An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat. (N.S.)
Tomul LXII, 2016, f. 2, vol. 3

The axiom of spheres in Finsler geometry

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Received: 22.VII.2014 / Revised: 23.I.2015 / Accepted: 28.I.2015

Abstract Here, an axiom of spheres in Finsler geometry is proposed and it is proved that if a Finslerian manifold satisfies the axiom of spheres then it is of constant flag curvature.

Keywords (κ, μ)-contact metric manifold · η-Einstein manifold · Quasi-conharmonically flat manifold · ϕ-conharmonically flat manifold · ξ-conharmonically flat manifold · Conharmonically semi-symmetric manifold

Mathematics Subject Classification (2010) 53C25 · 53D15

1 Introduction

In Riemannian geometry, E. Cartan defined an axiom of r-planes as follows. *A Riemannian manifold M of dimension $n \geq 3$ satisfies the axiom of r-planes, where r is a fixed integer $2 \leq r < n$, if for each point p of M and any r-dimensional subspace S of the tangent space T_pM there exists an r-dimensional totally geodesic submanifold V containing p such that $T_pV = S$.* He proved that if M satisfies the axiom of r-planes for some r, then M has constant sectional curvature (see [7]). The axiom of r-spheres in Riemannian geometry was proposed by Leung and Nomizu [9] as follows. *For each point p of M and any r-dimensional subspace S of T_pM, there is an r-dimensional umbilical submanifold V with parallel mean curvature vector field such that $p \in V$ and $T_pV = S$.* They proved that if a Riemannian manifold M of dimension $n \geq 3$ satisfies the axiom of r-spheres for some $r, 2 \leq r < n$, then M has constant sectional curvature (see [9]). In [2], Akbar-Zadeh extends the Cartan's axiom of 2-planes to Finsler geometry as follows. *A Finslerian manifold M of dimension $n \geq 3$ satisfies the axiom of 2-planes if for each point $p \in M$ and every subspace E_2 of dimension two of T_pM there exists a totally geodesic surface S passing through p such that $T_pS = E_2$.* He proved that every Finsler manifold satisfying the axiom of 2-planes is of constant flag curvature (see [2], page 182).

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Recently, a definition of circle in Finsler spaces is introduced by one of the present authors in a joint work with Shen [5]. Based on the definition of a circle we will show later that a connected submanifold of a Finsler manifold is an *extrinsic sphere* if and only if its circles coincide with circles of the ambient manifold. The proof will appear elsewhere.

The present authors in a previous work have proved that if a forward geodesically complete Finsler manifold admits a circle preserving change of metric then its indicatrix is conformally diffeomorphic to the Euclidean sphere (see [11]).

In the present work, we propose in a natural way, the following axiom of r-spheres in Finsler geometry.

Axiom of r-spheres

Let (M, F) be a Finsler manifold of dimension $n \geq 3$. For each point x in M and any r-dimensional subspace E_r of T_xM, there exists an r-dimensional umbilical submanifold S with parallel mean curvature vector field such that $x \in S$ and $T_xS = E_r$.

We shall prove the following theorem.

Theorem 1.1 *If a Finsler manifold of dimension $n \geq 3$ satisfies the axiom of r-spheres for some r, $2 \leq r < n$, then M has constant flag curvature.*

2 Notations and preliminaries on Finsler submanifolds

Let M be a real $n-$dimensional manifold of class C^∞. We denote by TM the tangent bundle of tangent vectors, by $p : TM_0 \longrightarrow M$ the fiber bundle of non-zero tangent vectors and by $p^*TM \longrightarrow TM_0$ the pull back tangent bundle. Let (x, U) be a local chart on M and (x^i, y^i) the induced local coordinates on $p^{-1}(U)$. A *Finsler structure* on M is a function $F : TM \longrightarrow [0, \infty)$, with the following properties:

(i) F is differentiable C^∞ on TM_0;

(ii) F is positively homogeneous of degree one in y, that is, $F(x, \lambda y) = \lambda F(x, y)$, for all $\lambda > 0$;

