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Geometry of warped product pseudo-slant submanifolds in nearly Kaehler manifolds

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Abstract Non-existence of warped product pseudo-slant submanifolds of nearly Kaehler manifolds was proved under some conditions in [16]. In this paper, we continue the study of such warped products for their existence. We characterise pseudo-slant submanifolds of a nearly Keahler manifold to be locally warped products. Also, we obtain a geometric inequality for the squared norm of the second fundamental form of a mixed geodesic warped product submanifold in terms of the warping function and the slant angle. The equality case is also considered.

Keywords Warped product · Slant submanifolds · Pseudo-slant submanifolds · Warped product pseudo-slant submanifolds · Nearly Kaehler manifolds

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1 Introduction

Nearly Kaehler manifolds are exactly the Tachibana manifolds initially studied in [15]. Nearly Kaehler manifolds form an interesting class of manifolds admitting a metric connection with parallel totally anti-symmetric torsion (see [1]). The best known example of a nearly Kaehler non-Kaehler manifold is S^6, a six dimensional sphere.

On the other hand, slant submanifolds of almost Hermitian manifolds were introduced by Chen in [5-6] as a generalization of both holomorphic and totally real submanifolds. In [12], Papaghiuc has introduced another class of submanifolds in almost

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Hermitian manifolds, called the semi-slant submanifolds, which are natural generalizations of CR and slant submanifolds. On the similar line of thought, Sahin [14] has introduced the idea of pseudo-slant submanifolds of Kaehler manifolds to study their warped products. Initially, these submanifolds were introduced by Carriazo [4] in almost contact settings.

Recently, warped product pseudo-slant submanifolds of nearly Kaehler manifolds were studied by the first author in [16]. In this paper, we continue this study for the existence of warped product pseudo-slant submanifolds. The paper is organized as follows: in Section 2, we recall some basic formulas and definitions. In Section 3, we study pseudo-slant submanifolds and their warped products of the forms $N_\perp \times_f N_\theta$ and $N_\theta \times_f N_\perp$ of a nearly Kaehler manifold $\widetilde{M}$, where $N_\perp$ and N_θ are totally real and proper slant submanifolds of $\widetilde{M}$, respectively. In the beginning of this section we obtain some basic results for later use and then obtain a characterization. In Section 4, we derive an inequality for the squared norm of the second fundamental form. The equality case is also discussed.

2 Preliminaries

Let $(\widetilde{M}, g)$ be an almost Hermitian manifold with almost complex structure J and a Riemannian metric g such that

$$(a)\quad J^2 = -I, \qquad\qquad (b)\quad g(JX, JY) = g(X, Y), \tag{2.1}$$

for all vector fields X, Y on $\widetilde{M}$.

Further, let $\Gamma(T\widetilde{M})$ denote the set of all vector fields on $\widetilde{M}$ and $\widetilde{\nabla}$, the covariant differential operator on $\widetilde{M}$ with respect to g. Then according to Gray [9], if the almost complex structure J satisfies

$$(\widetilde{\nabla}_X J)X = 0, \tag{2.2}$$

for any $X \in \Gamma(T\widetilde{M})$, then the manifold $\widetilde{M}$ is called a *nearly Kaehler manifold*. Equation (2.2) is equivalent to $(\widetilde{\nabla}_X J)Y + (\widetilde{\nabla}_Y J)X = 0$, for any $X, Y \in \Gamma(T\widetilde{M})$.

Let $\widetilde{M}$ be an almost Hermitian manifold with almost complex structure J, and M a Riemannian manifold isometrically immersed in $\widetilde{M}$. Then M is called *holomorphic* (or complex) if $J(T_xM) \subseteq T_xM$, for any $x \in M$ where T_xM denotes the tangent space of M at x, and *totally real* if $J(T_xM) \subseteq T_x^\perp M$ for any $x \in M$, where $T_x^\perp M$ denotes the normal space of M at x. There are some other important classes of submanifolds of an almost Hermitian manifold determined by the behaviour of tangent bundle of the submanifold under the action of almost complex structure J of the ambient manifold. Some of them are following:

(i) A submanifold M of an almost Hermitian manifold $\widetilde{M}$ is said to be a *CR-submanifold* (see [13]) of $\widetilde{M}$ if there exist a differentiable distribution $\mathcal{D} : x \to \mathcal{D}_x \subset T_xM$ such that $\mathcal{D}$ is holomorphic distribution and the orthogonal complementary distribution $\mathcal{D}^\perp$ is totally real.

(ii) A submanifold M of an almost Hermitian manifold $\widetilde{M}$ is said to be *slant* (see [5]) if for each non-zero vector X tangent to M the angle $\theta(X)$ between JX and T_xM is a constant, i.e., it does not depend on the choice of $x \in M$ and $X \in T_xM$.

(iii) The submanifold M is called *semi-slant* (see [12]) if there exist a pair of orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^\theta$ such that $\mathcal{D}$ is holomorphic and $\mathcal{D}^\theta$ is proper slant, i.e., the angle $\theta(X)$ between JX and $\mathcal{D}^\theta_x$ is constant and it is neither 0 nor $\pi/2$ for any $X \in \mathcal{D}^\theta_x$.

