces of a para-Sasakian space form

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Received: 13.XII.2014 / Last revision: 13.VIII.2015 / Accepted: 25.IX.2015

Abstract In this paper, we study semi-symmetric, Ricci semi-symmetric and semi–parallel lightlike hypersurfaces of a para-Sasakian space form.

Keywords Para-Sasakian space form · Semi-symmetric lightlike hypersurfaces · Ricci semi-symmetric lightlike hypersurfaces

Mathematics Subject Classification (2010) 53C25 · 53C40

1 Introduction

It is well known that if the induced metric on the submanifold of semi-Riemannian manifold is degenerate, the study becomes more different from the study of non-degenerate submanifolds. The main difference between the lightlike submanifolds and non-degenerate submanifolds arises due to the fact in the first case the normal vector bundle has non-trivial intersection with the tangent vector bundle and moreover in a lightlike hypersurface the normal vector bundle is contained in the tangent vector bundle. Lightlike submanifolds of semi-Riemannian manifolds were introduced by Duggal and Bejancu in [10]. Since then many authors studied lightlike hypersurfaces of semi-Riemannian manifolds and especially of indefinite Sasakian manifolds (for differential geometry of lightlike submanifolds we refer the book [11]).

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A Riemannian manifold is called locally symmetric if $\nabla R = 0$, where R is the Riemannian curvature tensor of M (see [5]). Locally symmetric Riemannian manifolds can be considered as a generalization of manifolds of constant curvatures. As a generalization of locally symmetric Riemannian manifolds, semi-symmetric Riemannian manifolds were defined by the condition $R(X,Y) \cdot R = 0$, where $R(X,Y)$ is the curvature operator and acts as a derivative on R. It is known that locally symmetric manifolds are semi-symmetric manifolds but the converse is not true (see [22]). Semi-symmetric hypersurfaces of Euclidean spaces were classified by Nomizu [19] and a general study of semi-symmetric Riemannian manifolds was made by Szabo [20]. In [21], Şahin introduced the notion of semi-symmetric lightlike hypersurfaces of a semi-Riemannian manifold and obtained some new results. Later, many authors have studied such hypersurfaces in different kinds of semi-Riemannian manifolds (see [15–18, 24]).

A semi-Riemannian manifold is said to be Ricci semi-symmetric (see [7]), if the following condition is satisfied: $R(X,Y) \cdot Ric = 0$. It is clear that every semi-symmetric manifold is Ricci semi-symmetric; however the converse is not true in general.

In the theory of submanifolds of a space form, conditions analogous to local symmetry and semi-symmetry have been introduced and studied quite intensively. Ferus [13] and others [4, 23] introduced the concept of parallel immersions, i.e. immersions with parallel second fundamental form, and classified such immersions. On the other hand, Deprez [8, 9] introduced the concept of semi-parallel immersions, i.e. immersions such that the curvature tensor annihilates the second fundamental form.

In 1985, Kaneyuki and Konzai [14] defined the almost paracontact structure on pseudo-Riemannian manifold $\bar{M}$ of dimension $(2n+1)$ and constructed the almost paracomplex structure on $\bar{M}^{2n+1} \times \mathbb{R}$. Recently, Zamkovoy [26] studied paracontact metric manifolds and some remarkable subclasses like para-Sasakian manifolds. Especially, in the recent years, many authors [3, 6, 12] have pointed out the importance of paracontact geometry, and in particular of para-Sasakian geometry, by several papers giving the relationships with the theory of para-Kähler manifolds and its role in pseudo-Riemannian geometry and mathematical physics. The lightlike geometry of para-Sasakian manifolds were first studied in [1, 2, 25], where the authors obtained some basic results and gave important characterizations with examples.

The purpose of the present paper is to study lightlike hypersurfaces of a para-Sasakian space form tangent to the structure vector field, satisfying some symmetry conditions, particularly semi-symmetry and Ricci semi-symmetry. Also, a characterization for semi-parallel lightlike hypersurfaces of a para-Sasakian space form is given.

2 Preliminaries

A $(2n+1)$-dimension pseudo-Riemannian manifold $\bar{M}^{2n+1}$ has an almost paracontact structure $(\bar{\phi}, \xi, \eta)$ if it is equipped with a tensor field $\bar{\phi}$ of type $(1,1)$, a vector field ξ, a 1-form η satisfying the following conditions (see [14]):

$$\bar{\phi}^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \bar{\phi}\xi = 0, \quad \eta \circ \bar{\phi} = 0. \tag{2.1}$$

If we set $D = Ker\ \eta = \left\{X \in \Gamma\left(T\bar{M}\right) : \eta\left(X\right) = 0\right\}$, then the tensor field $\bar{\phi}$ induces an almost paracomplex structure (see [14]) on each fibre on D.

