An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat. (N.S.)
Tomul LXII, 2016, f. 2, vol. 3

A variation on strongly lacunary ward continuity

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Received: 23.VII.2014 / Revised: 13.VII.2015 / Accepted: 7.VIII.2015

Abstract A function f defined on a subset A of a cone normed space X is strongly lacunary ward continuous if it preserves strongly lacunary quasi-Cauchy sequences of points in A; that is, $(f(x_k))$ is a strongly lacunary quasi-Cauchy sequence whenever (x_k) is strongly lacunary quasi-Cauchy. In this paper, not only strongly lacunary ward continuity, but also some other kinds of continuities are investigated in cone normed spaces.

Keywords Strongly lacunary convergence · Cone normed spaces · Quasi-Cauchy sequences · Continuity

Mathematics Subject Classification (2010) 40A35 · 40A05 · 26A15 · 40A30

1 Introduction

A basic task in image processing is comparing images and computing some measure of the images. The choice of a suitable definition of measure is not all that easy; what is small in one norm can be very huge in another one. This naturally leads to an environment in which many possible norms can be considered simultaneously and cone norm spaces lend themselves to this requirement. One specific instance of this is in the analysis of the structural similarity (SSIM) index of images (see [3], [4] and [27]). SSIM is used to improve the measuring of visual distortion between images and is also used in fractal-based approximation using entropy maximization and sparsity constraints (see [28]). In both of these contexts the difference between two images is

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calculated using multiple criteria, which leads in a natural way to consider vector-valued distances.

Investigation of cone metric spaces and cone normed spaces goes back to papers published by several Russian authors in the mid-20th century. In 1966, Vandergraft [37], in 1980 Rzepecki [31], in 1987 Lin [29], and in 1997 Zabrejko [38] studied such spaces, calling them K-metric spaces, and K-normed spaces. In 2007, Long-Guang and Xian [23] announced the notion of a cone metric space by replacing real numbers with an ordered Banach space, which is the same as either the definition of Vandergraft, Rzepecki, Lin or of Zabrejko (see also [21], [32] and [34]). In [35], Sonmez and Cakalli studied main properties of cone normed spaces and proved theorems in cone normed spaces and complete cone normed spaces. Abdeljawad, Turkoglu and Abuloha [1] studied some properties of cone Banach spaces. In fact, main properties of cone normed spaces seem to be first studied independently by Karapinar [25], Sonmez [33] and Abdeljawad, Turkoglu and Abuloha [1].

Using the main idea in the definition of sequential continuity, many kinds of continuities were introduced and investigated, and not all but some of them are in [5, 10, 16, 17, 19, 36]. The concept of N_θ-convergence was introduced in [22] and further studied in [24]. Strongly lacunary ward continuity of a real function was introduced by Cakalli in [10] and further studied in [13, 14].

The aim of this paper is to investigate strongly lacunary ward continuity in cone normed spaces, and prove interesting theorems.

2 Preliminaries

A subset P of a real Banach space E is called a cone if (C1) P is closed, nonempty, and $P \neq \{0\}$, (C2) $a, b \in R, \quad a, b \geq 0, \quad x, y \in P \quad \Rightarrow \quad ax + by \in P$, (C3) $x \in P$ and $-x \in P \quad \Rightarrow \quad x = 0$. One can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \overset{\circ}{P}$ where $\overset{\circ}{P}$ denotes the interior of P. The cone P is called normal if there is a constant number $K > 0$ such that for all $x, y \in E$ $0 \preceq x \preceq y \quad$ implies $\quad \|x\| \leq K\|y\|$. The least positive number satisfying the above inequality is called the normal constant of P. In this paper, we always suppose that E is a Banach space, P is a normal cone in E with $\overset{\circ}{P} \neq \emptyset$ and $\preceq$ is partial ordering with respect to P. A mapping d from a cartesian product of a vector space X into E is called a cone metric if satisfies the following conditions: (CM1) $0 \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$; (CM2) $d(x, y) = d(y, x)$ for all $x, y \in X$; (CM3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$. Then (X, d) is called a cone metric space. Now we recall the following definitions:

(a) If for every $c \in \overset{\circ}{P}$ there is a positive integer N such that for all $n > N$, $d(x_n, x) \ll c$, then (x_n) is said to be convergent and (x_n) converges to x, and x is called the limit of (x_n) (see [23]). We denote this by $\lim_n x_n = x$ or $x_n \to x$ $(n \to \infty)$.