For a submanifold M of a Riemannian manifold $\widetilde{M}$, the Gauss, Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \qquad \widetilde{\nabla}_X N = -A_N X + \nabla^\perp_X N, \tag{2.3}$$

for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where $\Gamma(TM)$ is the Lie algebra of vector fields in M and $\Gamma(T^\perp M)$ is the set of all vector fields normal to M and ∇ is the induced Riemannian connection on M, h is the second fundamental form of M, $\nabla^\perp$ is the normal connection in the normal bundle $T^\perp M$ and A_N is the shape operator of the second fundamental form. They are related as $g(A_N X, Y) = g(h(X,Y), N)$, where g denotes the Riemannian metric on $\widetilde{M}$ as well as the metric induced on M. The mean curvature vector H of M is given by $H = \frac{1}{m}\sum_{i=1}^m h(e_i, e_i)$, where m is the dimension of M and $\{e_1, e_2, \ldots, e_m\}$ is a local orthonormal frame of vector fields on M. A submanifold M of an almost Hermitian manifold $\widetilde{M}$ is said to be *totally umbilical* if the second fundamental form satisfies $h(X,Y) = g(X,Y)H$, for all $X, Y \in \Gamma(TM)$. The submanifold M is *totally geodesic* if $h(X,Y) = 0$, for all $X, Y \in \Gamma(TM)$ and minimal if $H = 0$.

Now, let $\{e_1, \ldots, e_m\}$ be an orthonormal basis of tangent space TM and e_r belongs to the orthonormal basis $\{e_{m+1}, \ldots, e_2 n\}$ of the normal bundle $T^\perp M$, we put

$$h^r_{ij} = g(h(e_i, e_j), e_r) \quad \text{and} \quad \|h\|^2 = \sum_{i,j=1}^m g(h(e_i, e_j), h(e_i, e_j)). \tag{2.4}$$

For a differentiable function φ on M, the gradient $\boldsymbol{\nabla}\varphi$ is defined by

$$g(\boldsymbol{\nabla}\varphi, X) = X\varphi, \tag{2.5}$$

for any $X \in \Gamma(TM)$. As a consequence, we have

$$\|\boldsymbol{\nabla}\varphi\|^2 = \sum_{i=1}^m (e_i(\varphi))^2. \tag{2.6}$$

For any $X \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, the transformations JX and JN are decomposed into tangential, normal part as

$$(a)\;\; JX = TX + FX, \qquad (b)\;\; JN = BN + CN. \tag{2.7}$$

On a slant submanifold M of an almost Hermitian manifold $\widetilde{M}$, we have

$$T^2 X = -\cos^2\theta X, \tag{2.8}$$

where θ is the slant angle of M in $\widetilde{M}$ (see [6]). As a consequence of the relation (2.8), we have

$$g(TX, TY) = \cos^2\theta g(X, Y), \tag{2.9}$$

$$g(FX, FY) = \sin^2\theta g(X, Y), \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$. Also, for a slant sumanifold from (2.7) (a) and (2.8), we have

$$BFX = -\sin^2\theta X \quad \text{and} \quad CFX = -FTX, \tag{2.11}$$

for any $X \in \Gamma(TM)$.

Now, denote by $\mathcal{P}_X Y$ and $\mathcal{Q}_X Y$ the tangential and normal parts of $(\widetilde{\nabla}_X J)Y$, i.e.,

$$(\widetilde{\nabla}_X J)Y = \mathcal{P}_X Y + \mathcal{Q}_X Y, \tag{2.12}$$

for all $X, Y \in \Gamma(TM)$. For the properties of $\mathcal{P}$ and $\mathcal{Q}$ we refer [11], which we enlist here for later use.
(p_1) (i) $\mathcal{P}_{X+Y} W = \mathcal{P}_X W + \mathcal{P}_Y W,$ (ii) $\mathcal{Q}_{X+Y} W = \mathcal{Q}_X W + \mathcal{Q}_Y W,$
(p_2) (i) $\mathcal{P}_X(Y + W) = \mathcal{P}_X Y + \mathcal{P}_X W,$ (ii) $\mathcal{Q}_X(Y + W) = \mathcal{Q}_X Y + \mathcal{Q}_X W,$
(p_3) (i) $g(\mathcal{P}_X Y,\ W) = -g(Y, \mathcal{P}_X W),$ (ii) $g(\mathcal{Q}_X Y,\ N) = -g(Y, \mathcal{P}_X N),$
(p_4) $\mathcal{P}_X JY + \mathcal{Q}_X JY = -J(\mathcal{P}_X Y + \mathcal{Q}_X Y),$
for all $X, Y, W \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$.

On a Riemannian submanifold M of a nearly Kaehler manifold $\widetilde{M}$, by equations (2.2) and (2.12), we have

$$(a)\ \ \mathcal{P}_X Y + \mathcal{P}_Y X = 0, \quad (b)\ \ \mathcal{Q}_X Y + \mathcal{Q}_Y X = 0, \tag{2.13}$$

for any $X,\ Y \in \Gamma(TM)$.