If a manifold $\bar{M}$ with an almost paracontact structure $(\bar{\phi}, \xi, \eta)$ admits a pseudo-Riemannian metric $\bar{g}$ such that (see [26])

$$\bar{g}(\bar{\phi}X, \bar{\phi}Y) = -\bar{g}(X, Y) + \eta(X)\eta(Y), \quad X, Y \in \Gamma(T\bar{M}), \tag{2.2}$$

then we say that $\bar{M}$ has an almost paracontact metric structure and $\bar{g}$ is called compatible metric. Any compatible metric $\bar{g}$ with a given almost paracontact structure is necessarily of signature $(n+1, n)$.

Setting $Y = \xi$, we have

$$\eta(X) = \bar{g}(X, \xi). \tag{2.3}$$

The fundamental 2-form of $\bar{M}$ is defined by

$$\Phi(X, Y) = \bar{g}(X, \bar{\phi}Y). \tag{2.4}$$

Definition 2.1 *If $\bar{g}(X, \bar{\phi}Y) = d\eta(X, Y)$ (where $d\eta(X, Y) = \frac{1}{2}\{X\eta(Y) - Y\eta(X) - \eta([X, Y])\}$) then η is a paracontact form and the almost paracontact metric manifold $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ is said to be paracontact metric manifold.*

An almost paracontact metric structure $(\bar{\phi}, \xi, \eta, \bar{g})$ is a para-Sasakian manifold if and only if (see [26])

$$(\bar{\nabla}_X \bar{\phi})Y = -\bar{g}(X, Y)\xi + \eta(Y)X, \tag{2.5}$$

where $X, Y \in \Gamma(T\bar{M})$ and $\bar{\nabla}$ is a Levi-Civita connection on $\bar{M}$.

From (2.5), it can be seen that $\bar{\nabla}_X \xi = -\bar{\phi}X$.

Theorem 2.2 ([27]) *Let $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ be a para-Sasakian manifold of dimension $(2m+1)$, $m > 1$. If the paraholomorphic sectional curvature is independent of the paraholomorphic section at a point, the curvature tensor has form*

$$\bar{g}(\bar{R}(X, Y)Z, W) = \frac{k-3}{4}\left(\bar{g}(Y, Z)\bar{g}(X, W) - \bar{g}(X, Z)\bar{g}(Y, W)\right) + \frac{k+1}{4}\begin{pmatrix} \bar{g}(X, Z)\eta(Y)\eta(W) + \bar{g}(Y, W)\eta(X)\eta(Z) \\ -\bar{g}(X, W)\eta(Y)\eta(Z) - \bar{g}(Y, Z)\eta(X)\eta(W) \\ +\bar{g}(Y, \bar{\phi}Z)\bar{g}(\bar{\phi}X, W) - \bar{g}(X, \bar{\phi}Z)\bar{g}(\bar{\phi}Y, W) \\ +2\bar{g}(\bar{\phi}X, Y)\bar{g}(\bar{\phi}Z, W) \end{pmatrix}. \tag{2.6}$$

A para-Sasakian manifold $\bar{M}$ is said to be of constant paraholomorphic curvature or para-Sasakian space form if the paraholomorphic sectional curvature is constant on the manifold. We denote a para-Sasakian space form by $\bar{M}(k)$.

Let $(\bar{M}, \bar{g})$ be a semi-Riemannian manifold. We denote its curvature operator $\bar{R}(X, Y) = \bar{\nabla}_X\bar{\nabla}_Y - \bar{\nabla}_Y\bar{\nabla}_X - \bar{\nabla}_{[X,Y]}$, for $X, Y \in \Gamma(T\bar{M})$, where $\bar{\nabla}$ denotes the Levi-Civita connection on $\bar{M}$. Then the Riemannian Christoffel curvature tensor R and Ricci tensor Ric are defined by

$$\bar{R}(X, Y, Z, W) = \bar{g}(\bar{R}(X, Y)Z, W), \tag{2.7}$$

$$Ric(X, Y) = \text{trace}\{Z \to R(X, Y)Z\}, \tag{2.8}$$

respectively.

For a $(0,k)$-tensor field T on $\bar{M}$, $k \geq 1$, the $(0,k+2)$ tensor field $\bar{R} \cdot T$ is called curvature conditions of semi-symmetry type (see [7]) and given by

$$(\bar{R} \cdot T)(X_1, ..., X_k, X, Y) = -T(\bar{R}(X,Y)X_1, X_2, ..., X_k) - ... - T(X_1, X_2, ..., \bar{R}(X,Y)X_k), \tag{2.9}$$

for $X_1, ..., X_k, X, Y \in \Gamma(T\bar{M})$.

