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Warped product pseudo-slant submanifolds of cosymplectic manifolds

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Abstract In this paper, first we study pseudo-slant submanifolds of cosymplectic manifolds in detail and then we discuss their warped products. To give the answer of the question that: Is there any warped product submanifold of almost contact metric manifolds with slant factor? We study warped product pseudo-slant submanifolds of cosymplectic manifolds. We obtain a characterization and establish an inequality for the squared norm of the second fundamental form in terms of the warping function for such types of warped product submanifolds. The equality case is also discussed.

Keywords Warped product · Slant submanifolds · Semi-slant submanifolds · Pseudo-slant submanifolds · Mixed geodesic · Cosymplectic manifolds

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1 Introduction

It is well known that the odd dimensional counter part of Kaehler manifolds are cosymplectic manifolds. Let us recall that an almost contact metric structure (φ, ξ, η, g) on manifold $\widetilde{M}$ is cosymplectic if it is integrable and 1-form η and the fundamental 2-form of the structure are closed (see [2]). The canonical example of compact cosymplectic manifold is given by the product of a compact Kaehler manifold with a circle S^1 or a real line $\mathbb{R}$.

Recently, warped product pseudo-slant submanifolds of Kaehler manifolds were studied by B. Sahin under the name of Hemi-slant wapred products (see [14]). He proved that the warped product pseudo-slant submanifolds of the form $N_\perp \times_f N_\theta$ do not exist and he obtained a characterization and an inequality for the existence of warped products of the type $N_\theta \times_f N_\perp$ of a Kaehler manifold. In this paper, we extend this study to the warped product submanifolds of cosymplectic manifolds which are in an important class of almost contact metric manifolds. Also, the warped products for different spaces were studied in (see [13], [15], [16]).

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The paper is organized as follows: In Section 2, we provide some preliminaries formulas and some basic results for later use. In Section 3, we study pseudo-slant submanifolds in detail. First we find the integrability conditions for the anti-invariant and slant distributions involved in the definition of pseudo-slant submanifolds and then we investigate the geometry of the leaves of both distributions. In Section 4, first we obtain basic lemmas for later use and then we obtain a characterization for mixed geodesic warped product pseudo-slant submanifolds of cosymplectic manifolds. In Section 5, we establish an inequality for the squared norm of the second fundamental form of a mixed geodesic warped product pseudo-slant submanifold M in terms of the warping function and the slant angle of M. The equality case is also considered. To this end, we give an example of warped product cosymplectic manifolds.

2 Preliminaries

Let $(\widetilde{M}, \varphi, \xi, \eta, g)$ be an almost contact metric manifold. Then we have

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\varphi\widetilde{X}, \varphi\widetilde{Y}) = g(\widetilde{X}, \widetilde{Y}) - \eta(\widetilde{X})\eta(\widetilde{Y}), \tag{2.1}$$

for any $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\widetilde{M})$, the Lie algebra of vector fields on $\widetilde{M}$ and I being the identity transformation on $\mathfrak{X}(\widetilde{M})$. As a consequence, the dimension of $\widetilde{M}$ is odd ($= 2m + 1$), $\varphi(\xi) = 0 = \eta \circ \varphi$ and $\eta(\widetilde{X}) = g(\widetilde{X}, \xi)$. The fundamental 2-form Φ of $\widetilde{M}$ is defined $\Phi(\widetilde{X}, \widetilde{Y}) = g(\widetilde{X}, \varphi\widetilde{Y})$. An almost contact metric manifold $(\widetilde{M}, \varphi, \xi, \eta, g)$ is said to be *cosymplectic* if $[\varphi, \varphi] = 0$ and $d\eta = 0$, $d\Phi = 0$, where $[\varphi, \varphi](\widetilde{X}, \widetilde{Y}) = \varphi^2[\widetilde{X}, \widetilde{Y}] + [\varphi\widetilde{X}, \varphi\widetilde{Y}] - \varphi[\varphi\widetilde{X}, \widetilde{Y}] - \varphi[\widetilde{X}, \varphi\widetilde{Y}]$ and d is an exterior differential operator. In terms of the covariant derivative of φ and η the cosymplectic structure is characterized by