3 Pseudo-slant submanifolds and their warped products

In this section, we study pseudo-slant submanifolds and their warped products in a nearly Kaehler manifold $\widetilde{M}$. Pseudo-slant submanifolds were introduced by Carriazo under the name anti-slant (see [4]). Later on, these submanifolds were studied by Sahin under the name hemi-slant submanifolds of Kaehler manifolds for their warped products (see [14]). We define these submanifolds as follows:

Definition 3.1 *A submanifold M of an almost Hermitian manifold $\widetilde{M}$ is said to be a pseudo-slant submanifold if there exist a pair of orthogonal distributions $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ such that:*

(i) TM admits the orthogonal direct decomposition $TM = \mathcal{D}^\perp \oplus \mathcal{D}^\theta$;
(ii) the distribution $\mathcal{D}^\perp$ is totally real, i.e., $J\mathcal{D}^\perp \subset T^\perp M$;
(iii) the distribution $\mathcal{D}^\theta$ is slant with slant angle $\theta \neq 0$ or $\frac{\pi}{2}$.

From the definition, it is clear that if $\theta = 0$, then M is a CR-submanifold. Also, if we denote the dimensions of $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ by q and p, respectively. Then, we have:

(i) If $p = 0$, then M is totally real.

(ii) If $q = 0$ and $\theta = 0$, then M is holomorphic.
(iii) If $q = 0$ and $\theta \neq 0, \pi/2$, then M is a proper slant submanifold with slant angle θ.
(iv) If $\theta = \pi/2$, then M is again a totally real submanifold.

We say that a pseudo-slant submanifold is proper if neither $q = 0$ nor $\theta = 0, \pi/2$. The normal space of a pseudo-slant submanifold M is decomposed as $T^\perp M = J\mathcal{D}^\perp \oplus F\mathcal{D}^\theta \oplus \mu$, where μ is the invariant normal subbundle of M with respect to J. On a pseudo-slant submanifold of a nearly Kaehler manifold we have the following results for later use.

Lemma 3.2 *Let M be a pseudo-slant submanifold of a nearly Kaehler manifold $\widetilde{M}$. Then $g(\nabla_X Y, Z) = \sec^2\theta\{g(A_{JZ}X, TY) - g(A_{FTY}X, Z) - g(\mathcal{Q}_X Z, FY) - g(\mathcal{Q}_X Y, JZ)\}$, for any $X, Y \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$.*

Proof. For any $X, Y \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$, we have $g(\nabla_X Y, Z) = g(\widetilde{\nabla}_X Y, Z)$. Using (2.1) (b), we get

$$\begin{aligned} g(\nabla_X Y, Z) &= g(J\widetilde{\nabla}_X Y, JZ) \\ &= g(\widetilde{\nabla}_X JY, JZ) - g((\widetilde{\nabla}_X J)Y, JZ). \end{aligned}$$

Then from (2.7) (a) and (2.12), we obtain

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\widetilde{\nabla}_X TY, JZ) + g(\widetilde{\nabla}_X FY, JZ) - g(\mathcal{Q}_X Y, JZ) \\ &= g(h(X, TY), JZ) - g(\widetilde{\nabla}_X JFY, Z) + g((\widetilde{\nabla}_X J)FY, Z) \\ &\quad - g(\mathcal{Q}_X Y, JZ) \\ &= g(h(X, TY), JZ) - g(\widetilde{\nabla}_X BFY, Z) - g(\widetilde{\nabla}_X CFY, Z) \\ &\quad + g(\mathcal{P}_X FY, Z) - g(\mathcal{Q}_X Y, JZ). \end{aligned}$$

Using (2.11) and the properties of $\mathcal{P}$-$\mathcal{Q}$, (p_3) (ii), we arrive at $g(\nabla_X Y, Z) = g(h(X, TY), JZ) + \sin^2\theta g(\nabla_X Y, Z) + g(\widetilde{\nabla}_X FTY, Z) - g(\mathcal{Q}_X Z, FY) - g(\mathcal{Q}_X Y, JZ)$. Thus the result follows from the above relation by using (2.3). □

The following corollary is a consequence of the above lemma.

Corollary 3.3 *On a pseudo-slant submanifold M of a nearly Kaehler manifold $\widetilde{M}$, the slant distribution $\mathcal{D}^\theta$ defines a totally geodesic foliation if and only if $A_{JZ}TX - A_{FTX}Z \in \Gamma(D^\perp)$ and $\mathcal{Q}_X U \in \Gamma(\mu)$, for any $X \in \Gamma(\mathcal{D}^\theta)$, $Z \in \Gamma(\mathcal{D}^\perp)$ and $U \in \Gamma(TM)$.*

Now, we discuss the warped product submanifolds of a nearly Kaehler manifold. The warped product manifolds were studied by Bishop and O'Neill [3]. They defined these manifolds as follows: Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds and f, a positive differentiable function on N_1. Then their warped product $M = N_1 \times_f N_2$ is the product manifold $N_1 \times N_2$ equipped with the Riemannian structure such that $g = g_1 + f^2 g_2$. The function f is called the warping function on M. It was proved in [3] that for any $X \in \Gamma(TN_1)$ and $Z \in \Gamma(TN_2)$, the following relation holds

$$\nabla_X Z = \nabla_Z X = (X \ln f)Z, \tag{3.1}$$

where ∇ denotes the Levi-Civita connection on M. A warped product manifold $M = N_1 \times_f N_2$ is said to be *trivial* (or *Riemannian product*) if the warping function f is constant. If $M = N_1 \times {}_f N_2$ be a warped product manifold then N_1 and N_2 are totally geodesic and totally umbilical submanifolds of M, respectively [3,7].

