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On 2-absorbing and strongly 2-absorbing ideals of commutative semigroups

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Abstract In this paper, S denotes a commutative multiplicative semigroup with 1 and 0. A nonempty subset I of S is called an ideal of S if I satisfies the condition $SI \subseteq I$. We introduce and investigate the concepts of 2-absorbing ideals and strongly 2-absorbing ideals of S. Also, we study the relationships between 2-absorbing ideals and strongly 2-absorbing ideals of S.

Keywords 2-absorbing ideals · Strongly 2-absorbing ideals · Commutative semigroups

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1 Introduction

Throughout the paper, we suppose that S is a commutative multiplicative semigroup with 1 and 0. It is well known that S is commutative if it satisfies $xy = yx$ for every $x, y \in S$. A subsemigroup I of S is a nonempty subset of S if for every $x, y \in I$, $xy \in I$. A nonempty subset I of S is called an ideal of S if $SI \subseteq I$. It can be easily seen that $I \cup J$, $I \cap J$ and $IJ = \{xy : x \in I, y \in J\}$ are ideals of S for every ideals I, J of S. For an element $a \in S$, $(a) = Sa = \{sa : s \in S\}$ is an ideal of S generated by a. The ideal (a) is said to be a principal ideal of S. Let I and J be two ideals of S. We define

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the residual of I by J as $(I : J) = \{s \in S : sJ \subseteq I\}$ and we define the radical of J as $\sqrt{J} = \{s \in S : s^n \in J$ for some positive integer $n\}$.

An ideal I of S is said to be a proper ideal if it is a proper subset of S, that is, $I \neq S$. A proper ideal P of S is said to be prime if whenever $ab \in P$ for any $a, b \in S$, then either $a \in P$ or $b \in P$. Note that, if P is a prime ideal, then for every ideals I, J of S, $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. It is clear that $\sqrt{J}$ is the intersection of prime ideals containing J. A proper ideal I of S is called a primary ideal of S if for any $a, b \in S$, $ab \in I$ implies either $a \in I$ or $b^n \in I$ for some positive integer n. If I is a primary ideal of S, then $P = \sqrt{I}$ is a prime ideal of S. In this case I is called a P-primary ideal of S. A proper ideal M of S is called a maximal ideal if there is no other proper ideal M' of S with $M \subset M'$. For any maximal ideal M and a positive integer n, M^n is M-primary.

Let q be a congruence of S, and $S/q = \{xq : x \in S\}$ be the set of all congruence classes of q. By $(xq)(yq) = (xy)q$, we define the quotient semigroup on the domain S/q. For a congruence q, we define a mapping $f : S \to S/q$ as $f(x) = xq$. Note that f is a surjective natural homomorphism of S.

Let $f : S \to T$ be a semigroup homomorphism. The kernel of f is defined as $\ker f = \{(a, b) \in S \times S : f(a) = f(b)\}$.

An alphabet is an intangible set of symbols (letters). Let W be an alphabet. A word on W is a finite sequence $(w_1, \ldots, w_n)$ where every term w_i is a letter from W. In this case W^* is said to be the set of all words on W. The multiplication of two words $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in W^*$ is defined as $(a_1, \ldots, a_n)(b_1, \ldots, b_n) = (a_1, \ldots, a_n, b_1, \ldots, b_n)$. For a word $(w_1, \ldots, w_n)$, a subword is defined as $(w_i, \ldots, w_j)$ where $1 \leq i \leq j \leq n$.

Throughout the paper, R is a commutative ring with identity. Badawi in [3] generalized the concept of prime ideals in a different way. He defined a proper ideal I of R to be a 2-absorbing ideal of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Anderson and Badawi [2] generalized the concept of 2-absorbing ideals to n-absorbing ideals. According to their definition, a proper ideal I of R is called an n-absorbing (resp. strongly n-absorbing) ideal if whenever $a_1 \cdots a_{n+1} \in I$ for $a_1, \ldots, a_{n+1} \in R$ (resp. $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \ldots, I_{n+1}$ of R), then there are n of the a_i's (resp. n of the I_i's) whose product is in I. Thus a strongly 1-absorbing ideal is just a prime ideal. Clearly, a strongly n-absorbing ideal of R is also an n-absorbing ideal of R. Anderson and Badawi conjectured that these two concepts are equivalent, e.g., they proved that an ideal I of a Prüfer domain R is strongly n-absorbing if and only if I is an n-absorbing ideal of R, [2, Corollary 6.9]. Later, Darani and Puczylowski [5] said that a commutative semigroup S is 2-absorbing if for arbitrary elements $s_1, s_2, s_3 \in S$ satisfying $s_1 s_2 s_3 = 0$ there are $1 \leq i \neq j \leq 3$ such that $s_i s_j = 0$. Generally, they said that S is strongly 2-absorbing if for arbitrary ideals I_1, I_2, I_3 of S satisfying $I_1 I_2 I_3 = 0$ there are $1 \leq i \neq j \leq 3$ such that $I_i I_j = 0$. For more studies concerning 2-absorbing ideals (2-absorbing elements) we refer to [4, 7].