(b) If for every $c \in \overset{\circ}{P}$ there is a positive integer N such that for all $n, m > N$, $d(x_n, x_m) \ll c$, then (x_n) is called a Cauchy sequence in X (see [33]).

(c) If every Cauchy sequence is convergent in X, then X is called a complete cone metric space (see [35]).

Let X be a real vector space and E be a real Banach space with a cone P. Suppose that the mapping $|||.||| : X \to P$ satisfies (CN1) $|||x||| = 0 \Leftrightarrow x = 0$; (CN2) $|||\alpha x||| = |\alpha||||x|||$ for any scalar α and any $x \in X$; (CN3) $|||x + y||| \preceq |||x||| + |||y|||$ for all $x, y \in X$. Then $|||.|||$ is called a cone norm on X, and we call $(X, |||.|||)$ a cone normed space ([2], [9], [12], [13], [14]) . One can see that the condition (CN3) means $|||x||| + |||y||| - |||x+y||| \in P$ and $|||x||| \in P$ for all $x \in X$. It is clear that $d(x, y) = |||x-y|||$ is a cone metric. A cone Banach space X is a complete cone normed space. Throughout the paper, X will denote a cone Banach space. We use the symbol $||.||$ for the norm of E and $|||.|||$ for the cone norm of X. We say that the series $\sum x_n$ with $x_n \in X$, converges to $x \in X$ if $(s_n) = (x_1, x_1 + x_2, x_1 + x_2 + x_3, ...)$ converges to x, i.e., $|||s_n - x||| \to 0$ as $n \to \infty$. Then we write $\sum_{n=1}^{\infty} x_n = x$. A series $\sum x_n$ is called absolutely convergent if $\sum |||x_n|||$ is convergent in E. Take any series $\sum \alpha_n$ and $\sum \beta_n$ in E. If $0 \preceq \alpha_n \preceq \beta_n$ for all n and the series $\sum \beta_n$ is convergent, then the series $\sum \alpha_n$ is convergent. In [2], Asadi and Soleimani give a counterexample in a non-normal cone metric space that the preceding inequality does not hold.

3 Strongly lacunary continuity

A lacunary sequence $\theta = (k_r)$ is an increasing sequence of positive integers such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and the ratio k_r/k_{r-1} will be abbreviated by q_r, and $q_1 = k_1$ for convenience. A sequence (x_k) of points in X is called strongly lacunary convergent, or N_θ-convergent to an element ℓ of X if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |||x_k - \ell||| = 0,$$

and it is denoted by $N_\theta - \lim_{k\to\infty} x_k = \ell$. A sequence (x_k) of points in X is called lacunary statistical convergent, or S_θ-convergent to an element ℓ of X if

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : c - |||x_k - \ell||| \notin \overset{\circ}{P}\}| = 0$$

for every $c \in \overset{\circ}{P}$ and it is denoted by $S_\theta - \lim_{k\to\infty} x_k = \ell$. In the sequel, we will use the word "N_θ" instead of "strongly lacunary", and "S_θ" instead of "lacunary statistically". A sequence $\mathbf{x} = (x_k)$ of points in X is called lacunary statistically Cauchy, or S_θ-Cauchy if there exists a subsequence $(x_{k'(r)})$ of the sequence $\mathbf{x}$ such that $\lim_{r\to\infty} x_{k'(r)} = \ell$ and

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : c - |||x_k - x_{k'(r)}||| \notin \overset{\circ}{P}\}| = 0,$$

for every $c \in \overset{\circ}{P}$. Since any cone normed space is a first countable Hausdorff topological group, it follows from Theorem 6 of [6] that S_θ-Cauchyness coincides with S_θ-convergence, i.e. the set of S_θ-Cauchy sequences is equal to the set of S_θ-convergent sequences (see also [11]).