In this paper we discuss the warped product pseudo-slant submanifolds of a nearly Kaehler manifold $\widetilde{M}$ those are either in the form $N_\perp \times {}_f N_\theta$ or $N_\theta \times {}_f N_\perp$, where $N_\perp$ and N_θ are totally real and proper slant submanifolds of $\widetilde{M}$, respectively. These two types of warped products are the products between the totally real and proper slant submanifolds of $\widetilde{M}$, we call such types of warped products as warped product pseudo-slant submanifolds in the same sense of warped product CR-submanifolds (see [7-8]).

Proposition 3.4 *Let $M = N_\perp \times {}_f N_\theta$ be a warped product submanifold of a nearly Kaehler manifold $\widetilde{M}$, where $N_\perp$ and N_θ are totally real and proper slant submanifolds of $\widetilde{M}$, respectively. Then:*

(i) $2g(h(X,Y), JZ) = g(h(X,Z), FY) + g(h(Y,Z), FX)$,
(ii) $2g(h(Z,W), FX) = g(h(X,Z), JW) + g(h(X,W), JZ)$,

for any $X, Y \in \Gamma(TN_\theta)$ and $Z, W \in \Gamma(TN_\perp)$.

Proof. For any $X, Y \in \Gamma(TN_\theta)$ and $Z \in \Gamma(TN_\perp)$, we have

$$g(h(X,Z), FY) = g(\widetilde{\nabla}_X Z, JY) - g(\widetilde{\nabla}_X Z, TY).$$

Using the property of Riemannian metric g, (2.3) and (3.1), we get

$$g(h(X,Z), FY) = g((\widetilde{\nabla}_X J)Z, Y) - g(\widetilde{\nabla}_X JZ, Y) - (Z \ln f) g(X, TY).$$

Then from (2.12), we obtain $g(h(X,Z),\ FY) = g(\mathcal{P}_X Z,\ Y) + g(A_{JZ}X, Y)) - (Z \ln f) g(X, TY)$. Then by property (p_3) of $\mathcal{P}$, we arrive at

$$g(h(X,Z), FY) = g(h(X,Y), JZ) - g(\mathcal{P}_X Y, Z) - (Z \ln f) g(X, TY). \tag{3.2}$$

Interchanging X and Y in (3.2), we obtain

$$g(h(Y,Z), FX) = g(h(X,Y), JZ) - g(\mathcal{P}_Y X, Z) + (Z \ln f) g(X, TY). \tag{3.3}$$

Thus, the first part of the lemma follows from (3.2) and (3.3). Now, for any $Z, W \in \Gamma(TN_\perp)$ and $X \in \Gamma(TN_\theta)$ we have $g(h(Z,W), FX) = g(\widetilde{\nabla}_Z W, JX) - g(\widetilde{\nabla}_Z W, TX)$. Using the property of Riemannian metric g, (2.3) and (3.1), we get $g(h(Z,W), FX) = g((\widetilde{\nabla}_Z J)W, X) - g(\widetilde{\nabla}_Z JW, X) + (Z \ln f) g(TX, W)$. Then from (2.12), we obtain

$$\begin{aligned} g(h(Z,W), FX) &= g(\mathcal{P}_Z W, X) + g(A_{JW} Z, X) \\ &= g(\mathcal{P}_Z W, X) + g(h(Z,X), JW). \end{aligned} \tag{3.4}$$

Interchanging Z and W in (3.4), we obtain

$$g(h(Z,W), FX) = g(\mathcal{P}_W Z, X) + g(h(W,X), JZ). \tag{3.5}$$

Then from (3.4), (3.5) and (2.13) (a), we derive (ii). This completes the proof. $\square$

Now, we discuss the warped product pseudo-slant submanifolds of the form $N_\theta \times_f N_\perp$ of a nearly Kaehler manifold. First, we give the following two results for later use.

Lemma 3.5 ([2]) *Let $M = N_\theta \times {}_f N_\perp$ be a warped product submanifold of a nearly Kaehler manifold $\widetilde{M}$, where $N_\perp$ and N_θ are totally real and proper slant submanifolds of $\widetilde{M}$, respectively. Then $2g(h(X,Y),JZ) = g(h(X,Z),FY) + g(h(Y,Z),FX)$, for any $X, Y \in \Gamma(TN_\theta)$ and $Z \in \Gamma(TN_\perp)$.*

Lemma 3.6 *Let $M = N_\theta \times {}_f N_\perp$ be a warped product submanifold of a nearly Kaehler manifold $\widetilde{M}$, where $N_\perp$ and N_θ are totally real and proper slant submanifolds of $\widetilde{M}$, respectively. Then*

(i) $2g(h(Z,W),FX) = g(h(X,Z),JW) + g(h(X,W),JZ) + 2(TX \ln f)g(Z,W)$,

(ii) $2g(h(Z,W),FTX) = g(h(TX,Z),JW) + g(h(TX,W),JZ) - 2\cos^2\theta(X \ln f)g(Z,W)$,

for any $X \in \Gamma(TN_\theta)$ and $Z, W \in \Gamma(TN_\perp)$.