Let R be a ring and $S(R)$ be the multiplicative semigroup with 1 and 0 of R. Among many results in this paper, it is shown (Theorem 2.6) that a proper ideal I of R is a 2-absorbing ideal of $S(R)$ if and only if it is a strongly 2-absorbing ideal of $S(R)$. It is proved (Theorem 2.7) that, for a strongly 2-absorbing ideal I of a multiplicative semigroup S, $\sqrt{I}$ is a strongly 2-absorbing ideal of S and $(\sqrt{I})^2 \subseteq I$. Let S be a multiplicative subsemigroup of a ring R and let B be an ideal of S such that $B \neq \sqrt{B} = P_1 \cap P_2$ where P_1 and P_2 are the only distinct prime ideals of S that are minimal over B. Moreover, assume that P_1, P_2 are two subgroups of the additive group of R. In Theorem 2.16 we show that the following statements are equivalent:

1. B is a strongly 2-absorbing ideal of S;
2. $P_1P_2 \subseteq B$ and $(B : C)$ is a prime ideal of S for every ideal C of S with $C \subseteq \sqrt{B}$ and $C \nsubseteq B$;
3. $(B : C)$ is a prime ideal of S for every ideal C of S with $C \nsubseteq B$ such that either $C \subseteq P_1$ or $C \subseteq P_2$.

2 Properties of 2-absorbing and strongly 2-absorbing ideals

In this section, we introduce the concepts of 2-absorbing ideals and strongly 2-absorbing ideals of commutative semigroups.

Definition 2.1 *A proper ideal I of S is said to be a 2-absorbing ideal of S if for arbitrary elements $s_1, s_2, s_3 \in S$, $s_1s_2s_3 \in I$ implies $s_1s_2 \in I$ or $s_1s_3 \in I$ or $s_2s_3 \in I$.*

Definition 2.2 *A proper ideal I of S is called a strongly 2-absorbing ideal of S if for every ideals I_1, I_2, I_3 of S, $I_1I_2I_3 \subseteq I$ implies $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$.*

It is evident that any strongly 2-absorbing ideal of S is a 2-absorbing ideal of S. But the converse is not correct. We show this with an example:

Example 2.1 Note that a zero element can be adjoined to any semigroup as in [6]. Let W be the free commutative semigroup generated by k, l, m and contains a zero element. Also, assume that I be the ideal of W generated by kl, lm, km. If $w_1, w_2, w_3 \in W$ and $w_1w_2w_3 \in I$, then one of the words kl, lm, km must be a subword of w_iw_j for some $1 \leq i \neq j \leq 3$. Hence $w_iw_j \in I$. Thus I is a 2-absorbing ideal of W. Note that $(k, l)(l, m)(m, k) \subseteq I$ but I does not contain the product of any two of the ideals $(k, l), (l, m), (m, k)$. Consequently, I is not a strongly 2-absorbing ideal of W.

Let R be a ring and S be a multiplicative subsemigroup of R. The following example shows that an ideal I of S that is a subgroup (subsemigroup) of the additive group of R need not be an ideal of R.

Example 2.2 Consider the multiplicative subsemigroup $S = \mathbb{Z}$ of the ring $R = \mathbb{Q}$. Then $I = 2\mathbb{Z}$ is an ideal of S, also I is a subgroup (subsemigroup) of the additive group of R. However, clearly I is not an ideal of R.

Let R be a ring and $S(R)$ be the multiplicative semigroup with 1 and 0 of R. Now, we give a lemma which is important in the proof of Theorem 2.6 showing that a proper ideal I of R is a 2-absorbing ideal of $S(R)$ if and only if it is a strongly 2-absorbing ideal of $S(R)$.

Lemma 2.3 *Let R be a ring. Suppose that I and A are respectively a subsemigroup and a subgroup of the additive group of R, and X, Y are two subsets of R. If for every element $a \in A$, $aX \subseteq I$ or $aY \subseteq I$, then $AX \subseteq I$ or $AY \subseteq I$.*

Proof. Assume that there are a,$b \in A$ such that $aX \nsubseteq I$ and $bY \nsubseteq I$. Hence $aY \subseteq I$ and $bX \subseteq I$, by the assumption. Set $c := a - b$. Clearly $c \in A$. Then $cX \subseteq I$ or $cY \subseteq I$. Let the first case hold. Choose an element $x \in X$. Then $ax = (a - b + b)x = (a - b)x + bx \in I$. So $aX \subseteq I$, a contradiction. Similarly, in the case when $cY \subseteq I$ we reach a contradiction. □

Corollary 2.4 *Let S be a multiplicative subsemigroup of a ring R and I be an ideal of S. Suppose that I and A are respectively a subsemigroup and a subgroup of the additive group of R. If $(I : a)$ is a prime ideal of S for every $a \in A$, then $(I : A)$ is a prime ideal of S.*

Proof. Let $kl \in (I : A)$ for some elements $k, l \in S$. Then $klA \subseteq I$. Thus $kla \in I$ for every $a \in A$. Fix an arbitrary element $a \in A$. Since $kl \in (I : a)$ and $(I : a)$ is a prime ideal of S, then we have that $k \in (I : a)$ or $l \in (I : a)$. Hence $ak \in I$ or $al \in I$.Therefore $aSk \subseteq I$ or $aSl \subseteq I$. By Lemma 2.3, $ASk \subseteq I$ or $ASl \subseteq I$, that is, $Sk \subseteq (I : A)$ or $Sl \subseteq (I : A)$. Thus we obtain that $k \in (I : A)$ or $l \in (I : A)$. Consequently $(I : A)$ is a prime ideal of S. □