A semi-Riemannian manifold $\bar{M}$ is said to be semi-symmetric if it satisfies the condition $\bar{R} \cdot \bar{R} = 0$. Thus from (2.9) and using the properties of curvature tensor, we have

$$(\bar{R}(X,Y) \cdot \bar{R})(U,V)W = \bar{R}(X,Y)\bar{R}(U,V)W - \bar{R}(\bar{R}(X,Y)U,V)W - \bar{R}(U,\bar{R}(X,Y)V)W - \bar{R}(U,V)\bar{R}(X,Y)W = 0, \tag{2.10}$$

for any $X, Y, U, V, W \in \Gamma(T\bar{M})$.

A semi-Riemannian manifold $\bar{M}$ is said to be Ricci semi-symmetric if it satisfies the condition $\bar{R} \cdot Ric = 0$, i.e.,

$$(\bar{R}(X,Y) \cdot Ric)(U,V) = -Ric(\bar{R}(X,Y)U,V) - Ric(U,\bar{R}(X,Y)V) = 0, \tag{2.11}$$

for any $X, Y, U, V \in \Gamma(T\bar{M})$.

We recall that a hypersurface M of a semi-Riemannian manifold $\bar{M}$ is said to be semi-parallel (see [8]) if the following condition is satisfied for all $p \in M$ and vector fields $X, Y, Z, W \in \Gamma(T\bar{M})$:

$$(\bar{R}(X,Y) \cdot h)(Z,W) = -h(\bar{R}(X,Y)Z,W) - h(Z,\bar{R}(X,Y)W) = 0, \tag{2.12}$$

where h is the second fundamental form and $\bar{R}$ is the curvature tensor field of $\bar{M}$.

Let (M,g) be a hypersurface of $(2n+1)$-dimensional semi-Riemannian manifold $(\bar{M},\bar{g})$ with index q, $0 < q < 2n+1$ and $g = \bar{g} \mid_M$. Then M is lightlike hypersurface of $\bar{M}$ if g is of constant rank $2n-1$ and the normal bundle $TM^{\perp}$ is a distribution of rank 1 on M (see [10]). A non-degenerate complementary distribution $S(TM)$ of rank $2n-1$ to $TM^{\perp}$ in TM, that is, $TM = TM^{\perp} \perp S(TM)$, is called screen distribution.

The following result has an important role in studying the geometry of a lightlike hypersurface (see [10]).

Theorem 2.3 ([10]) *Let $(M,g,S(TM))$ be a lightlike hypersurface of a semi- Riemannian manifold $\bar{M}$. Then there exists a unique rank one vector subbundle $ltr(TM)$ of $T\bar{M}$, with base space M, such that for any non-zero section E of $RadTM$ on a coordinate neighborhood $U \subset M$, there exists a unique section N of $ltr(TM)$ on U satisfying:*

$$\bar{g}(N,E) = 1, \bar{g}(N,N) = 0, \bar{g}(N,W) = 0, \textit{for all } W \in \Gamma(S(TM) \mid_U . \tag{2.13}$$

$ltr(TM)$ is called the lightlike transversal vector bundle of M with respect to $S(TM)$.

By the previous Theorem, one can state the following decompositions:

$$TM = S(TM) \perp RadTM, \tag{2.14}$$
$$T\bar{M} = TM \oplus ltr(TM) = S(TM) \perp \{RadTM \oplus ltr(TM)\}. \tag{2.15}$$

Let $\bar{\nabla}$ be the Levi-Civita connection on $(\bar{M}, \bar{g})$, then by using decomposition given by (2.15) and considering a normalizing pair $\{E, N\}$ as in Theorem 2.3, we have Gauss and Weingarten formulas

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.16}$$
$$\bar{\nabla}_X N = -A_N X + \nabla^{lt}_X N, \tag{2.17}$$

respectively, for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(ltr(TM))$,where $\nabla_X Y$, $A_N X \in \Gamma(TM)$ and $h(X, Y)$, $\nabla^{lt}_X N \in \Gamma(ltr(TM))$. Here ∇ is an induced symmetric linear connection on the vector subbundle $ltr(TM)$, h is a $ltr(TM)-$valued symmetric bilinear form and A_N is shape operator of M.

Locally let E, N and U be as in Theorem 2.3. For any $X, Y \in \Gamma(TM\mid_U)$, by putting $B(X, Y) = \bar{g}(h(X, Y), E)$ and $\tau(X) = \bar{g}(\nabla^{lt}_X N, E)$, we can write