Proof. For any $Z, W \in \Gamma(TN_\perp)$ and $X \in \Gamma(TN_\theta)$, we have

$$g(h(Z,W),FX) = g(\widetilde{\nabla}_Z W, JX) - g(\widetilde{\nabla}_Z W, TX).$$

Then by the property of Riemannian metric g and covariant derivative of J, we obtain

$$g(h(Z,W),FX) = g((\widetilde{\nabla}_Z J)W, X) - g(\widetilde{\nabla}_Z JW, X) + g(\widetilde{\nabla}_Z TX, W).$$

Using (2.3), (2.12) and (3.1), we get

$$\begin{aligned} g(h(Z,W),FX) &= g(\mathcal{P}_Z W, X) + g(A_{JW}Z, X) + (TX \ln f)g(Z,W) \\ &= g(\mathcal{P}_Z W, X) + g(h(Z,X),JW) + (TX \ln f)g(Z,W). \end{aligned} \tag{3.6}$$

Interchanging Z and W in (3.6), we derive

$$\begin{aligned} g(h(Z,W),FX) = g(\mathcal{P}_W Z, X) &+ g(h(W,X),JZ) \\ &+ (TX \ln f)g(Z,W). \end{aligned} \tag{3.7}$$

First part follows from (3.6) and (3.7). If we interchange X by TX in (i) we get (ii), which proves the lemma completely. □

Now, we prove the following characterization theorem for pseudo-slant submanifolds.

Theorem 3.7 *Let M be a proper pseudo-slant submanifold of a nearly Kaehler manifold $\widetilde{M}$ such that $\mathcal{P}_Z W \in \Gamma(\mathcal{D}^\perp)$, for any $Z, W \in \Gamma(\mathcal{D}^\perp)$ and $\mathcal{Q}_U V \in \Gamma(\mu)$, for any $U, V \in \Gamma(TM)$ where $\mathcal{D}^\perp$ and μ are totally real distribution and invariant normal subbundle of M, respectively. Then M is locally a mixed geodesic warped product submanifold if and only if*

$$A_{JZ}X = 0, \quad A_{FTX}Z = -(X\lambda)\cos^2\theta Z, \tag{3.8}$$

for any $X \in \Gamma(\mathcal{D}^\theta)$, where λ is a differentiable function on M satisfying $W'\lambda = 0$, for any $W' \in \Gamma(\mathcal{D}^\perp)$.

Proof. Let $M = N_\theta \times_f N_\perp$ be a mixed geodesic warped product submanifold of a nearly Kaehler manifold $\widetilde{M}$ such that $N_\perp$ and N_θ are totally real and proper slant submanifolds of $\widetilde{M}$, respectively. Then by Lemma 3.2 and Lemma 3.3, we get (3.8).

Conversely, if M is a proper pseudo-slant submanifold of a nearly Kaehler manifold $\widetilde{M}$ such that $\mathcal{P}_Z W \in \Gamma(\mathcal{D}^\perp)$, for any $Z, W \in \Gamma(\mathcal{D}^\perp)$ and $\mathcal{Q}_U V \in \Gamma(\mu)$, for any $U, V \in \Gamma(TM)$. Then by Lemma 3.1 and the relation (3.8), we get $g(\nabla_X Y, Z) = 0$, which means that the leaves of $\mathcal{D}^\theta$ are totally geodesic in M. On the other hand, for any $Z, W \in \Gamma(\mathcal{D}^\perp)$ and any $X \in \Gamma(\mathcal{D}^\theta)$, we have

$$\begin{aligned} g([Z,W],X) &= g(J\widetilde{\nabla}_Z W, JX) - g(J\widetilde{\nabla}_W Z, JX) \\ &= g(\widetilde{\nabla}_Z JW, JX) - g((\widetilde{\nabla}_Z J)W, JX) - g(\widetilde{\nabla}_W JZ, JX) \\ &+ g((\widetilde{\nabla}_W J)Z, JX) \\ &= -g(JW, \widetilde{\nabla}_Z JX) - g(\mathcal{P}_Z W, TX) - g(\mathcal{Q}_Z W, FX) \\ &+ g(JZ, \widetilde{\nabla}_W JX) + g(\mathcal{P}_W Z, TX) + g(\mathcal{Q}_W Z, FX). \end{aligned}$$