Proposition 2.5 *Let I be a proper ideal of S. Then I is a strongly 2-absorbing ideal of S if and only if for every ideals I_1, I_2, I_3 of S that are generated by at most two elements, $I_1I_2I_3 \subseteq I$ implies that there exist $1 \leq i \neq j \leq 3$ such that $I_iI_j \subseteq I$.*

Proof. (⇒) Is clear.
(⇐) Assume that I is not strongly 2-absorbing. Hence, there exist three ideals X, Y, Z of S such that $XYZ \subseteq I$, but $XY \nsubseteq I$, $XZ \nsubseteq I$ and $YZ \nsubseteq I$. Then there are $x_1, x_2 \in X$, $y_1, y_2 \in Y$ and $z_1, z_2 \in Z$ such that $x_1y_1 \notin I$, $x_2z_1 \notin I$ and $y_2z_2 \notin I$. Let $I_1 := (x_1, x_2)$, $I_2 := (y_1, y_2)$ and $I_3 := (z_1, z_2)$. Clearly $I_1I_2I_3 \subseteq I$, but $I_iI_j \nsubseteq I$ for arbitrary $1 \leq i \neq j \leq 3$, which contradicts our assumption. □

Theorem 2.6 *Let $S(R)$ be the multiplicative semigroup with 1 and 0 of a ring R. For a proper ideal I of R the following statements are equivalent:*

1. *I is a 2-absorbing ideal of R;*
2. *I is a 2-absorbing ideal of $S(R)$;*
3. *I is a strongly 2-absorbing ideal of $S(R)$;*
4. *If $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of R, then $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$.*

Proof. (1)⇔(2) It is clear.
(1)⇔(4) By [3, Theorem 2.13].
(3)⇒(2) It is evident.
(2)⇒(3) By Proposition 2.5, we show that if I_1,I_2,I_3 are ideals of $S(R)$ generated by at most two elements and $I_1I_2I_3 \subseteq I$, then $I_iI_j \subseteq I$ for some $1 \leq i \neq j \leq 3$. Let n be the sum of the minimal number of generators of I_i. Clearly $3 \leq n \leq 6$. If each I_i (i = 1,2,3) is a principal ideal of $S(R)$, i.e., $n = 3$, then the result holds, since I is a 2-absorbing ideal of $S(R)$. Suppose that $n \geq 4$ and set $I_1 := (s, t)$, also assume that the result holds if the sum of the numbers of generators is $\leq n - 1$. If $I_2I_3 \subseteq I$, then we are done. Suppose that $\widehat{I_1}$ is the subgroup of the additive group of R generated by I_1. Since for each $k \in \widehat{I_1}$, $S(R)kI_2I_3 \subseteq I$, our assumption on n implies that $kI_2 \subseteq I$ or $kI_3 \subseteq I$. From Lemma 2.3, we obtain that $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq I$. Consequently, I is a strongly 2-absorbing ideal of $S(R)$. □

Theorem 2.7 *Let I be a strongly 2-absorbing ideal of S. Then the following statements hold:*

1. $A^2 \subseteq I$ *for every subset* $A \subseteq \sqrt{I}$.
2. $\sqrt{I}$ *is a strongly 2-absorbing ideal of* S.
3. *If* $A, D \subseteq \sqrt{I}$ *for ideals* A, D *of* S, *then* $AD \subseteq I$.
4. $(\sqrt{I})^2 \subseteq I$.

Proof. (1) Let A be a subset of $\sqrt{I}$ and $x, y \in A$. Then $x, y \in \sqrt{I}$. Since I is a 2-absorbing ideal, then $x^2, y^2 \in I$. Also, we obtain that $(x)\big((x) \cup (y)\big)(y) \subseteq I$. Therefore $(x)^2 \cup (x)(y) \subseteq I$ or $(x)(y) \cup (y)^2 \subseteq I$ or $(x)(y) \subseteq I$, since I is a strongly 2-absorbing ideal. Each of these cases implies $(x)(y) \subseteq I$, so $xy \in I$. Consequently, $A^2 \subseteq I$.

(2) Let K, L, N be ideals of S with $KLN \subseteq \sqrt{I}$. Then, by part (1) we get $(KLN)^2 = K^2L^2N^2 \subseteq I$. Since I is a strongly 2-absorbing ideal, we have that $K^2L^2 \subseteq I$ or $K^2N^2 \subseteq I$ or $L^2N^2 \subseteq I$. Without loss of the generality we may assume that $(KL)^2 = K^2L^2 \subseteq I$. Then $KL \subseteq \sqrt{I}$. Hence $\sqrt{I}$ is a strongly 2-absorbing ideal of S.

(3) Let $A, D \subseteq \sqrt{I}$ for two ideals A, D of S. Then by part (1), $A^2 \subseteq I$ and $D^2 \subseteq I$, so $A(A \cup D)D \subseteq I$. As I is a strongly 2-absorbing ideal, it follows that $AD \subseteq I$.