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2.18}$$
$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.19}$$

where B, A_N, τ and ∇ are called the local second fundamental form, the local shape operator, the transversal differential 1-form and induced linear torsion free connection, respectively, on $TM\mid_U$. It is important to mention that the second fundamental form B of M is independent of the choice of screen distribution and

$$B(\cdot, E) = 0. \tag{2.20}$$

Let $P : \Gamma(TM) \to \Gamma(S(TM))$ be the projection morphism with respect to the orthogonal decomposition of TM. We have, for any $X, Y \in \Gamma(TM)$ and $E \in \Gamma(TM^{\perp})$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)E, \tag{2.21}$$
$$\nabla_X E = -A^*_E X - \tau(X)E, \tag{2.22}$$

where $\nabla^*_X PY$, $A^*_E X$ belong to $\Gamma(S(TM))$. C, A^*_E and ∇^* are called the second fundamental form, the local shape operator and induced connection on $S(TM)$, respectively. The induced linear connection ∇ is not a metric connection and we have

$$(\nabla_X g)(Y, Z) = B(X, Y)\theta(Z) + B(X, Z)\theta(Y), \tag{2.23}$$

for any $X, Y \in \Gamma(TM\mid_U)$, where θ is a differential 1-form locally defined on M by $\theta(\cdot) = \bar{g}(N, \cdot)$.

The local second fundamental forms B and C, respectively, of M and on $S(TM)$ are related to their shape operators by

$$g(A^*_E X, PY) = B(X, PY), \;\; g\ (A^*_E X, N) = 0, \tag{2.24}$$
$$g(A_N X, PY) = C(X, PY), \;\; g(A_N X, N) = 0. \tag{2.25}$$

If $\bar{R}$ and R are the curvature tensors of $\bar{M}$ and M, then using (2.18), we obtain

$$\begin{aligned}\bar{R}(X,Y)Z = &R(X,Y)Z + B(X,Z)A_N Y - B(Y,Z)A_N X \\ &+ \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ &+ \tau(X)B(Y,Z) - \tau(Y)B(X,Z)\}N,\end{aligned} \tag{2.26}$$

for any $X, Y, Z \in \Gamma(TM\mid_U)$.

3 Semi-symmetric and Ricci semi-symmetric lightlike hypersurfaces of a para-Sasakian space form

For $X \in \Gamma(TM)$, we write

$$\bar{\phi}X = \phi X + u(X)N, \tag{3.1}$$

where ϕX is the tangential parts of $\bar{\phi}X$ and u is a differential 1-form locally defined on M by $u(X) = -g(X, \bar{\phi}E)$.

Definition 3.1 *Let M be a lightlike hypersurface of a $(2m+1)$-dimensional para-Sasakian space form $\bar{M}(k)$. We say that M is semi-symmetric if the following condition is satisfied*

$$(R(X,Y)\cdot R)(U,V,W,Z) = 0, \tag{3.2}$$

for $X, Y, U, V, W, Z \in \Gamma(TM)$. We note that $(R(X,Y)\cdot R)(U,V,W,E) = 0$, for $E \in \Gamma(TM^{\perp})$. Therefore equation (3.2) reduces to

$$(R(X,Y)\cdot R)(U,V,W,PZ) = 0. \tag{3.3}$$

So, we have

Lemma 3.2 *Let M be a lightlike hypersurface of para-Sasakian space form $(\bar{M}(k), \bar{\phi}, \xi, \eta, \bar{g})$. Then the equation of Gauss of M is given by*

$$\begin{aligned}R(X,Y)Z = &\frac{k-3}{4}(\bar{g}(Y,Z)X - \bar{g}(X,Z)Y) \\ &+ \frac{k+1}{4}\begin{pmatrix}\bar{g}(X,Z)\eta(Y)\xi + \bar{g}(Y,Z)\eta(X)\xi \\ +\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ +\bar{g}(\bar{\phi}X,Z)\phi Y - \bar{g}(\bar{\phi}Y,Z)\phi X \\ -2\bar{g}(X,\bar{\phi}Y)\phi Z\end{pmatrix} \\ &- B(X,Z)A_N Y + B(Y,Z)A_N X,\end{aligned} \tag{3.4}$$

for any $X, Y, Z \in \Gamma(TM)$.

Proof. Since $(\bar{M}(k), \bar{\phi}, \xi, \eta, \bar{g})$ is a para-Sasakian space form, from (2.6) and (2.26) we obtain

$$\begin{aligned}R(X,Y)Z = &\frac{k-3}{4}(\bar{g}(Y,Z)X - \bar{g}(X,Z)Y) \\ &+ \frac{k+1}{4}\begin{pmatrix}\bar{g}(X,Z)\eta(Y)\xi + \bar{g}(Y,Z)\eta(X)\xi \\ +\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ +\bar{g}(\bar{\phi}X,Z)\phi Y - \bar{g}(\bar{\phi}Y,Z)\phi X \\ -2\bar{g}(X,\bar{\phi}Y)\phi Z\end{pmatrix} \\ &- A_{h(X,Z)}Y + A_{h(Y,Z)}X - (\nabla_X h)(Y,Z) + (\nabla_Y h)(X,Z).\end{aligned}$$