$$\widetilde{\nabla}_{\widetilde{X}}\varphi = 0 \quad \text{and} \quad \widetilde{\nabla}_{\widetilde{X}}\eta = 0, \tag{2.2}$$

where $\widetilde{\nabla}$ is the Levi-Civita connection of g. From the formula $\widetilde{\nabla}_{\widetilde{X}}\varphi = 0$, it follows that $\widetilde{\nabla}_{\widetilde{X}}\xi = 0$. From the geometric and topological points of view the cosymplectic manifolds were studied in ([7], [10]).

Let M be a submanifold of an almost contact metric manifold $\widetilde{M}$ with induced metric g and if ∇ and $\nabla^{\perp}$ are the induced connections on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively then Gauss-Weingarten formulas are given by

$$(a) \ \ \widetilde{\nabla}_U V = \nabla_U V + h(U, V), \quad (b) \ \ \widetilde{\nabla}_U N = -A_N U + \nabla^{\perp}_U N, \tag{2.3}$$

for any vector field U, V tangent to M and N normal to M, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N) respectively for the immersion of M into $\widetilde{M}$. They are related as

$$g(h(U, V), N) = g(A_N U, V), \tag{2.4}$$

where g denotes the Riemannian metric on $\widetilde{M}$ as well as the one induced on M.

For any $U \in TM$ and $N \in T^{\perp}M$, the tangential and normal components of φU and φN are decomposed by

$$(a)\ \ \varphi U = TU + FU, \quad (b)\ \ \varphi N = tN + fN. \tag{2.5}$$

Furthermore, the covariant derivatives of the tensor fields T and F are defined as

$$(a)\ \ (\nabla_U T)V = \nabla_U TV - T\nabla_U V, \quad (b)\ \ (\nabla_U F)V = \nabla^{\perp}_U FV - F\nabla_U V, \tag{2.6}$$

for all $U, V \in TM$. A submanifold M of an almost contact metric manifold $\widetilde{M}$ is said to be *invariant* (or *anti-invariant*) if F (or T) is identically zero.

Let M be a submanifold tangent to the structure vector field ξ isometrically immersed into an almost contact metric manifold $\widetilde{M}$. Then M is said to be a contact CR-submanifold if there exists a pair of orthogonal distributions $\mathcal{D} : p \to \mathcal{D}_p$ and $\mathcal{D}^{\perp} : p \to \mathcal{D}^{\perp}_p$, $\forall p \in M$ such that

(i) $TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \langle\xi\rangle$, where $\langle\xi\rangle$ is the 1-dimensional distribution spanned by the structure vector field ξ.
(ii) $\mathcal{D}$ is invariant, i.e., $\varphi\mathcal{D} = \mathcal{D}$.
(iii) $\mathcal{D}^{\perp}$ is anti-invariant, i.e., $\varphi\mathcal{D}^{\perp} \subseteq T^{\perp}M$.

Invariant and anti-invariant submanifolds are the special cases of a contact CR-submanifold. If we denote the dimensions of the distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ by d_1 and d_2, respectively. Then M is *invariant* (resp. *anti-invariant*) if $d_2 = 0$ (resp. $d_1 = 0$).

There is another class of submanifolds that is called the slant submanifold. For each non zero vector U tangent to M at p, such that U is not proportional to ξ_p, we denote by $0 \le \theta(U) \le \frac{\pi}{2}$, the angle between φU and T_pM is called the *Wirtinger angle*. If the angle $\theta(U)$ is constant for all nonzero $U \in T_pM - \langle\xi_p\rangle$ and $p \in M$, then M is said to be a *slant* submanifold (see [5]) and the angle θ is slant angle of M. Obviously if $\theta = 0$, M is invariant and if $\theta = \frac{\pi}{2}$, M is an anti-invariant submanifold. A slant submanifold is said to be *proper slant* if it is neither invariant nor anti-invariant.