(4) It is clear from part (1). □

Theorem 2.8 *Let B be a strongly 2-absorbing ideal of S and P and P' be prime ideals of S. If $\sqrt{B} = P \cap P'$, then for all ideals $C \nsubseteq P \cup P'$, $(B : C)$ is a strongly 2-absorbing ideal of S and $\sqrt{(B : C)} = P \cap P'$.*

Proof. Assume that C is an ideal of S with $C \nsubseteq P \cup P'$. We show that $(B : C)$ is strongly 2-absorbing. Let K, L, N be ideals of S with $KLN \subseteq (B : C)$. Hence $K(LN)C \subseteq B$ and so either $KC \subseteq B$ or $(LN)C \subseteq B$ or $K(LN) \subseteq B$. If $KC \subseteq B$ or $(LN)C \subseteq B$, then we are done. Assume that $KLN \subseteq B$. Then $KL \subseteq B$ or $LN \subseteq B$ or $KN \subseteq B$. Hence $KLC \subseteq B$ or $LNC \subseteq B$ or $KNC \subseteq B$. Thus $(B : C)$ is a strongly 2-absorbing ideal. Now, we claim that $\sqrt{(B : C)} \subseteq P \cap P'$. Choose an element $x \in (B : C)$. Hence $xC \subseteq B$. Thus $(x)C \subseteq B \subseteq P$. Since P is a prime ideal, we get that $(x) \subseteq P$ or $C \subseteq P$. By our assumption, we know that $C \nsubseteq P$. Hence $(x) \subseteq P$ and so $x \in P$. In a similar way, we obtain that $x \in P'$. Thus $x \in P \cap P'$. Therefore $(B : C) \subseteq P \cap P'$, and so

$$\sqrt{(B : C)} \subseteq \sqrt{P \cap P'} = \sqrt{P} \cap \sqrt{P'} = P \cap P'.$$

On the other hand $P \cap P' = \sqrt{B} \subseteq \sqrt{(B : C)} \subseteq P \cap P'$, which implies that $\sqrt{(B : C)} = P \cap P'$. □

Remark 2.1 Similar to Theorem 2.8, we can prove that if $\sqrt{B} = P$, then for all ideals $C \nsubseteq P$, $(B : C)$ is a strongly 2-absorbing ideal of S and $\sqrt{(B : C)} = P$.

Theorem 2.9 *Let B be a P-primary ideal of S. Then B is a strongly 2-absorbing ideal of S if and only if $P^2 \subseteq B$. In particular, M^2 is a strongly 2-absorbing ideal of S for every maximal ideal M of S.*

Proof. Let B be a strongly 2-absorbing ideal of S. Since $\sqrt{B}=P$, then, by Theorem 2.7(4), $P^2 \subseteq B$. Conversely, assume that $P^2 \subseteq B$, and $KLN \subseteq B$. If $K \subseteq B$ or $LN \subseteq B$, then we are done. Assume that $K \nsubseteq B$ and $LN \nsubseteq B$. Then there are some elements $k \in K$, $l \in L$ and $n \in N$ such that $k \notin B$ and $ln \notin B$. We claim that $K \subseteq P$. Choose an element $x \in K$. Then $xln \in KLN \subseteq B$. Since B is a P-primary ideal of S and $ln \notin B$, then we have that $x \in P$. Hence $K \subseteq P$. Now, assume that $y \in LN$. Thus $ky \in KLN \subseteq B$. So $y \in P$, because $k \notin B$ and B is P-primary. Then $LN \subseteq P$, and so $L \subseteq P$ or $N \subseteq P$. Thus $K, L \subseteq P$ or $K, N \subseteq P$. Since $P^2 \subseteq B$, then it follows that either $KL \subseteq B$ or $KN \subseteq B$. Therefore B is a strongly 2-absorbing ideal. Let M be a maximal ideal of S. Since M^2 is M-primary, then M^2 is a strongly 2-absorbing ideal of S. □

Theorem 2.10 *Let B be a strongly 2-absorbing ideal of S with $B \neq \sqrt{B} = P$. Then the following statements hold:*

1. *$(B : C)$ is a prime ideal of S and $P \subseteq (B : C)$ for every ideal C of S with $C \subseteq \sqrt{B}$ and $C \nsubseteq B$.*
2. *$(B : C) \subseteq (B : D)$ or $(B : D) \subseteq (B : C)$ for all ideals C, D of S with $C, D \subseteq \sqrt{B}$ and $C, D \nsubseteq B$.*

Proof. (1) Let C be an ideal of S with $C \subseteq P$ and $C \nsubseteq B$. By Theorem 2.7(4), $P^2 \subseteq B$. Thus $P \subseteq (B : C)$. Let d, k be elements of S such that $dk \in (B : C)$. If $d \in P$ or $k \in P$, then either $d \in (B : C)$ or $k \in (B : C)$. Assume that $d \notin P$ and $k \notin P$. Then $dk \notin B$. Since $dk \in (B : C)$, then we obtain that $(d)(k)C \subseteq B$. Thus $(d)C \subseteq B$ or $(k)C \subseteq B$. Hence $d \in (B : C)$ or $k \in (B : C)$. Thus $(B : C)$ is prime.