Using (3.1) and comparing the tangential parts, we obtain (3.4). □

Theorem 3.3 *Let M be a lightlike hypersurface of para-Sasakian space form $\bar{M}(k)$. Then we have*

$$
\begin{aligned}
Ric(X,Y) = &-\left(\frac{k-3}{4}\right)(2m-1)g(X,Y) \\
&+\left(\frac{k+1}{4}\right)\begin{bmatrix} g(\phi X,\bar{\phi}Y)+g(E,\bar{\phi}Y)g(\phi X,N) \\ +2g(\bar{\phi}X,\phi Y)-2g(X,\bar{\phi}E)g(\phi Y,N) \\ g(X,Y)+(2m-2)\eta(X)\eta(Y) \end{bmatrix} \\
&+\sum_{i=1}^{2m-1}\epsilon_i\left(B(w_i,Y)C(X,w_i)-B(X,Y)C(w_i,w_i)\right),
\end{aligned}
\tag{3.5}
$$

where $\{w_i\}$, $i=1,2,...,(2m-1)$, is the orthogonal basis of $S(TM)$.

Proof. The Ricci tensor of a lightlike hypersurface is given by

$$
Ric(X,Y) = \sum_{i=1}^{2m-1}\epsilon_i g(R(X,w_i)Y,w_i)+\bar{g}(R(X,E)Y,N),
$$

for any $X,Y \in \Gamma(TM)$, $E \in \Gamma(TM^{\perp})$ and $N \in \Gamma(ltr(TM))$, where $\{w_i\}$, $i = 1,2,...,(2m-1)$ is the orthogonal basis of $S(TM)$. By using (2.13) and (2.20), from (3.4), we obtain (3.5). □

Proposition 3.4 *The Ricci tensor of a lightlike hypersurface M in a para-Sasakian space form $\bar{M}(k)$ is symmetric if $k=-1$and the shape operator of M is symmetric with respect to the second fundamental form of M.*

Proof. From (3.5), we have

$$
\begin{aligned}
&Ric(X,Y)-Ric(Y,X) \\
&=\left(\frac{k+1}{4}\right)\begin{bmatrix} g(\phi X,\bar{\phi}Y)-g(\bar{\phi}X,\phi Y) \\ +2g(\bar{\phi}X,\phi Y)-2g(\phi X,\bar{\phi}Y) \\ +g(E,\bar{\phi}Y)g(\phi X,N)-g(E,\bar{\phi}X)g(\phi Y,N) \\ -2g(X,\bar{\phi}E)g(\phi Y,N)+2g(Y,\bar{\phi}E)g(\phi X,N) \end{bmatrix} \\
&+\sum_{i=1}^{2m-1}\epsilon_i\left(B(w_i,Y)C(X,w_i)-B(w_i,X)C(Y,w_i)\right).
\end{aligned}
\tag{3.6}
$$

On the other hand, by using (2.24) and (2.25), we get

$$
\sum_{i=1}^{2m-1}\epsilon_i B(w_i,Y)C(X,w_i) = B(Y,A_N X).
\tag{3.7}
$$

From (3.6) and (3.7), we obtain

$$\begin{aligned} &Ric(X,Y) - Ric(Y,X) \\ &= \left(\frac{k+1}{4}\right)\begin{bmatrix} g(\phi X,\bar{\phi}Y) - g(\bar{\phi}X,\phi Y) + 2g(\bar{\phi}X,\phi Y) \\ -2g(\phi X,\bar{\phi}Y) + g(E,\bar{\phi}Y)g(\phi X,N) \\ -g(E,\bar{\phi}X)g(\phi Y,N) - 2g(X,\bar{\phi}E)g(\phi Y,N) \\ +2g(Y,\bar{\phi}E)g(\phi X,N) \end{bmatrix} \\ &+ B(Y,A_N X) - B(X,A_N Y), \end{aligned} \tag{3.8}$$

which completes the proof. □

Theorem 3.5 *Let M be a Ricci semi-symmetric lightlike hypersurface of a para-Sasakian space form $\bar{M}(k)$. If $k=-1$, then, either M is totally geodesic or $Ric(E, A_N E) = 0$, where $E \in \Gamma(TM^{\perp})$, Ric is the Ricci tensor of M and A denotes the shape operator of M.*