Lemma 3.1 *Any N_θ-convergent sequence is S_θ-convergent, i.e. $N_\theta \subset S_\theta$ for any lacunary sequence θ.*

Proof. Let $c \in \overset{\circ}{P}$, $\varepsilon > 0$ and $(x_k) \in N_\theta$. Write $N_\theta - \lim_{k\to\infty} x_k = \ell$. Then $c|\{k \in I_r : c - |||x_k - \ell||| \notin \overset{\circ}{P}\}| \preceq \sum_{k\in I_r, c-|||x_k-\ell|||\notin \overset{\circ}{P}} |||x_k - \ell||| \preceq \sum_{k\in I_r} |||x_k - \ell|||$. Denote $\rho_r = |\{k \in I_r : c - |||x_k - \ell||| \notin \overset{\circ}{P}\}|$. Now because $(x_k) \in N_\theta$, there exist a positive integer R such that if $r \geq R$,

$$\varepsilon c - \frac{1}{h_r}\sum_{k\in I_r} |||x_k - \ell||| \in \overset{\circ}{P}.$$

But due to the above inequality, we have

$$\varepsilon c - \frac{1}{h_r}\sum_{k\in I_r} |||x_k - \ell||| + \frac{1}{h_r}(\sum_{k\in I_r} |||x_k - \ell||| - c\rho_r) \in \overset{\circ}{P}.$$

That is $(\varepsilon - \frac{1}{h_r}\rho_r)c \in \overset{\circ}{P}$. Therefore $\frac{1}{h_r}\rho_r < \varepsilon$. This completes the proof. □

We recall that a method G is regular if every convergence sequence $\mathbf{x} = (x_n)$ is G-convergent with $G(\mathbf{x}) = \lim\ \mathbf{x}$, and that a sequential method is called subsequential if a sequence (x_k) is summable by the sequential method to ℓ, then there exists a convergent subsequence (x_{k_n}) with ordinary limit ℓ (see [7] and [9]). Now we are going to prove that the sequential method N_θ is regular and subsequential.

Theorem 3.2 *The sequential method N_θ is regular and subsequential.*

Proof. To prove that the sequential method N_θ is regular, take any convergent sequence (x_k) of points in X with the ordinary limit ℓ, i.e. $\lim_{k\longrightarrow\infty} x_k = \ell$. Let $c \in \overset{\circ}{P}$. Then there exists a positive integer $n_0 \in \mathbb{N}$ such that $|||x_k - \ell||| \ll c$ for $k \geq n_0$. Thus for $r \geq n_0$

$$\frac{1}{h_r}\sum_{k\in I_r} |||x_k - \ell||| \ll \frac{1}{h_r}\sum_{k\in I_r} v_k = \frac{1}{h_r}[k_r - (k_{r-1}+1)+1]c = c,$$

where $v_k = c$ for every $k \in I_r$. Thus, $(x_k) \in N_\theta$, so the sequential method N_θ is regular. For the proof of the subsequentiality of N_θ, take any element $\mathbf{x} = (x_k)$ of N_θ. By Lemma 3.1, we have $\mathbf{x} \in S_\theta$. Since a cone normed space is a Hausdorff topological group which satisfies the first axiom of countability, applying Corollary 7 of [6], we find a convergent subsequence of $\mathbf{x}$. This completes the proof of the theorem. □

Definition 3.3 *A subset A of X is called N_θ-sequentially compact if any sequence of points in A has an N_θ-convergent subsequence with an N_θ-limit in A.*

We note that union of two N_θ-sequentially compact subsets of X is N_θ-sequentially compact, intersection of any N_θ-sequentially compact subsets is N_θ-sequentially compact, any compact subset of X is N_θ-sequentially compact, and any finite subset of X is N_θ-sequentially compact. Sum of N_θ-sequentially compact subsets of X is N_θ-sequentially compact where sum of two subsets A and B is defined as $A + B = \{a + b : a \in A, b \in B\}$. Now we give the following characterization of N_θ-sequentially compactnness of a subset of X.

Theorem 3.4 *A subset of X is N_θ-sequentially compact if and only if it is sequentially compact in the ordinary sense.*

Proof. The proof follows from Corollary 5 in [7], so is omitted. □

Definition 3.5 *A function f is said to be strongly lacunary sequentially continuous, or N_θ-sequentially continuous at a point x_0 of X if $(f(x_k))$ is an N_θ-convergent sequence to $f(x_0)$ whenever (x_k) is an N_θ-convergent to x_0 sequence of points in X. If f is N_θ-sequentially continuous at every points of a subset A of X, then it is said to be strongly lacunary sequentially continuous, or N_θ-sequentially continuous on A.*

We note the that sum of two N_θ-sequentially continuous functions at a point x_0 of X is N_θ-sequentially continuous at x_0, and the composite of an N_θ-sequentially continuous function g at a point x_0 of X, and an N_θ-sequentially continuous function f at the point $g(x_0)$ is N_θ-sequentially continuous at x_0.