(2) Let C, D be two ideals of S such that $C, D \subseteq P$ and $C, D \nsubseteq B$, but $(B : C) \nsubseteq (B : D)$. So there exists an element k in S such that $k \in (B : C)$ and $k \notin (B : D)$. By (1), $P \subseteq (B : D)$. Thus $k \notin P$. We show that $(B : D) \subseteq (B : C)$. Let $a \in (B : D)$. If $a \in P$, then $a \in (B : C)$, because $P \subseteq (B : C)$. So, assume that $a \notin P$. Since $(k)(C \cup D)(a) \subseteq B$, $(k)(a) \nsubseteq B$ and $(k)D \nsubseteq B$, we have $(C \cup D)(a) \subseteq B$. Hence $(a)C \subseteq B$. Therefore $(a) \subseteq (B : C)$. Consequently $(B : D) \subseteq (B : C)$. □

Theorem 2.11 *Let B be an ideal of S with $B \neq \sqrt{B}$ and $\sqrt{B}$ be prime ideal of S. Then the following statements are equivalent:*

1. *B is a strongly 2-absorbing ideal of S;*
2. *$(B : C)$ is a prime ideal of S for every ideal C of S with $C \subseteq \sqrt{B}$ and $C \nsubseteq B$.*

Proof. (1)⇒(2) By Theorem 2.10(1).

(2)⇒(1) Let C, A, D be ideals of S and let $CAD \subseteq B \subseteq \sqrt{B}$. Since $\sqrt{B}$ is prime, then we may assume that $C \subseteq \sqrt{B}$. If $C \subseteq B$, then we are done. Now, let $C \nsubseteq B$. Then by part (2), $(B : C)$ is prime and so either $CA \subseteq B$ or $CD \subseteq B$. Thus B is a strongly 2-absorbing ideal of S. □

Lemma 2.12 *Let P be a prime ideal of S and I be an ideal of S with $I \subseteq P$. Then the following statements are equivalent:*

1. *P is a minimal prime ideal of I;*
2. *$S \backslash P$ is a multiplicatively closed set that is maximal with respect to being disjoint from I;*

3. *For every element $p \in P$, there is an element $s \in S \backslash P$ and a nonnegative integer n such that $sp^n \in I$.*

Proof. (1)⇒(2) Let T be a multiplicatively closed set containing $S \backslash P$ and maximal with respect to being disjoint from I. If Q is an ideal containing I that is maximal with respect to being disjoint from T, then Q is prime (see [1, Lemma 3.2]). Note that Q is disjoint from $S \backslash P$ which implies that $Q = P$. Thus $S \backslash P = T$.

(2)⇒(3) Choose a nonzero element p in P. Define

$$T = \{sp^n \ : \ s \in S \backslash P \text{ and } n = 0, 1, 2, \dots\}.$$

Note that T is a multiplicatively closed set that properly contains $S \backslash P$. So $I \cap T \neq \emptyset$. Thus there exists $s \in S \backslash P$ and a nonnegative integer n such that $sp^n \in I$.

(3)⇒(1) Let (3) be true. Assume that (1) does not hold. Then we can find a prime ideal P' such that $I \subseteq P' \subset P$. So there exists an element $p \in P \backslash P'$. Then by (3), $sp^n \in I \subseteq P'$ for some $s \in S \backslash P$ and a nonnegative integer n. Therefore $s \in P'$ or $p \in P'$ which is a contradiction. Hence P is a minimal prime ideal of I. □

Theorem 2.13 *Let S be a multiplicative subsemigroup of a ring R and I be a proper ideal of S that all of its minimal prime ideals are subgroups of the additive group of R. If I is a strongly 2-absorbing ideal of S, then there are at most two prime ideals of S that are minimal over I.*

Proof. Let $\Omega = \{P_i : P_i \text{ is a prime ideal of } S \text{ that is minimal over } I\}$. Assume that Ω has at least three elements. Let $P_1, P_2 \in \Omega$ be two distinct prime ideals. Then there are two elements a, b of S such that $a \in P_1 \backslash P_2$ and $b \in P_2 \backslash P_1$. Let $A = (a)$ and $B = (b)$. First, we show that $AB \subseteq I$. By Lemma 2.12, there are elements $c_1 \in S \backslash P_1$ and $c_2 \in S \backslash P_2$ such that $c_1 A^n \subseteq I$ and $c_2 B^m \subseteq I$ for some positive integers n, m. As $a, b \notin P_1 \cap P_2$ and I is a strongly 2-absorbing ideal of S, we obtain that $C_1 A \subseteq I$ and $C_2 B \subseteq I$ where $C_1 := (c_1)$ and $C_2 := (c_2)$. Hence $(C_1 \cup C_2)AB \subseteq I$. Note that $C_1 \cup C_2 \nsubseteq P_1$ and $C_1 \cup C_2 \nsubseteq P_2$. Therefore $(C_1 \cup C_2)A \nsubseteq P_2$ and $(C_1 \cup C_2)B \nsubseteq P_1$. Then, neither $(C_1 \cup C_2)A \subseteq I$ nor $(C_1 \cup C_2)B \subseteq I$. Hence $AB \subseteq I$. Now assume that there is $P_3 \in \Omega$ distinct from P_1 and P_2. Since P_i's are subgroups of the additive group of R, we can take two elements $y_1 \in P_1 \backslash (P_2 \cup P_3)$ and $y_2 \in P_2 \backslash (P_1 \cup P_3)$. By the former argument $(y_1)(y_2) \subseteq I$. Since $I \subseteq P_3$ and $(y_1)(y_2) \subseteq I$, we get that either $(y_1) \subseteq P_3$ or $(y_2) \subseteq P_3$, a contradiction. Thus Ω has at most two elements. □