We recall the following result for slant submanifolds.

Theorem 2.1 ([5]) *Let M be a submanifold of an almost contact metric manifold $\widetilde{M}$, such that $\xi \in TM$. Then M is slant if and only if there exists a constant $\delta \in [0,1]$ such that*

$$T^2 = \delta(-I + \eta \otimes \xi). \tag{2.7}$$

Furthermore, if θ is slant angle, then $\delta = \cos^2\theta$.

Following relations are straightforward consequence of equation (2.7)

$$g(TU, TV) = \cos^2\theta[g(U,V) - \eta(U)\eta(V)], \tag{2.8}$$

$$g(FU, FV) = \sin^2\theta[g(U,V) - \eta(U)\eta(V)], \tag{2.9}$$

for any U, V tangent to M.

For a slant submanifold M tangent to the structure vector field ξ of an almost contact metric manifold $\widetilde{M}$, we can prove that

$$(a)\ \ tFU = \sin^2\theta(-U + \eta(U)\xi), \quad (b)\ \ fFU = -FTU, \tag{2.10}$$

for any $U \in TM$.

A natural generalization of CR-submanifolds of almost Hermitian manifolds in terms of slant distribution was given by Papaghiuc [12]. These submanifolds are known as semi-slant submanifolds. The semi-slant submanifolds of almost contact metric manifolds are defined and studied by Cabrerizo et al. [4], they defined these submanifolds as: A Riemannian submanifold M of an almost contact manifold $\widetilde{M}$ is said to be a *semi-slant submanifold* if there exist two orthogonal distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ such that $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \langle\xi\rangle$, the distribution $\mathcal{D}_1$ is invariant i.e., $\varphi\mathcal{D}_1 = \mathcal{D}_1$ and the distribution $\mathcal{D}_2$ is slant with slant angle $\theta \neq \frac{\pi}{2}$.

If we denote the dimension of $\mathcal{D}_i$ by d_i, for $i = 1, 2$, then it is clear that contact CR-submanifolds and slant submanifolds are semi-slant submanifolds with $\theta = \frac{\pi}{2}$ and $d_1 = 0$, respectively.

3 Pseudo-slant submanifolds

Pseudo-slant submanifolds were defined by Carriazo in [6] under the name of *anti-slant submanifolds* as a particular class of bi-slant submanifolds. However, the term "anti-slant" seems that there is no slant part, which is not a case, as one can see the following definition:

Definition 3.1 *A submanifold M of an almost contact metric manifold $\widetilde{M}$ is said to be a pseudo-slant submanifold if there exists a pair of orthogonal distributions $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ on M such that*

(i) TM admits the orthogonal direct decomposition $TM = \mathcal{D}^\perp \oplus \mathcal{D}^\theta \oplus \langle\xi\rangle$.
(ii) The distribution $\mathcal{D}^\perp$ is anti-invariant, i.e., $\varphi(\mathcal{D}^\perp) \subset T^\perp M$.
(iii) The distribution $\mathcal{D}^\theta$ is slant with angle $\theta \neq \frac{\pi}{2}$.

In this case, we call the angle θ the slant angle of the submanifold M. If we denote the dimensions of $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ by d_1 and d_2, respectively, then we have the following:

(i) If $d_1 = 0$ and $\theta = 0$, then M is an invariant submanifold.
(ii) If $d_2 = 0$, then M is an anti-invariant submanifold.
(iii) If $d_1 = 0$ and $\theta \neq 0, \frac{\pi}{2}$, then M is a proper slant submanifold with slant angle θ.
(iv) If $\theta = 0$, then M is a contact CR-submanifold.
(v) We say that M is *proper pseudo-slant submanifold* if $d_1.d_2 \neq 0$ and $\theta \in (0, \frac{\pi}{2})$.