(iii) The Finsler *metric tensor* g defined by the Hessian matrix of F^2, $(g_{ij}) = (\frac{1}{2}[\frac{\partial^2}{\partial y^i \partial y^j} F^2])$, is positive definite on TM_0. A *Finsler manifold* is a pair (M, F) consisting of a differentiable manifold M and a Finsler structure F on M. We denote by TTM_0, the tangent bundle of TM_0 and by ρ, the canonical linear mapping $\rho : TTM_0 \longrightarrow p^*TM$, where, $\rho = p_*$. There is the horizontal distribution HTM such that we have the Whitney sum $TTM_0 = HTM_0 \oplus VTM_0$. This decomposition permits to write a vector field $\hat{X} \in \chi(TM_0)$ into the horizontal and vertical parts in a unique manner, namely $\hat{X} = H\hat{X} + V\hat{X}$. In the sequel, we decorate the vector fields on TM_0 by hat, i.e. $\hat{X}$ and $\hat{Y}$ and the corresponding sections of p^*TM by $X = \rho(\hat{X})$ and $Y = \rho(\hat{Y})$, respectively, unless otherwise specified (see [2]). For all $X \in p^*TM$ we denote by ${}^h\hat{X}$ the horizontal lift of X defined by the bundle morphism $\beta : p^*TM \longrightarrow HTM$ where, $\beta(\frac{\partial}{\partial x^i}) = \frac{\delta}{\delta x^i}$ (see [1]). For another approach on geometry of Finslerian manifolds, one can refer to [6].

2.1 Finsler geometry of submanifolds

Let (M, F) be a Finsler manifold and $S \subset M$ a k-dimensional submanifold defined by the immersion $i : S \longrightarrow M$. We identify any point $x \in S$ by its image $i(x)$ and any tangent vector $X \in T_xS$ by its image $i_*(X)$, where i_* is the linear tangent mapping. Thus T_xS becomes a sub-space of T_xM. Let TS_0 be the fiber bundle of non-zero tangent vectors on S. TS_0 is a sub-vector bundle of TM_0 and the restriction of p to TS_0 is denoted by $q : TS_0 \longrightarrow S$. We denote by $\bar{T}(S) = i^*TM$, the pull back induced vector bundle of TM by i. The Finslerian metric g on TM_0 induces a Finslerian metric on TS_0, where, we denote it again by g. At a point $x = qz \in S$, where $z \in TS_0$, the orthogonal complement of $T_{qz}S$ in $\bar{T}_{qz}S$ is denoted by $N_{qz}S$, namely, $\bar{T}_x(S) = T_x(S) \oplus N_{qz}S$, where $T_x(S) \cap N_{qz}S = 0$. We have the following decomposition:

$$q^*\bar{T}S = q^*TS \oplus N, \tag{2.1}$$

where, N is called the normal fiber bundle. If TTS_0 is the tangent vector bundle to TS_0, we denote by ϱ, the canonical linear mapping $\varrho : TTS_0 \longrightarrow q^*TS$. Let $\hat{X}$ and $\hat{Y}$ be the two vector fields on TS_0. For $z \in TS_0$, $(\nabla_{\hat{X}}Y)_z$ belongs to $\bar{T}_{qz}S$. Attending to (2.1) we have

$$\nabla_{\hat{X}}Y = \bar{\nabla}_{\hat{X}}Y + \alpha(\hat{X}, Y), \quad Y = \varrho(\hat{Y}), \quad X = \varrho(\hat{X}), \tag{2.2}$$

where, ∇ is the covariant derivative of Cartan connection and $\alpha(\hat{X}, Y)$ the second fundamental form of the submanifold S. It belongs to N and is bilinear in $\hat{X}$ and Y. It results from (2.2) that the induced connection $\bar{\nabla}$ is a metric compatible covariant derivative with respect to the induced metric g in the vector bundle $q^*TS \longrightarrow TS_0$.