Theorem 2.14 *Let S be a multiplicative subsemigroup of a ring R and I be a proper ideal of S that all of its minimal prime ideals are subgroups of the additive group of R. If I is a strongly 2-absorbing ideal of S, then one of the following statements holds:*

1. *$\sqrt{I} = P$ is a prime ideal of S with $P^2 \subseteq I$.*
2. *$\sqrt{I} = P_1 \cap P_2$, $P_1 P_2 \subseteq I$ and $(\sqrt{I})^2 \subseteq I$ where P_1, P_2 are the only distinct prime ideals of S that are minimal over I.*

Proof. By Theorem 2.13, either $\sqrt{I} = P$ where P is a prime ideal of S or $\sqrt{I} = P_1 \cap P_2$ where P_1, P_2 are the only distinct prime ideals of S that are minimal over I. By Theorem 2.7(4), $(\sqrt{I})^2 \subseteq I$. Thus, it remains in the second case we show that $P_1 P_2 \subseteq I$. Let c_1 and c_2 be elements of S such that $c_1 \in P_1 \backslash P_2$ and $c_2 \in P_2 \backslash P_1$. Then $c_1 c_2 \in I$, by the proof of Theorem 2.13. Let k_1 and k_2 be elements of S such that $k_1 \in \sqrt{I}$ and $k_2 \in P_2 \backslash P_1$. Choose an element $l_1 \in P_1 \backslash P_2$. Then $l_1 k_2 \in I$, by the proof of Theorem 2.13. Moreover, $k_1 + l_1 \in P_1$. If $k_1 + l_1 \in P_2$, then $l_1 \in P_2$ because P_2 is a subgroup of the additive group of R. This contradiction shows that $k_1 + l_1 \in P_1 \backslash P_2$. Thus $k_1 k_2 + l_1 k_2 = (k_1 + l_1) k_2 \in I$, again by the proof of Theorem 2.13. Hence $k_1 k_2 \in I$. Let $t_1 \in \sqrt{I}$ and $t_2 \in P_1 \backslash P_2$. In a similar way, we have $t_1 t_2 \in I$. Hence $P_1 P_2 \subseteq I$. $\square$

Theorem 2.15 *Let S be a multiplicative subsemigroup of a ring R. Let B be a strongly 2-absorbing ideal of S such that $B \neq \sqrt{B} = P_1 \cap P_2$ where P_1 and P_2 are the only distinct prime ideals of S that are minimal over B. Moreover, assume that P_1, P_2 are two subgroups of the additive group of R. Then the following statements hold:*

1. *$(B : C)$ is a prime ideal of S such that $P_1 \subseteq (B : C)$ and $P_2 \subseteq (B : C)$ for every ideal C of S with $C \subseteq \sqrt{B}$ and $C \nsubseteq B$.*
2. *$(B : D) \subseteq (B : C)$ or $(B : C) \subseteq (B : D)$ for every ideals C, D of S with $C, D \subseteq \sqrt{B}$ and $C, D \nsubseteq B$.*

Proof. (1) Suppose that C is an ideal of S with $C \subseteq \sqrt{B}$ and $C \nsubseteq B$. Notice that Theorem 2.14 implies $P_1 P_2 \subseteq B$. Then it is obvious that $CP_1 \subseteq B$ and $CP_2 \subseteq B$. Thus $P_1 \subseteq (B : C)$ and $P_2 \subseteq (B : C)$. Suppose that $a, d \in S$ and $ad \in (B : C)$. Then we have $(a)(d)C \subseteq B$. Since B is strongly 2-absorbing, we have $(a)C \subseteq B$ or $(d)C \subseteq B$ or $(a)(d) \subseteq B$. If $(a)C \subseteq B$ or $(d)C \subseteq B$, then $a \in (B : C)$ or $d \in (B : C)$, respectively. If $(a)(d) \subseteq B$, then $a \in (B : C)$ or $d \in (B : C)$, because $B \subseteq P_1 \subseteq (B : C)$ and $B \subseteq P_2 \subseteq (B : C)$. Therefore, $(B : C)$ is prime.

(2) Assume that $(B : D) \nsubseteq (B : C)$. So there exists $k \in (B : D) \backslash (B : C)$. By part (1) we have $P_1 \cup P_2 \subseteq (B : C)$ and $P_1 \cup P_2 \subseteq (B : D)$. We show that $(B : C) \subset (B : D)$. Choose an element $a \in (B : C)$. Since $(k)(C \cup D)(a) \subseteq B$, then $(k)(C \cup D) \subseteq B$ or $(C \cup D)(a) \subseteq B$ or $(k)(a) \subseteq B$. In the first case we have that $k \in (B : C)$ which is a contradiction. In the second case we have that $a \in (B : C)$ as desired. In the case when $(k)(a) \subseteq B$, we get that $k \in P_1 \cup P_2 \subseteq (B : C)$, a contradiction; or $a \in P_1 \cup P_2 \subseteq (B : D)$. Therefore $a \in (B : D)$. Consequently $(B : C) \subset (B : D)$. $\square$