Example 3.1 Let M be a submanifold of $\mathbb{R}^9$ given by $x(u, v, w, s, z) = 2(u,\ 0,\ w,\ 0,\ 0,\ v,\ s\cos\theta, s\sin\theta, z)$. with $\theta \in [0, \pi/2)$. Then M is a 5-dimensional pseudo-slant submanifold with the slant angle θ such that the cosymplectic sturcture (φ, ξ, η, g) is given by

$$\varphi\left\{\sum_{i=1}^{4}\left(X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i}\right) + Z\frac{\partial}{\partial z}\right\} = \sum_{i=1}^{4}\left(-Y_i\frac{\partial}{\partial x^i} + X_i\frac{\partial}{\partial y^i}\right),$$

$$\xi = 2\frac{\partial}{\partial z}, \quad \eta = \frac{1}{2}dz \quad \text{and} \quad g = \eta\otimes\eta + \frac{1}{4}\sum_{i=1}^{4}(dx^i\otimes dx^i + dy^i\otimes dy^i).$$

It is easy to see that a local orthonormal frame of TM is given by

$$e_1 = 2\left(\frac{\partial}{\partial x^1}\right), \quad e_2 = 2\left(\frac{\partial}{\partial y^2}\right), \quad e_3 = 2\left(\frac{\partial}{\partial x^3}\right)$$
$$e_4 = \cos\theta\left(2\frac{\partial}{\partial y^3}\right) + \sin\theta\left(2\frac{\partial}{\partial y^4}\right), \quad e_5 = 2\left(\frac{\partial}{\partial z}\right) = \xi,$$

providing an anti-invariant distribution $\mathcal{D}^\perp = \{e_1, e_2\}$ and a slant distribution $\mathcal{D}^\theta = \{e_3, e_4\}$.

A pseudo-slant submanifold M of an almost contact metric manifold $\widetilde{M}$ is said to be *mixed geodesic* if $h(X, Z) = 0$, for any $X \in \mathcal{D}^\theta \oplus \langle\xi\rangle$ and $Z \in \mathcal{D}^\perp$.

Let M be a pseudo-slant submanifold of an almost contact metric manifold $\widetilde{M}$ and if we denote the projections on $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ by P_1 and P_2, respectively, then for any vector field $U \in TM$, we get

$$U = P_1U + P_2U + \eta(U)\xi. \tag{3.1}$$

Then from (2.5), we obtain

$$TU = TP_2U \quad \text{and} \quad FU = \varphi P_1U + FP_2U. \tag{3.2}$$

On a pseudo-slant submanifold M of an almost contact metric manifold $\widetilde{M}$, the normal bundle $T^\perp M$ can be decomposed as $T^\perp M = \varphi\mathcal{D}^\perp \oplus F\mathcal{D}^\theta \oplus \mu$, where μ is the normal invariant subbundle under φ.

Now, we obtain some preparatory results for the next section.

Lemma 3.2 *On a pseudo-slant submanifold M of a cosymplectic manifold $\widetilde{M}$, we have $A_{\varphi Z}W = A_{\varphi W}Z$, for any $Z, W \in \mathcal{D}^\perp$.*

Proof. For any $Z, W \in \mathcal{D}^\perp$ and $U \in TM$, where $U = P_1U + P_2U + \eta(U)\xi$, we have $g(\widetilde{\nabla}_U\varphi Z, W) = g(\varphi\widetilde{\nabla}_U Z, W) = -g(\widetilde{\nabla}_U Z, \varphi W)$. Using (2.3) and (2.4), we get the desired result. □

For the integrability of the involved distributions in the definition of a pseudo-slant submanifold, we have the following results.

Lemma 3.3 *Let M be a pseudo-slant submanifold of a cosymplectic manifold $\widetilde{M}$. Then the anti-invariant distribution is always integrable.*

Proof. From (2.1) and (2.2), we have $g([Z, W], \varphi U) = -g(\varphi[Z, W], U) = g(\widetilde{\nabla}_W\varphi Z - \widetilde{\nabla}_Z\varphi W, U)$, for any $Z, W \in \mathcal{D}^\perp$ and $U \in TM$. Then from (2.3) and Lemma 3.1, the right hand side of the above equation will be zero and hence by (3.2), we derive $g(TP_2U, [Z, W]) = 0$, which implies the integrability of $\mathcal{D}^\perp$. Thus the proof is complete. □