2.2 Shape operator or Weingarten formula

Let S be an immersed submanifold of (M, F). For any $\hat{X} \in \chi(TS_0)$ and $W \in \Gamma(N)$ we set

$$\nabla_{\hat{X}}W = -A_W\hat{X} + \bar{\nabla}^{\perp}_{\hat{X}}W, \tag{2.3}$$

where, $A_W\hat{X} \in \Gamma(q^*TS)$ and $\bar{\nabla}^{\perp}_{\hat{X}}W \in \Gamma(N)$ and we have partially used notations of [4]. It follows that $\bar{\nabla}^{\perp}$ is a linear connection on the normal bundle N. We also consider the bilinear map $A : \Gamma(N) \otimes \Gamma(TTS_0) \to \Gamma(q^*TS), A(W, \hat{X}) = A_W\hat{X}$.

For any $W \in \Gamma(N)$, the operator $A_W : \Gamma(TTS_0) \to \Gamma(q^*TS)$ is called the *shape operator* or the *Weingarten map* with respect to W. Finally, (2.3) is said to be the *Weingarten formula* for the immersion of S in M. We have $g(\alpha({}^h\hat{X}, Y), W) = g(A_W{}^h\hat{X}, Y)$, where, g is the Finslerian metric of M, $X, Y \in \Gamma(q^*TS)$ and ${}^h\hat{X}$ is the horizontal lift of X (see [1]).

2.3 Totally umbilical submanifolds in Finsler spaces

The *mean curvature* vector field η of the isometric immersion $i : S \longrightarrow M$ is defined by

$$\eta = \frac{1}{n} tr_g \alpha({}^h\hat{X}, Y), \tag{2.4}$$

where, $X, Y \in \Gamma(q^*TS)$ and ${}^h\hat{X}$ is the horizontal lift of X (see [1]). We say that the mean curvature vector field η is parallel in all directions if $\bar{\nabla}^{\perp}_{{}^h\hat{X}}\eta = 0$ for all $X \in \Gamma(q^*TS)$.

Definition 2.1 ([1]) *A submanifold of a Finsler manifold is said to be totally umbilical, or simply umbilical, if it is equally curved in all tangent directions.*

More precisely, let $i : S \longrightarrow M$ be an isometric immersion. Then i is called totally umbilical if there exists a normal vector field $\xi \in N$ along i such that its second fundamental form α with values in the normal bundle satisfies

$$\alpha({}^h\hat{X}, Y) = g(X, Y)\xi, \tag{2.5}$$

for all $X, Y \in \Gamma(q^*TS)$, where ${}^h\hat{X}$ is the horizontal lift of X. Equivalently, S is umbilical in M if $A_W = g(W, \xi)I$ for all $W \in \Gamma(N)$ where, I is the identity transformation (see [1]). To give an example of a totally umbilical submanifold in Finsler space, we refer to a theorem on totally umbilical submanifolds given in [8]. There is shown that if $(\tilde{M}^{n+1}, \tilde{\alpha} + \tilde{\beta})$ is a Randers space, where $\tilde{\alpha}$ is an Euclidean metric and $\tilde{\beta}$ is a closed 1-form, then any complete and connected n-dimensional totally umbilical submanifold of $(\tilde{M}^{n+1}, \tilde{\alpha} + \tilde{\beta})$ must be either a plane or an Euclidean sphere. The latter case happens only when there exist a point $\tilde{x}_0$ and a function $\lambda(\tilde{x})$ on M^{n+1} such that $\tilde{\beta} = \lambda(\tilde{x})d(\| \tilde{x} - \tilde{x}_0 \|^2_{\tilde{\alpha}})$ and the sphere is centered at $\tilde{x}_0$.