Theorem 3.6 *A function is N_θ-sequentially continuous if and only if it is continuous in the ordinary sense.*

Proof. If f is N_θ-sequentially continuous at a point x_0 of X, then it follows Theorem 13 in [9] that it is sequentially continuous in the ordinary sense at x_0. Conversely suppose that f is continuous at a point x_0 of X. Then it follows from Lemma 1 in [9] that the set of all G-sequentially closed subsets of X, coincides with the set of closed subsets of X obtained by ordinary closed subsets of X, so the remark given just after the proof of Corollary 9 in [9] implies that f is N_θ-sequentially continuous. This completes the proof of the theorem. □

Corollary 3.7 *Any slowly oscillating continuous function is N_θ-sequentially continuous.*

Proof. The proof follows from Theorem 3.1 of [16], so is omitted. □

4 Strongly lacunary ward continuity

Now we give the following definition of an N_θ-quasi-Cauchy sequence.

Definition 4.1 *A sequence (x_k) of points in X is called strongly lacunary quasi-Cauchy, or N_θ-quasi-Cauchy if (Δx_k) is N_θ-convergent to 0, that is*

$$N_\theta - \lim_{k\to\infty} \Delta x_k = 0,$$

where $\Delta x_k = x_{k+1} - x_k$.

Any slowly oscillating sequence is N_θ-quasi-Cauchy, so any convergent sequence is N_θ-quasi-Cauchy in X. Any Cauchy sequence is N_θ-quasi-Cauchy, but the converse is not always true. Sum of two N_θ-quasi-Cauchy sequence is N_θ-quasi-Cauchy. A subsequence of an N_θ-quasi-Cauchy sequence need not be N_θ-quasi-Cauchy. Now we give the definition of N_θ-ward compactness of a subset of X.

Definition 4.2 *A subset A of X is called N_θ-ward compact if any sequence of points in A has an N_θ-quasi-Cauchy subsequence.*

The union of two N_θ-ward compact subset of X is N_θ-ward compact, the intersection of any N_θ-ward compact subsets is N_θ-ward compact, the sum of two N_θ-ward compact subset of X is N_θ-ward compact, and any finite subset of X is N_θ-ward compact. These observations suggest us to give the following.

Theorem 4.3 *A subset of X is totally bounded if and only if it is N_θ-ward compact.*

Proof. To prove that totally boundedness implies N_θ-ward compactness take a totally bounded subset A of X. Let (x_n) be a sequence of points in A and $c \in \overset{\circ}{P}$. Since A can be covered by a finite number of subsets of X of diameter less than c, one of these sets , which we denote by A_1, must contain x_n for infinitely many values of n. We may choose a positive integer n_1 such that $x_{n_1} \in A_1$. Then A_1 is totally bounded and hence it can be covered by a finite number of subsets of A_1 of diameter less than $\frac{c}{2}$. One of these subsets of A_1, which we denote by A_2, contains x_n for infinitely many n. choose a positive integer n_2 such that $n_2 > n_1$ and $x_{n_2} \in A_2$. Since $A_2 \subset A_1$, it follows that $x_{n_2} \in A_1$ as well. Continuing in this way, we obtain, for any positive integer k, a subset A_k of A_{k-1} with *diameter* less than $\frac{c}{k}$ and a term $x_{n_k} \in A_k$ of the sequence (x_n), where $n_k > n_{k-1}$. Since all x_{n_k}, $x_{n_{k+1}}$, $x_{n_{k+2}}$,...,$x_{n_{k+j}}$,... lie in A_k and the diameter of A_k is less than $\frac{c}{k}$, it follows that (x_{n_k}) is an N_θ quasi-Cauchy subsequence of the sequence (x_n). To prove that N_θ-ward compactness implies totally boundedness, suppose that A is not totally bounded. Then there exists a $c \in \overset{\circ}{P}$ such that there does not exist a finite c-net. Take any $x_1 \in A$. By the assumption that A is not totally bounded, the open ball $B_A(x_1, c)$ is not equal to A, i.e. $B_A(x_1, c) \neq A$, so there exists an $x_2 \in A$ such that $c - |||x_1 - x_2||| \notin P$, i.e. $x_2 \notin B_A(x_1, c)$, and $x_2 \in A$ where $B_A(x_1, c) = B(x_1, c) \cap A$. Then $B_A(x_1, c) \cup B_A(x_2, c) \neq A$ otherwise $B_A(x_1, c) \cup B_A(x_2, c)$ would be a finite c-net in A. Let $x_3 \notin B_A(x_1, c) \cup B_A(x_2, c)$, i.e. $c - |||x_1 - x_2||| \notin P$, $c - |||x_1 - x_3||| \notin P$, and $c - |||x_2 - x_3||| \notin P$. Continuing in this manner, one can obtain a sequence (x_n) of points in A such that