Lemma 3.4 *On a proper pseudo-slant submanifold M of a cosymplectic manifold $\widetilde{M}$, the distribution $\mathcal{D}^\theta \oplus \langle\xi\rangle$ is integrable if and only if $g(\nabla_Y Z, X) = \sec^2\theta\{g(h(X, Z), FTY) - g(h(X, TY), \varphi Z)\}$, for $X, Y \in \mathcal{D}^\theta \oplus \langle\xi\rangle$ and $Z \in \mathcal{D}^\perp$.*

Proof. For any $X, Y \in \mathcal{D}^\theta \oplus \langle\xi\rangle$ and $Z \in \mathcal{D}^\perp$, we have $g([X,Y],Z) = g(\widetilde{\nabla}_X\varphi Y, \varphi Z) - g(\widetilde{\nabla}_Y X, Z) = g(\widetilde{\nabla}_X TY, \varphi Z) + g(\widetilde{\nabla}_X FY, \varphi Z) - g(\widetilde{\nabla}_Y X, Z) = g(h(X,TY),\varphi Z) - g(\widetilde{\nabla}_X \varphi FY, Z) - g(\widetilde{\nabla}_Y X, Z)$. Using (2.5) and then (2.10), we get $g([X,Y],Z) = g(h(X, TY),\varphi Z) + \sin^2\theta g(\widetilde{\nabla}_X Y,\ Z) + g(\widetilde{\nabla}_X FTY, Z) - g(\widetilde{\nabla}_Y X, Z)$. The above equation can be simplified as

$$\cos^2\theta g([X,Y],Z) = g(h(X,TY),\varphi Z) - g(A_{FTY}X,Z) - \cos^2\theta g(\widetilde{\nabla}_Y X, Z). \tag{3.3}$$

Thus the result follows from the last relation (3.3). □

Theorem 3.5 *Let M be a pseudo-slant submanifold of a cosymplectic manifold $\widetilde{M}$. Then the distribution $\mathcal{D}^\perp$ defines a totally geodesic foliation if and only if $g(A_{FTX}W - A_{\varphi W}TX, Z) = 0$ for $Z, W \in \mathcal{D}^\perp$ and $X \in \mathcal{D}^\theta \oplus \langle\xi\rangle$.*

Proof. For any $Z, W \in \mathcal{D}^\perp$ and $X \in \mathcal{D}^\theta \oplus \langle\xi\rangle$, we have $g(\nabla_Z W, X) = g(\widetilde{\nabla}_Z W, X) = g(\varphi\widetilde{\nabla}_Z W, \varphi X) + \eta(X) g(\widetilde{\nabla}_Z W, \xi)$.

Since W and ξ are orthogonal, hence the last term in the right hand side of the above equation is zero by using the cosymplectic character, then we derive $g(\nabla_Z W, X) = g(\widetilde{\nabla}_Z\varphi W, TX) + g(\widetilde{\nabla}_Z\varphi W, FX)$. Using (2.3)-(b) and the fact that the normal vector fields φW and FX are orthogonal, we get $g(\nabla_Z W, X) = -g(A_{\varphi W}Z, TX) - g(\varphi W, \widetilde{\nabla}_Z FX)$. Then from (2.4), (2.1) and (2.2), we obtain $g(\nabla_Z W, X) = -g(h(Z,TX), \varphi W) + g(W, \widetilde{\nabla}_Z\varphi FX) = -g(h(Z,TX), \varphi W) + g(W, \widetilde{\nabla}_Z tFX) + g(W, \widetilde{\nabla}_Z fFX)$. Using (2.10), we derive $g(\nabla_Z W, X) = -g(h(Z,TX), \varphi W) - \sin^2\theta g(W, \widetilde{\nabla}_Z X) - g(W, \widetilde{\nabla}_Z FTX)$.