Example 2.1 ([8]) Let $(\tilde{M}^{n+1}, \tilde{F})$ be a Randers space with $\tilde{F} = \tilde{\alpha} + \tilde{\beta}$, where,

$$\tilde{\alpha} = \sqrt{\sum_{k=1}^{n+1} (\tilde{y}^k)^2}, \quad \tilde{\beta} = \sum_{k=1}^{n+1} \frac{b\tilde{x}^k d\tilde{x}^k}{\sqrt{\sum_k (\tilde{x}^k)^2}}, \quad \forall (\tilde{x}, \tilde{y}) \in T\tilde{M}_0,$$

b is a constant and $0 <| b |< 1$. One can see that $d\tilde{\beta} = 0$. Let

$$M = \{\tilde{x} \in \tilde{M}^{n+1} : \sum_{k=1}^{n+1} (\tilde{x}^k - \tilde{x}^k_0)^2 = r^2\},$$

and $f : (M, F) \longrightarrow (\tilde{M}^{n+1}, \tilde{F})$ be an isometric immersion where $F = \alpha + \beta$ such that

$$\alpha = \sqrt{\sum_{k=1}^{n+1} \frac{\partial f^k}{\partial x^i} \frac{\partial f^k}{\partial x^j} y^i y^j}, \quad \beta = \sum_{k=1}^{n+1} \frac{\partial f^k}{\partial x^i} \frac{b\tilde{x}^k y^i}{\sqrt{\sum_k (\tilde{x}^k)^2}},$$

where, $(x, y) \in TM_0$. It is obvious that $\sum_{k=1}^{n+1}(f^k(x) - \tilde{x}^k_0)\frac{\partial f^k}{\partial x^i} = 0$ hence if $\tilde{x}_0 = 0$ then $\beta = 0$. On the other hand from theorem mentioned above we see that (M, F) is a totally umbilical submanifold of $(\tilde{M}^{n+1}, \tilde{F})$ if $\tilde{x}_0 = 0$. Therefore if $\tilde{x}_0 = 0$, the Euclidean sphere (M, α) is a totally umbilical submanifold of Randers space $(\tilde{M}^{n+1}, \tilde{\alpha} + \tilde{\beta})$. For more details on totally umbilical Finsler submanifolds one can refer to [10].

Remark 2.1 Let $i : S \longrightarrow M$ be an isometric immersion. If S is totally umbilical then the normal vector field ξ is equal to the mean curvature vector field η.

2.4 Codazzi equation for Finsler submanifolds

Consider a vector field $\hat{X} \in \Gamma(TTM_0)$. We have locally $\hat{X} = X^i \frac{\delta}{\delta x^i} + \dot{X}^i \frac{\partial}{\partial y^i}$ where, $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ are horizontal and vertical bases of TM. Then one can define $Q : \Gamma(TTM_0) \longrightarrow \Gamma(TTM_0); Q\hat{X} := \dot{X}^i \frac{\delta}{\delta x^i} + X^i \frac{\partial}{\partial y^i}$. By means of the Cartan connection ∇ on (M, F) and the operator Q, one can define a linear connection on the manifold TM_0 by $D_{\hat{X}}\hat{Y} := \nabla_{\hat{X}} Y + Q\nabla_{\hat{X}} Q(H\hat{Y})$, for all $\hat{X}, \hat{Y} \in \Gamma(TTM_0)$. D is called the *associated linear connection* to ∇ on TM_0. The torsion tensor field T^D of D is given by $T^D(\hat{X}, \hat{Y}) := \tau(\hat{X}, \hat{Y}) + Q(\nabla_{\hat{X}} Q(H\hat{Y}) - \nabla_{\hat{Y}} Q(H\hat{X}) - H[\hat{X}, \hat{Y}])$, where, τ is the torsion tensor field of ∇ (see [4]). Let R be the hh-curvature tensor of the Cartan connection ∇, $\bar{\nabla}$ the induced connection on the submanifold S, D the associated linear connection to the induced connection $\bar{\nabla}$ and $\bar{\nabla}^{\perp}$ the linear connection on the normal bundle N. Let A be the shape operator. One can define a covariant derivative ∇' of A as follows (see [4]).