Theorem 2.16 *Let S be a multiplicative subsemigroup of a ring R. Let B be an ideal of S such that $B \neq \sqrt{B} = P_1 \cap P_2$ where P_1 and P_2 are the only distinct prime ideals of S that are minimal over B. Moreover, assume that P_1, P_2 are two subgroups of the additive group of R. Then the following statements are equivalent:*

1. *B is a strongly 2-absorbing ideal of S;*
2. *$P_1 P_2 \subseteq B$ and $(B : C)$ is a prime ideal of S for every ideal C of S with $C \subseteq \sqrt{B}$ and $C \nsubseteq B$;*
3. *$(B : C)$ is a prime ideal of S for every ideal C of S with $C \nsubseteq B$ such that either $C \subseteq P_1$ or $C \subseteq P_2$.*

Proof. (1)⇒(2) By Theorem 2.14 and Theorem 2.15.

(2)⇒(3) Suppose that C is an ideal of S with $C \nsubseteq B$ such that either $C \subseteq P_1$ or $C \subseteq P_2$. Let $C \subseteq P_1$ and $C \nsubseteq P_2$. Since $P_1 P_2 \subseteq B$, we have $CP_2 \subseteq B$. Hence $(B : C) = P_2$ which is a prime ideal of S. Similarly, if $C \subseteq P_2$ and $C \nsubseteq P_1$, then $(B : C) = P_1$ which is a prime ideal of S. If $C \subseteq P_1$ and $C \subseteq P_2$, then $C \subseteq \sqrt{B}$ and so by condition (2), $(B : C)$ is a prime ideal of S.

(3)⇒(1) Suppose that (3) holds. Let $CAD \subseteq B$ for some ideals C, A, D of S. Since $CAD \subseteq P_1$, then we may assume that $C \subseteq P_1$. If $C \subseteq B$, then we are done. So, let $C \nsubseteq B$. Hence $AD \subseteq (B : C)$ and by part (3) either $A \subseteq (B : C)$ or $D \subseteq (B : C)$. Consequently either $AC \subseteq B$ or $DC \subseteq B$. Thus B is a strongly 2-absorbing ideal of S. □

Theorem 2.17 *Let S be a multiplicative subsemigroup of a ring R. Let B be a strongly 2-absorbing ideal of S such that $B \neq \sqrt{B}$ and each of the minimal prime ideals of B is a subgroup of the additive group of R. Then the following statements hold:*

1. *If A and C are ideals of S with $C \subseteq \sqrt{B}$, $C \nsubseteq B$ and $CA \nsubseteq B$, then $(B : CA) = (B : C)$.*
2. *If A and C are ideals of S with $C \subseteq \sqrt{B}$, $C \nsubseteq B$ and $(B : C) \subset (B : A)$, then $(B : I_1C \cup I_2A) = (B : C)$ for all ideals I_1, I_2 of S with $I_1I_2 \nsubseteq (B : C)$. In particular, $(B : C \cup A) = (B : C)$.*

Proof. (1) Let A, C be ideals of S with $C \subseteq \sqrt{B}$, $C \nsubseteq B$ and $CA \nsubseteq B$. Clearly $(B : C) \subseteq (B : AC)$. Let $d \in (B : AC)$. Then $(d)AC \subseteq B$. By Theorem 2.10 and Theorem 2.15, $(B : C)$ is a prime ideal of S. Therefore $(d) \subseteq (B : C)$ since $CA \nsubseteq B$. Hence $d \in (B : C)$. Thus $(B : CA) = (B : C)$.

(2) Suppose that A and C are ideals of S with $C \subseteq \sqrt{B}$, $C \nsubseteq B$ and $(B : C) \subset (B : A)$. Let I_1, I_2 be ideals of S with $I_1I_2 \nsubseteq (B : C)$. Clearly $(B : C) \subseteq (B : I_1C \cup I_2A)$. Assume that $(B : C) \neq (B : I_1C \cup I_2A)$. Then there is an element $k \in (B : I_1C \cup I_2A)$ such that $k \notin (B : C)$. So $(k)I_1C \subseteq B$ which means $(k)I_1 \subseteq (B : C)$. Since $(B : C)$ is prime, we have that either $(k) \subseteq (B : C)$ or $I_1 \subseteq (B : C)$, which each of these cases is a contradiction. Consequently $(B : I_1C \cup I_2A) = (B : C)$. In special case, set $I_1 = I_2 = S$ which implies that $(B : C \cup A) = (B : C)$. □

Theorem 2.18 *Let S and T be commutative multiplicative semigroups with 1 and 0. Let $f : S \to T$ be a homomorphism of semigroups.*

1. *If J is a 2-absorbing ideal of T, then $f^{-1}(J)$ is a 2-absorbing ideal of S.*
2. *Let I be a proper ideal of S such that $\{(x, y) \in \ker f : x \neq y\} \subseteq I \times I$. Then*
   (a) *If $f(I)$ is a 2-absorbing ideal of T, then I is a 2-absorbing ideal of S.*
   (b) *If f is surjective and I is a 2-absorbing ideal of S, then $f(I)$ is a 2-absorbing ideal of T.*

Proof. (1) Let $abc \in f^{-1}(J)$ for some $a, b, c \in S$. Then $f(abc) = f(a)f(b)f(c) \in J$. As J is a 2-absorbing ideal of T, then $f(a)f(b) \in J$ or $f(a)f(c) \in J$ or $f(b)f(c) \in J$. Thus $ab \in f^{-1}(J)$ or $ac \in f^{-1}(J)$ or $bc \in f^{-1}(J)$. Consequently, $f^{-1}(J)$ is a 2-absorbing ideal of S.