Again using (2.3)-(b) and then (2.4) and the orthogonality of the vector fields W and X, finally we get $\cos^2\theta g(\nabla_Z W, X) = g(h(Z,W), FTX) - g(h(Z,TX), \varphi W)$, which proves our assertion. □

Theorem 3.6 *On a proper pseudo-slant submanifold M of a cosymplectic manifold $\widetilde{M}$, the distribution $\mathcal{D}^\theta \oplus \langle\xi\rangle$ defines a totally geodesic foliation if and only if $g(A_{FTX}W - A_{\varphi W}TX, Y) = 0$ for any $X, Y \in \mathcal{D}^\theta \oplus \langle\xi\rangle$ and $W \in \mathcal{D}^\perp$.*

Proof. For any $X, Y \in \mathcal{D}^\theta \oplus \langle\xi\rangle$ and $W \in \mathcal{D}^\perp$, we have $g(\nabla_Y X, W) = g(\widetilde{\nabla}_Y X, W) = g(\varphi\widetilde{\nabla}_Y X, \varphi W)$. Then, using the covariant differentiation property of φ and the cosymplectic character with (2.5) and (2.3), we find $g(\nabla_Y X, W) = g(h(Y,TX),\varphi W) - g(\varphi\widetilde{\nabla}_Y FX, W)$. Again, from the structure equation of cosymplectic manifold, (2.5) and (2.10), we derive

$$g(\nabla_Y X, W) = g(A_{\varphi W}TX, Y) + \sin^2\theta\, g(\widetilde{\nabla}_Y X, W) + g(\widetilde{\nabla}_Y FTX, W).$$

Thus the result follows from the above relation by using (2.3) and (2.4). □

Thus from Theorem 3.1 and Theorem 3.2 we can state the following theorem.

Theorem 3.7 *Let M be a proper pseudo-slant submanifold of a cosymplectic manifold $\widetilde{M}$, then M is a locally Riemannian product manifold of $N_\perp$ and N_θ if and only if $A_{\varphi Z}TX = A_{FTX}Z$, for any $X \in \mathcal{D}^\theta \oplus \langle\xi\rangle$ and $Z \in \mathcal{D}^\perp$, where $N_\perp$ is an anti-invariant submanifold and N_θ is a proper slant submanifold tangent to the structure vector field ξ of $\widetilde{M}$.*

4 Warped product pseudo-slant submanifolds

The idea of warped product manifolds was given by Bishop and O'Neill [1]. They defined warped products as follows: Let N_1 and N_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and a positive differentiable function f on N_1. Consider the product manifold $N_1 \times N_2$ with its projections $\pi_1 : N_1 \times N_2 \to N_1$ and $\pi_2 : N_1 \times N_2 \to N_2$. Then their warped product manifold $M = N_1 \times_f N_2$ is the Riemannian manifold $N_1 \times N_2 = (N_1 \times N_2, g)$ equipped with the Riemannian structure such that

$$g(X, Y) = g_1(\pi_{1\star}X, \pi_{1\star}Y) + (f \circ \pi_1)^2 g_2(\pi_{2\star}X, \pi_{2\star}Y), \tag{4.1}$$

for any vector field X, Y tangent to M, where $\star$ is the symbol for the tangent maps. A warped product manifold $M = N_1 \times_f N_2$ is said to be *trivial* or simply *Riemannian product manifold* if the warping function f is constant.

We recall the following result for warped product manifolds.

Lemma 4.1 ([1]) *Let $M = N_1 \times_f N_2$ be a warped product manifold with the warping function f, then*

(i) $\nabla_X Y \in \Gamma(TN_1)$ *is the lift of* $\nabla_X Y$ *on* N_1,
(ii) $\nabla_X Z = \nabla_Z X = (X \ln f)Z$,
(iii) $\nabla_Z W = \nabla^{N_2}_Z W - (g(Z, W)/f) \operatorname{grad} f$,

for each $X, Y \in TN_1$ and $Z, W \in TN_2$, where $\operatorname{grad} f$ is the gradient of f defined as $g_1(\operatorname{grad} f, X) = Xf$ and ∇ and ∇^{N_2} denote the Levi-Civita connections on M and N_2, respectively.