$$(\nabla'_{\hat{X}} A)(W, \hat{Y}) := \bar{\nabla}_{\hat{X}}(A_W \hat{Y}) - A_{\bar{\nabla}^{\perp}_{\hat{X}} W} \hat{Y} - A_W(D_{\hat{X}} \hat{Y}), \tag{2.6}$$

for any $\hat{X}, \hat{Y} \in \Gamma(TTS_0)$ and $W \in \Gamma(N)$. The *A-Codazzi equation* for the Finsler submanifold S with respect to the connection ∇ on Finsler manifold (M, F) is written

$$\begin{aligned} g(R(X,Y)W, Z) = g((\nabla'_{H\hat{Y}} A)(W, H\hat{X}) - (\nabla'_{H\hat{X}} A)(W, H\hat{Y}), Z) \\ - g(A_W(T^D(H\hat{X}, H\hat{Y})), Z), \end{aligned} \tag{2.7}$$

where, $W \in \Gamma(N)$, $X = \varrho(\hat{X})$, $Y = \varrho(\hat{Y})$ and $X, Y, Z \in \Gamma(q^*TS)$ (see [4], page 84).

2.5 Sectional and flag curvatures

Let $G_2(M)$ be the fiber bundle of 2-planes on M. Denote by $\pi^{-1}G_2(M) \longrightarrow SM$ the fiber induced on SM by $\pi : SM \longrightarrow M$, where SM is the unit sphere bundle. Let $P \in \pi^{-1}G_2(M)$ be a 2-plane generated by vectors $X, Y \in T_xM$ linearly independent at $x = \pi z \in M$ where, $z \in SM$. By means of hh-curvature tensors of Berwald and Cartan connection Akbar-Zadeh defined two *sectional curvatures* denoted by K_1 and K_2 respectively. Here in this work we are dealing with Cartan connection and related sectional curvature $K_2 : \pi^{-1}G_2(M) \longrightarrow \mathbb{R}$ defined by

$$K_2(z, X, Y) = \frac{g(R(X,Y)Y, X)}{\| X \|^2 \| Y \|^2 - g(X,Y)^2}, \tag{2.8}$$

where, R is the hh-curvature tensor of Cartan connection. The scalar K_2 is called the *sectional curvature* at $z \in SM$. If the vector field Y is replaced by the canonical section v then sectional curvature is called flag curvature and does not depend on the choice of connection. If we denote the flag curvature by K then we have $K_2(z, v, X) = K(z, v, X)$, where, v is the canonical section (see [2], page 156).

Akbar-Zadeh as a *generalization of Schur's theorem* has proved the following theorem.

Theorem 2.2 ([2]) *$K_2(z, P)$ is independent of 2-plane $P(X, Y)$ ($\dim M > 2$) if and only if the curvature tensor R of the Cartan connection satisfies $R(X,Y)Z = K[g(Y,Z)X - g(X,Z)Y]$, where K is a constant and $X, Y, Z \in T_xM$.*

3 Main results

Lemma 3.1 *Let (M, F) be a Finsler manifold of dimension $n \geq 3$ satisfying the axiom of r-spheres for some r, $2 \leq r < n$, then $g(R(X,Y)Z, X) = 0$, where, $X, Y, Z \in T_xM$ are three orthonormal vectors.*

Proof. Let (M, F) be a Finsler manifold which satisfies the axiom of r-spheres. Consider the Cartan connection ∇ on the pull back bundle p^*TM, the induced connection $\bar{\nabla}$ on S and the normal connection $\bar{\nabla}^{\perp}$ on normal bundle. Let X, Y and Z be the three orthonormal vectors at $x = pz, z \in TM_0$. Consider the r-dimensional subspace E_r of T_xM which is normal to Z and contains X and Y. By assumption there exists an r-dimensional umbilical submanifold S with parallel mean curvature vector field η such that $x \in S$ and $T_xS = E_r$. It is well known for every point x in a Finsler manifold there is a sufficiently small neighborhood U on M such that every pair of points in U can be joined by a unique minimizing geodesic, see for instance [3], page 160. Hence there is a specific neighborhood U of x such that for each point $u \in U$ there exists a unique minimizing geodesic from x to u. Let $W_u \in N_uS$ be the normal vector at u which is parallel to Z with respect to the normal connection $\bar{\nabla}^{\perp}$ along the geodesic from x to u in U. The Finslerian metric g on TM_0 defined by F, induces a Finslerian metric on TS_0, where we denote it again by g. By means of metric compatibility of Cartan connection, along each geodesic γ from x to any point in U we have