$$x_n \notin B_A(x_1, c) \cup B_A(x_2, c) \cup ... \cup B_A(x_{n-1}, c), \ (n = 2, 3, ...)$$

$$i.e. \ c - |||x_i - x_n||| \notin P \ (i = 1, 2, ..., n-1) \ and \ (n = 1, 2, ...), \ n \neq i.$$

Hence the sequence (x_n) constructed in this manner has no N_θ-quasi-Cauchy subsequence. This contradiction completes the proof of the theorem. □

A subset A of X is slowly oscillating compact if any sequence of points in A has a slowly oscillating subsequence (see [16]).

Corollary 4.4 *A subset of X is slowly oscillating compact if and only if it is N_θ-ward compact.*

Proof. Since any slowly oscillating sequence is N_θ-quasi-Cauchy, any slowly oscillating compact subset of X is N_θ-ward compact. By Theorem 4.3, any N_θ-ward compact subset of X is totally bounded. It is easy to see that any sequence of points in a totally bounded subset of X has an N_θ-quasi-Cauchy subsequence (see Theorem 3 of [8] for an idea for a proof). This completes the proof. □

Definition 4.5 *A function defined on a subset A of X is called N_θ-ward continuous if it preserves N_θ-quasi-Cauchy sequences, i.e. $(f(x_k))$ is an N_θ-quasi-Cauchy sequence whenever (x_k) is.*

We note that the composite of two N_θ-ward continuous functions is N_θ-ward continuous, and the sum of two N_θ-ward continuous functions is N_θ-ward continuous.

Theorem 4.6 *N_θ-ward continuous image of any N_θ-ward compact subset of X is N_θ-ward compact.*

Proof. Assume that f is an N_θ-ward continuous function on a subset A of X, and B is an N_θ-ward compact subset of A. Let (y_n) be any sequence of points in $f(B)$ where $y_n = f(x_n)$, $\mathbf{x} = (x_n)$, $x_n \in B$, for each positive integer n. N_θ-ward compactness of B implies that there is a N_θ-quasi-Cauchy subsequence $(\gamma_k) = (x_{n_k})$ of $\mathbf{x}$. Write $(t_k) = (f(\gamma_k))$. Since f is N_θ-ward continuous, $(f(\gamma_k))$ is N_θ-quasi-Cauchy which is a subsequence of the sequence $f(\mathbf{x})$. This completes the proof of the theorem. □

Corollary 4.7 *N_θ-ward continuous image of any compact subset of X is N_θ-ward compact.*

Proof. The proof follows from the preceding theorem. □

Concerning N_θ-quasi-Cauchy sequences, N_θ convergent sequences, slowly oscillating sequences, and convergent sequences the problem arises to investigate the following types of continuity of functions on X. In the following N_θ, ΔN_θ, w and c will denote the set of N_θ-convergent sequence, the set of N_θ-quasi-Cauchy sequences, the set of slowly oscillating sequences, and the set of convergent sequences of points in X, respectively.

1. $(x_k) \in \Delta N_\theta \Rightarrow (f(x_k)) \in \Delta N_\theta$
2. $(x_k) \in \Delta N_\theta \Rightarrow (f(x_k)) \in c$
3. $(x_k) \in c \Rightarrow (f(x_k)) \in c$
4. $(x_k) \in c \Rightarrow (f(x_k)) \in \Delta N_\theta$
5. $(x_k) \in N_\theta \Rightarrow (f(x_k)) \in N_\theta$
6. $(x_k) \in \Delta N_\theta \Rightarrow (f(x_n))$ is quasi-Cauchy.
7. $(x_k) \in \Delta N_\theta \Rightarrow (f(x_n)) \in N_\theta$.
8. $(x_n) \in w \Rightarrow (f(x_n)) \in N_\theta$.