As a consequence, we have

$$\| \operatorname{grad} f\|^2 = \sum_{i=1}^{n}(e_i(f))^2, \tag{4.2}$$

for an orthonormal frame $\{e_1 \cdots , e_n\}$ on N_1.

In [9], we have proved the non-existence warped products semi-slant submanifolds of cosymplectic manifolds. In this section we study warped product pseudo-slant (anti-slant, in the sense of Cabrerizo) submanifolds of cosymplectic manifolds. If the manifolds $N_\perp$ and N_θ are anti-invariant and proper slant submanifolds of a cosymplectic manifold $\widetilde{M}$, respectively then their warped products are any one of the form $N_\perp \times_f N_\theta$ or $N_\theta \times {}_f N_\perp$. If we consider the structure vector field ξ tangent to the second factor (fiber) in both types of warped products, then we have already proved the non-existence of such warped products (see [9]). Also, on a warped product submanifold $M = N_\perp \times {}_f N_\theta$ of a cosymplectic manifold $\widetilde{M}$, we have the following non-existence result.

Theorem 4.2 ([17]) *There do not exist non-trivial warped product submanifolds $M = N_\perp \times {}_f N_\theta$ of a cosymplectic manifold $\widetilde{M}$ such that $\xi \in TN_\perp$, where $N_\perp$ and N_θ are anti-invariant and proper slant submanifolds of $\widetilde{M}$, respectively.*

Now, we consider the warped product pseudo-slant submanifolds of the form $N_\theta \times {}_fN_\perp$, where N_θ and $N_\perp$ are proper slant and anti-invariant submanifolds of a cosymplectic manifold $\widetilde{M}$, respectively. First, we prove the following lemmas for later use.

Lemma 4.3 *Let $M = N_\theta \times {}_fN_\perp$ be a warped product submanifold of a cosymplectic manifold $\widetilde{M}$ such that $\xi \in TN_\theta$, then*

$$g(h(X,Y),\varphi Z) = g(h(X,Z),FY), \tag{4.3}$$

for any $X, Y \in TN_\theta$ and $Z \in TN_\perp$.

Proof. From (2.3), we have $g(h(X,Y),\varphi Z)=g(\widetilde{\nabla}_XY,\varphi Z)$ for any $X,Y \in TN_\theta$ and $Z \in TN_\perp$. Thus using (2.1) and (2.2), we get $g(h(X,Y),\varphi Z) = -g(\widetilde{\nabla}_X\varphi Y,Z) = g(TY,\widetilde{\nabla}_XZ) - g(\widetilde{\nabla}_XFY,Z)$. Then from Lemma 4.1 (ii) and the relations (2.3) and (2.4), we get the desired result. □

If we replace X by TX and Y by TY in (4.3), then we get the following relations

$$g(h(TX,Y),\varphi Z) = g(h(TX,Z),FY), \tag{4.4}$$
$$g(h(X,TY),\varphi Z) = g(h(X,Z),FTY), \tag{4.5}$$
$$g(h(TX,TY),\varphi Z) = g(h(TX,Z),FTY). \tag{4.6}$$

Lemma 4.4 *Let $M = N_\theta \times {}_fN_\perp$ be a warped product submanifold of a cosymplectic manifold $\widetilde{M}$ such that $\xi \in TN_\theta$, then*

(i) $g(h(Z,W),FX) = g(h(X,Z),\varphi W) + (TX \ln f)g(Z,W)$,
(ii) $g(h(Z,W),FTX) = g(h(TX,Z),\varphi W) - (X \ln f)\cos^2\theta g(Z,W)$,

for any $Z,W \in TN_\perp$ and $X \in TN_\theta$.