Since $\mathcal{P}_Z W \in \Gamma(\mathcal{D}^\perp)$, for any $Z, W \in \Gamma(\mathcal{D}^\perp)$ and $\mathcal{Q}_U V \in \Gamma(\mu)$, for any $U, V \in \Gamma(TM)$. Then, the above relation will be

$$\begin{aligned} g([Z,W],X) &= -g(JW, \widetilde{\nabla}_Z TX) - g(JW, \widetilde{\nabla}_Z FX) \\ &+ g(JZ, \widetilde{\nabla}_W TX) + g(JZ, \widetilde{\nabla}_W FX) \\ &= g(\widetilde{\nabla}_Z JFX, W) - g((\widetilde{\nabla}_Z J)FX, W) - g(h(Z, TX), JW) \\ &- g(\widetilde{\nabla}_W JFX, Z) + g((\widetilde{\nabla}_W J)FX, Z) + g(h(W, TX), JZ). \end{aligned}$$

Using (2.7) (a) and (2.12), we get

$$\begin{aligned} g([Z,W],X) &= g(\widetilde{\nabla}_Z BFX, W) + g(\widetilde{\nabla}_Z CFX, W) - g(\mathcal{P}_Z FX, W) \\ &- g(A_{JW} TX, Z) - g(\widetilde{\nabla}_W BFX, Z) - g(\widetilde{\nabla}_W CFX, Z) \\ &+ g(\mathcal{P}_W FX, Z) + g(A_{JZ} TX, W). \end{aligned}$$

Using (2.11), property $(p_3)(ii)$ and (3.8), we obtain

$$\begin{aligned} g([Z,W],X) &= -\sin^2\theta g(\widetilde{\nabla}_Z X, W) - g(\widetilde{\nabla}_Z FTX, W) + g(\mathcal{Q}_Z W, FX) \\ &+ \sin^2\theta g(\widetilde{\nabla}_W X, Z) + g(\widetilde{\nabla}_W FTX, Z) - g(\mathcal{Q}_W Z, FX). \end{aligned}$$

Again using the given fact that $\mathcal{P}_Z W \in \Gamma(\mathcal{D}^\perp)$, for any $Z, W \in \Gamma(\mathcal{D}^\perp)$ and $\mathcal{Q}_U V \in \Gamma(\mu)$, for any $U, V \in \Gamma(TM)$ and then by (2.3), we arrive at

$$g([Z,W],X) = \sin^2\theta g([Z,W],X) + g(A_{FTX} Z, W) - g(A_{FTX} W, Z).$$

Since A_{FTX} is symmetric and $\theta \neq \pi/2$, then from above relation we get $\mathcal{D}^\perp$ is integrable. If we consider $N_\perp$ be a leaf of $\mathcal{D}^\perp$ in M and $h^\perp$ be the second fundamental form of $N_\perp$ in M. Then $g(h^\perp(Z,W),X) = g(\nabla_Z W, X) = g(\widetilde{\nabla}_Z W, X)$. Using (2.1), we get

$$\begin{aligned} g(h^\perp(Z,W),X) &= g(J\widetilde{\nabla}_Z W, JX) \\ &= g(\widetilde{\nabla}_Z JW, JX) - g((\widetilde{\nabla}_Z J)W, JX) \\ &= -g(JW, \widetilde{\nabla}_Z JX) - g(\mathcal{P}_Z W, TX) - g(\mathcal{Q}_Z W, FX). \end{aligned}$$

Again, from the fact that $\mathcal{P}_Z W \in \Gamma(\mathcal{D}^\perp)$, for any $Z, W \in \Gamma(\mathcal{D}^\perp)$ and $\mathcal{Q}_U V \in \Gamma(\mu)$, for any $U, V \in \Gamma(TM)$, we have

$$\begin{aligned} g(h^\perp(Z,W),X) &= -g(\widetilde{\nabla}_Z TX, JW) + g(\widetilde{\nabla}_Z JFX, W) - g((\widetilde{\nabla}_Z J)FX, W) \\ &= -g(h(Z,TX), JW) + g(\widetilde{\nabla}_Z BFX, W) + g(\widetilde{\nabla}_Z CFX, W) \\ &\quad - g(\mathcal{P}_Z FX, W). \end{aligned}$$

Using property $(p_3)(ii)$, (2.3) and (2.11), we obtain

$$\begin{aligned} g(h^\perp(Z,W),X) = &-g(A_{JW}TX, Z) + \sin^2\theta g(\widetilde{\nabla}_Z W, X) - g(A_{FTX}Z, W) \\ &+ g(\mathcal{Q}_Z W, FX). \end{aligned}$$

Thus, by the hypothesis of the theorem, we derive

$$\cos^2\theta g(h^\perp(Z,W),X) = (X\lambda)\cos^2\theta g(Z,W),$$

or equivalently $h^\perp(Z,W) = g(Z,W)\boldsymbol{\nabla}\lambda$, where $\boldsymbol{\nabla}\lambda$ is gradient of the function λ, which means that $N_\perp$ is totally umbilical in M with the mean curvature $H^\perp = \boldsymbol{\nabla}\lambda$. Also, since $W'\lambda = 0$, for all $W' \in \Gamma(\mathcal{D}^\perp)$, we can prove that $H^\perp$ is parallel corresponding to the normal connection $D^\#$ of $N_\perp$ in M (see [17]). Thus, $N_\perp$ is an extrinsic sphere in M. Hence, by a result of Hiepko [10], we conclude that M is a warped product submanifold. Thus, the proof is complete. □

4 Inequality for warped products $N_\theta \times_f N_\perp$

In this section, we obtain a geometric relationship between the squared norm of the second fundamental form of a warped product immersion in terms of the warping function. First, we construct the following orthonormal frame for a warped product pseudo-slant submanifold $M = N_\theta \times_f N_\perp$.