(2) (a) First assume that $f(I)$ is a 2-absorbing ideal of T. We show that $f^{-1}(f(I)) = I$. Let $x \in f^{-1}(f(I))$. Then $f(x) \in f(I)$ and so there is an element $l \in I$ such that $f(x) = f(l)$. Since $\{(x, y) \in \ker f : x \neq y\} \subseteq I \times I$, it implies that $x \in I$. Thus $f^{-1}(f(I)) = I$. So by part (1), we deduce that I is a 2-absorbing ideal of S.

(b) Let $f : S \to T$ be surjective and I be a 2-absorbing ideal of S. Let $a'b'c' \in f(I)$ for some $a', b', c' \in T$. Since f is surjective, there are elements $a, b, c \in S$ such that $f(a) = a'$, $f(b) = b'$ and $f(c) = c'$. Then $f(abc) = a'b'c' \in f(I)$ and so there is an element $x \in I$ such that $f(abc) = f(x)$. Again $\{(x, y) \in \ker f : x \neq y\} \subseteq I \times I$ implies that $abc \in I$. Since I is a 2-absorbing ideal of S, then $ab \in I$ or $ac \in I$ or $bc \in I$. Therefore $f(a)f(b) \in f(I)$ or $f(a)f(c) \in f(I)$ or $f(b)f(c) \in f(I)$. Consequently $f(I)$ is a 2-absorbing ideal of T. □

Theorem 2.19 *Let S and T be commutative multiplicative semigroups with 1 and 0.Let $f : S \to T$ be a homomorphism of semigroups.*

1. *If J is a strongly 2-absorbing ideal of T, then $f^{-1}(J)$ is a strongly 2-absorbing ideal of S.*
2. *Let I be a proper ideal of S such that $\{(x, y) \in \ker f : x \neq y\} \subseteq I \times I$. Then*
   (a) If $f(I)$ is a strongly 2-absorbing ideal of T, then I is a strongly 2-absorbing ideal of S.
   (b) If f is surjective and I is a strongly 2-absorbing ideal of S, then $f(I)$ is a strongly 2-absorbing ideal of T.

Proof. (1) Assume that J is a strongly 2-absorbing ideal of T.

Let I_1, I_2, I_3 be ideals of S with $I_1I_2I_3 \subseteq f^{-1}(J)$. Then $f(I_1I_2I_3) = f(I_1)f(I_2)f(I_3) \subseteq f(f^{-1}(J)) \subseteq J$. Hence $f(I_1I_2) \subseteq J$ or $f(I_1I_3) \subseteq J$ or $f(I_2I_3) \subseteq J$. Then we have that $I_1I_2 = f^{-1}(f(I_1I_2)) \subseteq f^{-1}(J)$ or $I_1I_3 = f^{-1}(f(I_1I_3)) \subseteq f^{-1}(J)$ or $I_2I_3 = f^{-1}(f(I_2I_3)) \subseteq f^{-1}(J)$. Thus $f^{-1}(J)$ is a strongly 2-absorbing ideal of S.

(2)(a) By part (1).

(b) Let I be a strongly 2-absorbing ideal of S. Suppose that $J_1J_2J_3 \subseteq f(I)$ for some ideals J_1, J_2, J_3 of T. Then

$$f^{-1}(J_1)f^{-1}(J_2)f^{-1}(J_3) \subseteq f^{-1}(J_1J_2J_3) \subseteq f^{-1}(f(I)) = I,$$

and so $f^{-1}(J_1)f^{-1}(J_2) \subseteq I$ or $f^{-1}(J_1)f^{-1}(J_3) \subseteq I$ or $f^{-1}(J_2)f^{-1}(J_3) \subseteq I$. We may assume that $f^{-1}(J_1)f^{-1}(J_2) \subseteq I$. By assumption f is surjective. Thus $J_1 = f(f^{-1}(J_1))$ and $J_2 = f(f^{-1}(J_2))$. Hence

$$J_1J_2 = f(f^{-1}(J_1))f(f^{-1}(J_2)) = f(f^{-1}(J_1)f^{-1}(J_2)) \subseteq f(I).$$

Consequently, $f(I)$ is a strongly 2-absorbing ideal of T. □

As a direct consequence of Theorem 2.18 and Theorem 2.19 the following result follows.

Corollary 2.20 *Let S and T be commutative multiplicative semigroups with 1 and 0.*

1. *Let $S \subseteq T$ be an extension of semigroups and J be a strongly 2-absorbing (resp. 2-absorbing) ideal of T. Then $J \cap S$ is a strongly 2-absorbing (resp. 2-absorbing) ideal of S.*
2. *Let $I \subseteq J$ be two ideals of S. Then J is a strongly 2-absorbing (resp. 2-absorbing) ideal of S if and only if J/I is a strongly 2-absorbing (resp. 2-absorbing) ideal of S/I.*

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