We see that (1) is N_θ-ward continuity of f, (5) is N_θ-sequentially continuity of f and (3) is the ordinary sequential continuity of f. It is easy to see that (2) implies (1), and (1) does not imply (2); and (1) implies (4), and since the function defined by $f(x) = x^2$ for every $x \in X$ is an example we deduce that (4) but does not imply (1), where X is real line (cone normed) space. (2) implies (3), and (3) does not imply (2). (6) implies (1), but (1) does not imply (6) since the identity function provides a counter example, (7) implies (1), but (1) does not imply (7), and (8) implies (7), but (7) does not imply (8). Now we give the implication (1) implies (5), that is, any N_θ-ward continuous function is N_θ-sequentially continuous.

Theorem 4.8 *If f is N_θ-ward continuous function on a subset A of X, then it is N_θ-sequentially continuous on A.*

Proof. Assume that f is an N_θ-ward continuous function on A. Let (x_n) be any N_θ-convergent sequence of points in A with $N_\theta - \lim_{k\to\infty} x_k = \ell$. Then the sequence $\mathbf{x} = (x_n)$ defined by

$$\mathbf{x} = (x_1, \ell, x_2, \ell, ..., \ell, x_n, \ell, ...)$$

is also N_θ-convergent to ℓ. Therefore it is an N_θ-quasi-Cauchy sequence. Since f is N_θ-ward continuous on A, the transformed sequence

$$\mathbf{y}{=}f(\mathbf{x}){=}(y_n) = (f(x_n)) = (f(x_1), f(\ell), f(x_2), f(\ell), ..., f(\ell), f(x_n), f(\ell), ...)$$

is N_θ-quasi-Cauchy. So

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |||f(x_k) - f(\ell)||| = 0.$$

It follows from this that the sequence $(f(x_k))$ is N_θ-convergent to $(f(\ell))$. □

The converse of this theorem is not valid in general, a counterexample can be easily constructed via the function $f(x, y) = (x^2, 0)$ on the cone normed space $\mathbb{R}^2$ with a suitable cone norm.

Corollary 4.9 *If f is an N_θ-ward continuous function on a subset A of X, then it is continuous on A.*

Proof. The proof follows immediately from the preceding theorem and Theorem 3.6, so is omitted. □

It is a well known result that any continuous function on a compact subset A of a normed space is uniformly continuous on A. It is also true for a regular subsequential method G that any N_θ-sequentially continuous function on a G-sequentially compact subset A of X is also uniformly continuous on A (see [9] for G sequentially compactness). Furthermore, for N_θ-ward continuous functions defined on a N_θ-ward compact subset of X, we have the following.

Theorem 4.10 *An N_θ-ward continuous function on an N_θ-ward compact subset of X is uniformly continuous.*

Proof. Let A be an N_θ-ward compact subset of X, and $f : A \longrightarrow X$ be an N_θ-ward continuous function. Suppose that f is not uniformly continuous on A so that there exists a $c_0 \in \overset{\circ}{P}$ such that for any $d \in \overset{\circ}{P}$ there are $x, y \in A$ with $|||x - y||| \ll d$ but $c_0 - |||f(x) - f(y)||| \notin \overset{\circ}{P}$. For each positive integer n, there exist x_n and y_n such that $|||x_n - y_n||| \ll \frac{c_0}{n}$, and $c_0 - |||f(x_n) - f(y_n)||| \notin \overset{\circ}{P}$. Since A is N_θ-ward compact, there exists an N_θ-quasi-Cauchy subsequence (x_{n_k}) of the sequence (x_n). It is clear that the corresponding subsequence (y_{n_k}) of the sequence (y_n) is also N_θ-quasi-Cauchy, since $(y_{n_{k+1}} - y_{n_k})$ is a sum of three N_θ-null sequences, i.e.

$$y_{n_{k+1}} - y_{n_k} = (y_{n_{k+1}} - x_{n_{k+1}}) + (x_{n_{k+1}} - x_{n_k}) + (x_{n_k} - y_{n_k}).$$