$$\frac{d}{dt} g(W, \eta) = g(\nabla_{{}^h\hat{\gamma}} W, \eta) + g(W, \nabla_{{}^h\hat{\gamma}} \eta), \tag{3.1}$$

where, ${}^h\hat{\gamma}$ is the horizontal lift of the tangent vector field $\dot{\gamma}$. By means of the Weingarten formula (2.3), rewrite (3.1) as follows

$$\begin{aligned} \frac{d}{dt} g(W, \eta) &= g(-A_W({}^h\hat{\gamma}) + \bar{\nabla}^{\perp}_{{}^h\hat{\gamma}} W, \eta) + g(W, -A_\eta({}^h\hat{\gamma}) + \bar{\nabla}^{\perp}_{{}^h\hat{\gamma}} \eta) \\ &= g(-A_W({}^h\hat{\gamma}), \eta) + g(\bar{\nabla}^{\perp}_{{}^h\hat{\gamma}} W, \eta) + g(W, -A_\eta({}^h\hat{\gamma})) + g(W, \bar{\nabla}^{\perp}_{{}^h\hat{\gamma}} \eta). \end{aligned} \tag{3.2}$$

Since $-A_W({}^h\hat{\gamma})$ and $-A_\eta({}^h\hat{\gamma})$ belong to T_xS and on the other hand η and W are normal to T_xS we have $g(-A_W({}^h\hat{\gamma}), \eta) = g(W, -A_\eta({}^h\hat{\gamma})) = 0$. By assumption the submanifold S has parallel mean curvature vector field, that is, $\bar{\nabla}^{\perp}_{{}^h\hat{\gamma}} \eta = 0$, hence $g(W, \bar{\nabla}^{\perp}_{{}^h\hat{\gamma}} \eta) = 0$. By definition the vector W is parallel along the geodesic γ with respect to the normal connection $\bar{\nabla}^{\perp}$, i.e. $\bar{\nabla}^{\perp}_{{}^h\hat{\gamma}} W = 0$, hence $g(\bar{\nabla}^{\perp}_{{}^h\hat{\gamma}} W, \eta) = 0$. Therefore by means of (3.2) we have $\frac{d}{dt} g(W, \eta) = 0$ and $g(W, \eta) = \lambda$ is constant along each geodesic. Keeping in mind S is a totally umbilical submanifold of M, we have $A_W = g(W, \eta)I = \lambda I$ at every point of U. Rewriting (2.6) for the horizontal lift ${}^h\hat{X}$ of X leads

$$(\nabla'_{{}^h\hat{X}} A)(W, \hat{Y}) = (\nabla^*_{{}^h\hat{X}} A_W)(\hat{Y}) - A_{\bar{\nabla}^{\perp}_{{}^h\hat{X}} W} \hat{Y}, \tag{3.3}$$

where, we have put, $(\nabla^*_{{}^h\hat{X}} A_W)(\hat{Y}) := \bar{\nabla}_{{}^h\hat{X}}(A_W \hat{Y}) - A_W(D_{{}^h\hat{X}} \hat{Y})$ which can be considered as a covariant derivative of A_W. Plugging $A_W = \lambda I$ in the last equation leads

$$\nabla^*_{{}^h\hat{X}} A_W = 0. \tag{3.4}$$

Similarly for the horizontal lift ${}^h\hat{Y}$ of Y we have

$$\nabla^*_{{}^h\hat{Y}} A_W = 0. \tag{3.5}$$

On the other hand, by means of metric compatibility of Cartan connection and the fact that $g(W,\eta)$ is constant we have $g(\nabla_{{}^h\hat{X}} W,\eta) + g(W,\nabla_{{}^h\hat{X}}\eta) = 0$. By means of the Weingarten formula (2.3) the last equation leads

$$g(-A_W({}^h\hat{X}),\eta) + g(\bar{\nabla}^{\perp}_{{}^h\hat{X}} W,\eta) + g(W,-A_\eta({}^h\hat{X})) + g(W,\bar{\nabla}^{\perp}_{{}^h\hat{X}}\eta) = 0. \tag{3.6}$$