Proof. Assume that M is Ricci semi-symmetric. Then from (2.11), we have

$$0 = Ric(R(X,Y)X_1,X_2) + Ric(X_1,R(X,Y)X_2). \tag{3.9}$$

Using (3.5) in (3.9), we get

$$\begin{aligned} &\left(\frac{k-3}{4}\right)(g(Y,X_1)Ric(X,X_2) - g(X,X_1)Ric(Y,X_2)) \\ &+\left(\frac{k+1}{4}\right)\begin{bmatrix} g(X,X_1)\eta(Y)Ric(\xi,X_2) - g(Y,X_1)\eta(X)Ric(\xi,X_2) \\ +Ric(Y,X_2)\eta(X_1)\eta(X) - Ric(X,X_2)\eta(X_1)\eta(Y) \\ +g(\bar{\phi}X,X_1)Ric(\phi Y,X_2) - g(\bar{\phi}Y,X_1)Ric(\phi X,X_2) \\ -2g(X,\bar{\phi}Y)Ric(\phi X_1,X_2) \end{bmatrix} \\ &-B(X,X_1)Ric(A_N Y,X_2) + B(Y,X_1)Ric(A_N X,X_2) \\ &+\left(\frac{k-3}{4}\right)(g(Y,X_2)Ric(X_1,X) - g(X,X_2)Ric(X_1,Y)) \\ &+\left(\frac{k+1}{4}\right)\begin{bmatrix} g(X,X_2)\eta(Y)Ric(X_1,\xi) - g(Y,X_2)\eta(X)Ric(X_1,\xi) \\ +Ric(X_1,Y)\eta(X_2)\eta(X) - Ric(X_1,X)\eta(X_2)\eta(Y) \\ +g(\bar{\phi}X,X_2)Ric(X_1,\phi Y) - g(\bar{\phi}Y,X_2)Ric(X_1,\phi X) \\ -2g(X,\bar{\phi}Y)Ric(X_1,\phi X_2) \end{bmatrix} \\ &-B(X,X_2)Ric(X_1,A_N Y) + B(Y,X_2)Ric(X_1,A_N X) = 0. \end{aligned} \tag{3.10}$$

Putting $X_1 = E$ in (3.10) and using (2.20), we have

$$\begin{aligned} &\left(\frac{k-3}{4}\right)(g(Y,X_2)Ric(E,X) - g(X,X_2)Ric(E,Y)) \\ &+\left(\frac{k+1}{4}\right)\begin{bmatrix} g(\bar{\phi}X,E)Ric(\phi Y,X_2) - g(\bar{\phi}Y,E)Ric(\phi X,X_2) \\ +g(X,X_2)\eta(Y)Ric(E,\xi) - g(Y,X_2)\eta(X)Ric(E,\xi) \\ +Ric(E,Y)\eta(X_2)\eta(X) - Ric(E,X)\eta(X_2)\eta(Y) \\ +g(\bar{\phi}X,X_2)Ric(E,\phi Y) - g(\bar{\phi}Y,X_2)Ric(E,\phi X) \\ -2g(X,\bar{\phi}Y)Ric(\phi E,X_2) - 2g(X,\bar{\phi}Y)Ric(E,\phi X_2) \end{bmatrix} \\ &-B(X,X_2)Ric(E,A_N Y) + B(Y,X_2)Ric(E,A_N X) = 0. \end{aligned} \tag{3.11}$$

By taking $Y = E$ in (3.11), we get

$$\left(\frac{k+1}{4}\right)\begin{bmatrix} 3g(\bar{\phi}X, E)Ric(\phi E, X_2) + g(\bar{\phi}X, X_2)Ric(E, \phi E) \\ -g(\bar{\phi}E, X_2)Ric(E, \phi X) - 2g(X, \bar{\phi}E)Ric(E, \phi X_2) \end{bmatrix}$$
$$-B(X, X_2)Ric(E, A_N E) = 0. \tag{3.12}$$

If $k = -1$ then from (3.12), we obtain $B(X, X_2)Ric(E, A_N E) = 0$. Therefore, if $B(X, X_2) = 0$, for any $X, X_2 \in \Gamma(TM)$, then M is totally geodesic. If M is not totally geodesic, it follows that $Ric(E, A_N E) = 0$. □

Theorem 3.6 *Let M be a totally geodesic lightlike hypersurface of para-Sasakian space form $\bar{M}(k)$. Then M is semi-symmetric if $k = -1$.*

Proof. Let M be a lightlike hypersurface of para-Sasakian space form $\bar{M}(k)$. By using (3.4), from (3.3) we have