Let $M = N_\theta \times_f N_\perp$ be an m-dimensional warped product pseudo-slant submanifold of a $2n$-dimensional nearly Kaehler manifold $\widetilde{M}$ such that N_θ and and $N_\perp$ are $2p$-dimensional slant and q-dimensional totally real submanifolds of $\widetilde{M}$, respectively. Let us denote by $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$, the tangent bundles of N_θ and $N_\perp$, respectively and let $\{e_1,\dots,e_p, e_{p+1} = \sec\theta Te_1,\dots,e_{2p} = \sec\theta Te_p\}$ and $\{e_{2p+1} = e_1^*,\dots,e_m = e_{2p+q} = e_q^*\}$ be the local orthonormal frames of $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$, respectively.

Then, the orthonormal frames of $F\mathcal{D}^\theta$, $J\mathcal{D}^\perp$ and μ, respectively are $\{e_{m+1} = \tilde{e}_1 = \csc\theta Fe_1$, …, $e_{m+p} = \tilde{e}_p = \csc\theta Fe_p$, $e_{m+p+1} = \tilde{e}_{p+1} = \csc\theta\sec\theta FTe_1$, …, $e_{m+2p} = \tilde{e}_{2p} = \csc\theta\sec\theta FTe_p\}$, $\{e_{m+2p+1} = Je_1^*$, …, $e_{2m} = Je_q^*\}$ and $\{e_{2m+1}$, …, $e_{2n}\}$. The dimensions of $F\mathcal{D}_\theta$, $J\mathcal{D}^\perp$ and μ, respectively are $2p$, q and $2n - 2m$.

Theorem 4.1 *Let $M{=}N_\theta \times {}_f N_\perp$ be a mixed geodesic warped product pseudo-slant submanifold of a nearly Kaehler manifold $\widetilde{M}$ such that $N_\perp$ and N_θ are totally real and proper slant submanifolds of $\widetilde{M}$, respectively. Then*

(*i*) *The squared norm of the second fundamental form h of M satisfies*

$$\|h\|^2 \geq q\cot^2\theta\|\boldsymbol{\nabla}\ln f\|^2, \tag{4.1}$$

where $\boldsymbol{\nabla}\ln f$ is the gradient of $\ln f$ and q is the dimension of $N_\perp$.

(*ii*) *If the equality holds in (4.1), then N_θ and $N_\perp$ are totally geodesic and totally umbilical submanifolds of $\widetilde{M}$, respectively.*

Proof. By definition, we have

$$\|h\|^2 = \sum_{i,j=1}^{m} g(h(e_i,e_j),h(e_i,e_j)) = \sum_{r=m+1}^{2n}\sum_{i,j=1}^{m} g(h(e_i,e_j),e_r)^2.$$

Now, decompose the above equation according to the frames of $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ as follows

$$\|h\|^2 = \sum_{r=m+1}^{2n}\sum_{i,j=1}^{2p} g(h(e_i,e_j),e_r)^2 + 2\sum_{r=m+1}^{2n}\sum_{i=1}^{2p}\sum_{j=1}^{q} g(h(e_i,e_j^*),e_r)^2 + \sum_{r=m+1}^{2n}\sum_{i,j=1}^{q} g(h(e_i^*,e_j^*),e_r)^2.$$

Since M is mixed geodesic, thus the second term in the right hand side of above equation is identically zero and then we break the above equation for the frames of $F\mathcal{D}^\theta$, $J\mathcal{D}^\perp$ and μ as follows

$$\begin{aligned}\|h\|^2 = &\sum_{r=1}^{2p}\sum_{i,j=1}^{2p} g(h(e_i,e_j),\tilde{e}_r)^2 + \sum_{r=1}^{q}\sum_{i,j=1}^{2p} g(h(e_i,e_j),Je_r^*)^2 \\ &+ \sum_{r=2m+1}^{2n}\sum_{i,j=1}^{2p} g(h(e_i,e_j),e_r)^2 + \sum_{r=1}^{2p}\sum_{i,j=1}^{q} g(h(e_i^*,e_j^*),\tilde{e}_r)^2 \\ &+ \sum_{r=1}^{q}\sum_{i,j=1}^{q} g(h(e_i^*,e_j^*),Je_r^*)^2 + \sum_{r=2m+1}^{2n}\sum_{i,j=1}^{q} g(h(e_i^*,e_j^*),e_r)^2.\end{aligned} \tag{4.2}$$