On the other hand, it follows from the equality $x_{n_{k+1}} - y_{n_k} = x_{n_{k+1}} - x_{n_k} + x_{n_k} - y_{n_k}$ that the sequence $(x_{n_{k+1}} - y_{n_k})$ is N_θ-convergent to 0. Hence the sequence

$$(x_{n_1}, y_{n_1}, x_{n_2}, y_{n_2}, x_{n_3}, y_{n_3}, ..., x_{n_k}, y_{n_k}, ...)$$

is N_θ-quasi-Cauchy. But the transformed sequence

$$(f(x_{n_1}), f(y_{n_1}), f(x_{n_2}), f(y_{n_2}), f(x_{n_3}), f(y_{n_3}), ..., f(x_{n_k}), f(y_{n_k}), ...)$$

is not N_θ-quasi-Cauchy. Thus f does not preserve N_θ-quasi-Cauchy sequences. This contradiction completes the proof of the theorem. □

Corollary 4.11 *If a function f is N_θ-ward continuous on a totally bounded subset A of X, then it is uniformly continuous on A.*

Proof. The proof follows from the preceding theorem and Theorem 4.3, so is omitted. □

Theorem 4.12 *Uniform limit of N_θ-ward continuous functions is N_θ-ward continuous.*

Proof. Let (f_n) be a sequence of N_θ-ward continuous functions on a subset A of X, (f_n) be uniformly convergent to a function f, (x_k) be any N_θ-quasi-Cauchy sequence of points in A, and $c \in \overset{\circ}{P}$. By uniform convergence of (f_n), there exists an $n_1 \in \mathbb{N}$ such that $|||f(x) - f_n(x)|||| \ll \frac{c}{3}$ for $n \geq n_1$ and every $x \in A$. Hence

$$\frac{1}{h_r}\sum_{k\in I_r} v_k \ll \frac{1}{h_r}(k_r - k_{r-1})\frac{c}{3} = \frac{c}{3},$$

for $r \geq n_1$ and every $x \in A$ where $v_k = |||f(x) - f_k(x)|||$ for every $k \in \mathbb{N}$. Since f_{n_1} is N_θ-ward continuous on A, there exists an $n_2 \in \mathbb{N}$ such that for $r \geq n_2$

$$\frac{1}{h_r}\sum_{k\in I_r} |||f_{n_1}(x_{k+1}) - f_{n_1}(x_k)||| \ll \frac{c}{3}.$$

Now write $n_0 = \max\{n_1, n_2\}$. Thus for $r \geq n_0$ we have

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r} |||f(x_{k+1}) - f(x_k)|||\\
&\preceq \frac{1}{h_r}\sum_{k\in I_r} |||f(x_{k+1}) - f_{n_1}(x_{k+1})||| + \frac{1}{h_r}\sum_{k\in I_r} |||f_{n_1}(x_{k+1}) - f_{n_1}(x_k)|||\\
&+ \frac{1}{h_r}\sum_{k\in I_r} |||f_{n_1}(x_k) - f(x_k)|||\\
&\ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c.
\end{aligned}$$

Hence

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{k\in I_r} |||f(x_{k+1}) - f(x_k)||| = 0.$$

Thus $(f(x_k))$ is an N_θ-quasi-Cauchy sequence. This completes the proof of the theorem. □

5 Conclusion

In this paper, we investigate strongly lacunary ward continuity and some other kinds of continuities defined via a lacunary sequence and we prove interesting theorems related to these kinds of continuities in cone normed spaces. The results in this paper are extensively deeper than existing related results, and most of the results are also new for the ordinary normed spaces some of them are also new for the real case. We note that the notion of a strongly lacunary quasi-Cauchy sequence coincides with the notion of a strongly lacunary convergent sequence in a complete non-Archimedean cone normed space, and so the set of strongly lacunary ward continuous functions coincides with the set of strongly lacunary sequentially continuous functions in a complete non-Archimedean cone normed space (see [30] for the related concepts in a normed field). For a further study, we suggest to investigate strongly lacunary quasi-Cauchy sequences of points for the fuzzy functions in a fuzzy cone normed space. However due to the change in settings, the definitions and methods of proofs will not always be analogous to those of the present work (see [12] and [26] for the definitions and related concepts in fuzzy setting). For another further study, we suggest to investigate strongly lacunary quasi-Cauchy double sequences (see [15] and [20] for the definitions and related concepts in double case).

Acknowledgements The authors would like to thank the referee for a careful reading and useful constructive comments that have improved the presentation of the results.

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