$$
\begin{aligned}
&g(R(X,Y)\cdot R)(U,V)W, PZ) = \\
&\left(\frac{k-3}{4}\right)(g(Y, R(U,V)W)g(X,PZ) - g(X, R(U,V)W)g(Y,PZ)) \\
&+\left(\frac{k+1}{4}\right)\begin{bmatrix} g(X, R(U,V)W)\eta(Y)\eta(PZ)) - g(Y, R(U,V)W)\eta(X)\eta(PZ)) \\ +g(Y,PZ)\eta(X)\eta(R(U,V)W) - g(X,PZ)\eta(Y)\eta(R(U,V)W) \\ +g(\bar{\phi}X, R(U,V)W)g(\phi Y, PZ) - g(\bar{\phi}Y, R(U,V)W)g(\phi X, PZ) \\ -2g(X,\bar{\phi}Y)g(\phi R(U,V)W, PZ) \end{bmatrix} \\
&- B(X, R(U,V)W)g(A_N Y, PZ) + B(Y, R(U,V)W)g(A_N X, PZ) \\
&-\left(\frac{k-3}{4}\right)(g(Y,U)g(R(X,V)W,PZ) - g(X,U)g(R(Y,V)W,PZ)) \\
&-\left(\frac{k+1}{4}\right)\begin{bmatrix} g(X,U)\eta(Y)g(R(\xi,V)W,PZ) - g(Y,U)\eta(X)g(R(\xi,V)W,PZ) \\ +\eta(X)\eta(U)g(R(Y,V)W,PZ) - \eta(Y)\eta(U)g(R(X,V)W,PZ) \\ +g(\bar{\phi}X,U)g(R(\phi Y,V)W,PZ) - g(\bar{\phi}Y,U)g(R(\phi X,V)W,PZ) \\ -2g(X,\bar{\phi}Y)g(R(\phi U,V)W,PZ) \end{bmatrix} \\
&+ B(X,U)g(R(A_N Y,V)W,PZ) - B(Y,U)g(R(A_N X,V)W,PZ) \\
&-\left(\frac{k-3}{4}\right)(g(Y,V)g(R(U,X)W,PZ) - g(X,V)g(R(U,Y)W,PZ)) \\
&-\left(\frac{k+1}{4}\right)\begin{bmatrix} g(X,V)\eta(Y)g(R(U,\xi)W,PZ) - g(Y,V)\eta(X)g(R(U,\xi)W,PZ) \\ +\eta(X)\eta(V)g(R(U,Y)W,PZ) - \eta(Y)\eta(V)g(R(U,X)W,PZ) \\ +g(\bar{\phi}X,V)g(R(U,\phi Y)W,PZ) - g(\bar{\phi}Y,V)g(R(U,\phi X)W,PZ) \\ -2g(X,\bar{\phi}Y)g(R(U,\phi V)W,PZ) \end{bmatrix} \\
&+ B(X,V)g(R(U,A_N Y)W,PZ) - B(Y,V)g(R(U,A_N X)W,PZ) \\
&-\left(\frac{k-3}{4}\right)(g(Y,W)g(R(U,V)X,PZ) - g(X,W)g(R(U,V)Y,PZ))
\end{aligned}
$$

$$-\left(\frac{k+1}{4}\right)\begin{bmatrix} g(X,W)\eta(Y)g(R(U,V)\xi,PZ)-g(Y,W)\eta(X)g(R(U,V)\xi,PZ) \\ +\eta(X)\eta(W)g(R(U,V)Y,PZ)-\eta(Y)\eta(W)g(R(U,V)X,PZ) \\ +g(\bar{\phi}X,W)g(R(U,V)\phi Y,PZ)-g(\bar{\phi}Y,W)g(R(U,V)\phi X,PZ) \\ -2g(X,\bar{\phi}Y)g(R(U,V)\phi W,PZ) \end{bmatrix}$$
$$+B(X,W)g(R(U,V)A_N Y,PZ)-B(Y,W)g(R(U,V)A_N X,PZ),$$

for any $X,Y,Z,U,V,W \in \Gamma(TM)$. Taking $Y=U=E\in\Gamma(TM^{\perp})$ in the above equation, considering the fact that M is totally geodesic and using (3.4) with (2.20), we obtain

$$g(R(X,E)\cdot R)(E,V)W,PZ)=\left(\frac{k+1}{4}\right)^2\cdot$$
$$\cdot\begin{bmatrix} g(W,\bar{\phi}E)g(\bar{\phi}X,V)g(\phi E,PZ)-g(\bar{\phi}X,\phi E)g(\bar{\phi}V,W)g(\phi E,PZ) \\ -g(\bar{\phi}E,W)g(\phi V,\bar{\phi}E)g(\phi X,PZ)-g(\bar{\phi}X,E)g(\phi V,PZ)g(\bar{\phi}(\phi E),W) \\ +g(\bar{\phi}X,E)g(\bar{\phi}V,W)g(\phi(\phi E),PZ)-g(\bar{\phi}X,V)g(\bar{\phi}E,W)g(\phi(\phi E),PZ) \\ -g(\phi E,V)g(\bar{\phi}(\phi X),W)g(\phi E,PZ)-g(\phi E,\bar{\phi}V)g(\bar{\phi}X,W)g(\phi E,PZ) \\ +g(\phi X,\bar{\phi}E)g(\bar{\phi}E,W)g(\phi V,PZ)-g(W,\bar{\phi}E)g(\bar{\phi}V,\phi X)g(\phi E,PZ) \\ -2g(\bar{\phi}V,E)g(\bar{\phi}X,\phi W)g(\phi E,PZ)+2g(\bar{\phi}V,E)g(\phi W,\bar{\phi}E)g(\phi X,PZ) \\ +2g(E,\bar{\phi}V)g(\bar{\phi}(\phi X),E)g(\phi W,PZ)-2g(X,\bar{\phi}E)g(\bar{\phi}(\phi V),W)g(\phi E,PZ) \\ +2g(\bar{\phi}X,W)g(E,\bar{\phi}V)g(\phi(\phi E),PZ)+2g(X,\bar{\phi}E)g(\bar{\phi}E,\phi W)g(\phi V,PZ) \\ -2g(X,\bar{\phi}E)g(\bar{\phi}V,\phi W)g(\phi E,PZ)+3g(X,\bar{\phi}E)g(\phi E,W)\eta(V)\eta(PZ) \\ +3g(\bar{\phi}E,V)g(\bar{\phi}E,W)g(\phi(\phi X),PZ)+3g(E,\bar{\phi}V)g(\phi E,PZ)\eta(X)\eta(W) \\ -4g(X,\bar{\phi}E)g(E,\bar{\phi}(\phi V))g(\phi W,PZ) \end{bmatrix}$$
$$+\left(\frac{k+1}{4}\right)\left(3\left(\frac{k-3}{4}\right)\left[g(\bar{\phi}X,E)g(\phi E,W)g(V,PZ)\right.\right.$$
$$\left.\left.-g(\bar{\phi}V,E)g(\phi E,PZ)g(X,W)\right]\right).$$