Then from Lemma 3.2, the second term of right hand side in (4.2) is zero for a mixed geodesic warped product submanifold. Since we couldn't find any relation in terms of the warping function for the first, third, fifth and sixth terms in the right hand side of (4.2). Then we shall leave these positive terms and only the fourth term is evaluated, thus the above equality for the constructed frame change into the following inequality

$$\begin{aligned}\|h\|^2 \geq &\sum_{r=1}^{p}\sum_{i,j=1}^{q} g(h(e_i^*,e_j^*),\csc\theta F e_r)^2 \\ &+ \sum_{r=1}^{p}\sum_{i,j=1}^{q} g(h(e_i^*,e_j^*),\csc\theta\sec\theta FTe_r)^2.\end{aligned}$$

Using Lemma 3.3 for mixed geodesic warped products, we derive

$$\|h\|^2 \geq \csc^2\theta \sum_{r=1}^{p}\sum_{i,j=1}^{q}(Te_r \ln f)^2 g(e_i^*, e_j^*)^2 + \cot^2\theta \sum_{r=1}^{p}\sum_{i,j=1}^{q}(e_r \ln f)^2 g(e_i^*, e_j^*)^2$$

$$= q\csc^2\theta \sum_{r=1}^{2p}(Te_r \ln f)^2 - q\csc^2\theta \sum_{r=p+1}^{2p}(Te_r \ln f)^2 + q\cot^2\theta \sum_{r=1}^{p}(e_r \ln f)^2$$

$$= q\csc^2\theta \|T\boldsymbol{\nabla} \ln f\|^2 - q\csc^2\theta \sum_{r=1}^{p} g(e_{r+p}, T\boldsymbol{\nabla} \ln f)^2$$

$$+ q\cot^2\theta \sum_{r=1}^{p}(e_r \ln f)^2$$

$$= q\cot^2\theta \|\boldsymbol{\nabla} \ln f\|^2 - q\csc^2\theta \sec^2\theta \sum_{r=1}^{p} g(Te_r, T\boldsymbol{\nabla} \ln f)^2$$

$$+ q\cot^2\theta \sum_{r=1}^{p}(e_r \ln f)^2.$$

Then from (2.9), we get $\|h\|^2 \geq q\cot^2\theta\|\boldsymbol{\nabla} \ln f\|^2$, which is inequality (i). If the equality holds in (4.1), then from the leaving first and third terms in (4.2), we conclude that

$$g(h(\mathcal{D}^\theta, \mathcal{D}^\theta), F\mathcal{D}^\theta) = 0 \quad \Rightarrow \quad h(\mathcal{D}^\theta, \mathcal{D}^\theta) \subset J\mathcal{D}^\perp \oplus \mu \tag{4.3}$$

and

$$g(h(\mathcal{D}^\theta, \mathcal{D}^\theta), \mu) = 0 \quad \Rightarrow \quad h(\mathcal{D}^\theta, \mathcal{D}^\theta) \subset J\mathcal{D}^\perp \oplus F\mathcal{D}^\theta. \tag{4.4}$$

Then from (4.3) and (4.4), we have

$$h(\mathcal{D}^\theta, \mathcal{D}^\theta) \subset J\mathcal{D}^\perp. \tag{4.5}$$

But, from Lemma 3.2 for a mixed geodesic warped product pseudo-slant submanifold, we have

$$h(\mathcal{D}^\theta, \mathcal{D}^\theta) \perp J\mathcal{D}^\perp. \tag{4.6}$$

Then, from (4.5) and (4.6), we get

$$h(\mathcal{D}^\theta, \mathcal{D}^\theta) = 0. \tag{4.7}$$

Since N_θ is totally geodesic in M (see [3]), with this fact (4.7) implies that N_θ is totally geodesic in $\widetilde{M}$. Similarly, from the leaving fifth and sixth terms, we conclude that

$$g(h(\mathcal{D}^\perp, \mathcal{D}^\perp), J\mathcal{D}^\perp) = 0 \quad \Rightarrow \quad h(\mathcal{D}^\perp, \mathcal{D}^\perp) \subset F\mathcal{D}^\theta \oplus \mu \tag{4.8}$$

and

$$g(h(\mathcal{D}^\perp, \mathcal{D}^\perp), \mu) = 0 \quad \Rightarrow \quad h(\mathcal{D}^\perp, \mathcal{D}^\perp) \subset J\mathcal{D}^\perp \oplus F\mathcal{D}^\theta. \tag{4.9}$$

Then from (4.8) and (4.9), we derive

$$h(\mathcal{D}^\perp, \mathcal{D}^\perp) \subset F\mathcal{D}^\theta. \tag{4.10}$$

Also, by Lemma 3.3, for a mixed geodesic warped product pseudo-slant submanifold, we have

$$g(h(Z,W), FTX) = -\cos^2\theta(X\ln f)g(Z,W), \tag{4.11}$$

for any $X \in \Gamma(\mathcal{D}^\theta)$ and $Z, W \in \Gamma(\mathcal{D}^\perp)$. Thus, $N_\perp$ is totally umbilical submanifold of $\widetilde{M}$ by using the fact that $N_\perp$ is totally umbilical in M (see [3]) with (4.10) and (4.11). This completes the proof of the theorem. □

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