Since $A_W({}^h\hat{X})$ and $A_\eta({}^h\hat{X})$ belong to T_xS and on the other hand η and W are normal to T_xS we have $g(-A_W({}^h\hat{X}),\eta) = g(W,-A_\eta({}^h\hat{X})) = 0$. By assumption the submanifold S has parallel mean curvature vector field, that is, $\bar{\nabla}^{\perp}_{{}^h\hat{X}}\eta = 0$, hence $g(W,\bar{\nabla}^{\perp}_{{}^h\hat{X}}\eta) = 0$. Therefore (3.6) reduces to $g(\bar{\nabla}^{\perp}_{{}^h\hat{X}} W,\eta) = 0$. By non-degeneracy of the metric tensor g at $x \in S$ we have

$$\bar{\nabla}^{\perp}_{{}^h\hat{X}} W = 0. \tag{3.7}$$

Similarly at $x \in S$ for vector Y we obtain

$$\bar{\nabla}^{\perp}_{{}^h\hat{Y}} W = 0. \tag{3.8}$$

Therefore plugging (3.4), (3.5), (3.7) and (3.8) in (3.3) at $x \in S$ we obtain $\nabla'_{{}^h\hat{X}} A = \nabla'_{{}^h\hat{Y}} A = 0$. Now the Codazzi equation (2.7) implies

$$g(R(X,Y)W,X) = -g(A_W(T^D({}^h\hat{X},{}^h\hat{Y})),X). \tag{3.9}$$

By assumption $A_W = g(W,\eta)I$. Thus we have

$$g(A_W(T^D({}^h\hat{X},{}^h\hat{Y})),X) = g(T^D({}^h\hat{X},{}^h\hat{Y}),X)g(W,\eta). \tag{3.10}$$

Plugging (3.10) in (3.9) we obtain

$$g(R(X,Y)W,X) + g(T^D({}^h\hat{X},{}^h\hat{Y}),X)g(W,\eta) = 0. \tag{3.11}$$

The first term $g(R(X,Y)W,X)$ is symmetric with respect to Y and W (see [2]). By means of the fact that η is normal to S we have $g(Y,\eta) = 0$. Therefore, we conclude $g(T^D({}^h\hat{X},{}^h\hat{Y}),X)g(W,\eta) = g(T^D({}^h\hat{X},{}^h\hat{W}),X)g(Y,\eta) = 0$. Thus (3.11) becomes $g(R(X,Y)W,X) = 0$. Hence for orthonormal vectors $X,Y \in T_xS$ and $Z \in N_xS$ we have $g(R(X,Y)Z,X) = 0$. This completes the proof. □

Lemma 3.2 *Let (M,F) be a Finsler manifold of dimension $n \geq 3$. If $g(R(X,Y)Z,X)=0$ whenever X,Y and Z are three orthonormal tangent vectors of M, then M has constant flag curvature.*

Proof. If we put $Y' = \frac{(Y+Z)}{\sqrt{2}}$, $Z' = \frac{(Y-Z)}{\sqrt{2}}$, then since X,Y and Z are orthonormal, the vectors X,Y' and Z' are again orthonormal. By means of assumption $g(R(X,Y')Z',X) = 0$. By replacing Y' and Z' we obtain

$$g(R(X,Y)Y,X) = g(R(X,Z)Z,X). \tag{3.12}$$

From which we can conclude from (2.8), $K_2(z,X,Y) = K_2(z,X,Z)$. Thus the sectional curvature K_2 does not depend on the 2-plane $P(X,Y)$. By generalization of Schur's Theorem 2.2, M has constant sectional curvature and hence constant flag curvature. This completes the proof. □

Proof of Theorem 1.1. Let (M, F) be a Finsler manifold which satisfies the axiom of r-spheres. By means of Lemmas 3.1 and 3.2 we conclude that M has constant flag curvature. □

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