If $k=-1$ we find

$$g((R(X,Y)\cdot R)(U,V)W,PZ)=0,$$

which completes the proof. □

4 Semi parallel lightlike hypersurfaces in a para-Sasakian space form

In this section, we give a characterization on semi-parallel lightlike hypersurface of a para-Sasakian space form.

Theorem 4.1 *Let M be a semi-parallel lightlike hypersurface of a para-Sasakian space form $\bar{M}(k)$. Then if $k=-1$, either M is totally geodesic or $C(E,A^*_E PX)=0$, for any $X\in\Gamma(TM)$ and $E\in\Gamma(TM^{\perp})$.*

Proof. Since M is a semi-parallel lightlike hypersurface, we get

$$h(R(X,Y)Z,W)+h(Z,R(X,Y)W)=0.$$

By using (3.4), we have

$$\begin{aligned}
&\frac{k-3}{4}\begin{pmatrix} g(Y,Z)B(X,W)-g(X,Z)B(Y,W) \\ +g(Y,W)B(Z,X)-g(X,W)B(Z,Y)\end{pmatrix} \\
&+\frac{k+1}{4}\begin{pmatrix} g(X,Z)\eta(Y)B(\xi,W)-g(Y,Z)\eta(X)B(\xi,W) \\ +g(X,W)\eta(Y)B(Z,\xi)-g(Y,W)\eta(X)B(Z,\xi) \\ +\eta(X)\eta(Z)B(Y,W)-\eta(Y)\eta(Z)B(X,W) \\ +\eta(X)\eta(W)B(Z,Y)-\eta(Y)\eta(W)B(Z,X) \\ +g(\bar{\phi}X,Z)B(\phi Y,W)-g(\bar{\phi}Y,Z)B(\phi X,W) \\ +g(\bar{\phi}X,W)B(Z,\phi Y)-g(\bar{\phi}Y,W)B(Z,\phi X) \\ -2\bar{g}(X,\bar{\phi}Y)B(\phi Z,W)-2\bar{g}(X,\bar{\phi}Y)B(Z,\phi W)\end{pmatrix} \\
&-B(X,Z)B(A_NY,W)+B(Y,Z)B(A_NX,W) \\
&-B(X,W)B(Z,A_NY)+B(Y,W)B(Z,A_NX)=0,
\end{aligned} \tag{4.1}$$

for any $X,Y,Z\in\Gamma(TM)$. Putting $X=E$ and using (2.20), we obtain

$$\begin{aligned}
&\frac{k+1}{4}\begin{pmatrix} g(\bar{\phi}E,Z)B(\phi Y,W)-g(\bar{\phi}Y,Z)B(\phi E,W) \\ +g(\bar{\phi}E,W)B(Z,\phi Y)-g(\bar{\phi}Y,W)B(Z,\phi E) \\ -2\bar{g}(E,\bar{\phi}Y)B(\phi Z,W)-2\bar{g}(E,\bar{\phi}Y)B(Z,\phi W)\end{pmatrix} \\
&+B(Y,Z)B(A_NE,W)+B(Y,W)B(Z,A_NE)=0.
\end{aligned}$$

Suppose that $k=-1$. Thus, for $W=Z$ we obtain $B(Y,Z)B(Z,A_NE)=0$. If $B(Y,Z)=0$, then M is totally geodesic. If $B(Y,Z)\neq 0$, then from (2.24) and (2.25), we have $C(E,A_E^*PZ)=0$, which proves the theorem. □

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