Proof. From (2.5)-(a), we have $g(h(Z,W),FX) = g(h(Z,W),\varphi X)$, for any $Z,W \in TN_\perp$ and $X \in TN_\theta$. Using (2.3)-(a) and the orthogonality of vector fields, we derive $g(h(Z,W),FX) = g(\widetilde{\nabla}_ZW,\varphi X) - g(\nabla_ZW,TX)$. Then from (2.1), (2.2) and the fact that W and TX are orthogonal vector fields, we get $g(h(Z,W),FX) = -g(\widetilde{\nabla}_Z\varphi W,X) + g(W,\nabla_ZTX)$. Hence, by Lemma 4.1 (ii) and relations (2.3)-(b) and (2.4), we obtain the first part of the lemma. If we replace X by TX in (i), then by Theorem 2.1 and the fact that $\xi \ln f = 0$, we get (ii), which completes the proof of the lemma. □

Theorem 4.5 *Let M be a proper pseudo-slant submanifold of a cosymplectic manifold $\widetilde{M}$. Then M is locally a mixed geodesic warped product submanifold of the form $N_\theta \times {}_fN_\perp$ if and only if*

$$A_{\varphi Z}X = 0, \quad A_{FX}Z = TX(\lambda)Z \quad \text{and} \quad A_{FTX}Z = -\cos^2\theta X(\lambda)Z, \tag{4.7}$$

for any $X \in \mathcal{D}^\theta \oplus \langle\xi\rangle$ and $Z \in \mathcal{D}^\perp$, where λ is a function on M such that $W(\lambda) = 0$, for any $W \in \mathcal{D}^\perp$.

Proof. If M is a mixed geodesic warped product submanifold, then the direct part follows from Lemmas 4.2 and 4.3. Let us prove the converse, let M be a pseudo-slant submanifolds with the slant distribution $\mathcal{D}^\theta \oplus \langle\xi\rangle$ and the anti-invariant distribution $\mathcal{D}^\perp$ with the given conditions of (4.7). By using (4.7) and Theorem 3.2, the distribution $\mathcal{D}^\theta \oplus\langle\xi\rangle$ defines a totally geodesic foliation. Also, as $\mathcal{D}^\perp$ is integrable (Lemma 3.2), consider $h^\perp$ be the second fundamental form of the leaf $N_\perp$ of $\mathcal{D}^\perp$ in M, then for any $X \in \mathcal{D}^\theta \oplus\langle\xi\rangle$ and $Z, W \in \mathcal{D}^\perp$, we have $g(h^\perp(Z,W), X) = g(\nabla_Z W, X) = g(\widetilde{\nabla}_Z W, X)$. Using (2.1), (2.2) and (2.5), we obtain $g(h^\perp(Z,W), X) = g(\widetilde{\nabla}_Z \varphi W, TX) + g(\widetilde{\nabla}_Z \varphi W, FX)$.

Then from (2.3)-(b) and the orthogonality of the normal vectors φW and FX, we derive $g(h^\perp(Z,W), X) = -g(A_{\varphi W} Z, TX) - g(\varphi W, \widetilde{\nabla}_Z FX)$. Again, using (2.1), (2.2) and (2.5)-(b) in the second term of right hand side, we get $g(h^\perp(Z,W), X) = -g(A_{\varphi W} TX, Z) + g(W, \widetilde{\nabla}_Z tFX) + g(W, \widetilde{\nabla}_Z fFX)$. By the given first condition of (4.7) and the relation (2.10), we derive $g(h^\perp(Z,W), X) = -\sin^2\theta g(W, \widetilde{\nabla}_Z X) - g(W, \widetilde{\nabla}_Z FTX)$. Then from (2.3)-(b) and the orthogonality of the vector fields X and W, we arrive at $g(h^\perp(Z,W), X) = \sin^2\theta g(\widetilde{\nabla}_Z W, X) + g(W, A_{FTX} Z)$. Thus on using (2.3)-(b) and the last relation of (4.7), we obtain

$$g(h^\perp(Z,W), X) - \sin^2\theta g(h^\perp(Z,W), X) = -\cos^2\theta X(\lambda) g(Z,W).$$

Then from the definition of gradient, we get

$$g(h^\perp(Z,W), X) = -g(Z,W) g(\operatorname{grad}\lambda, X),$$

which implies that $h^\perp(Z,W) = -g(Z,W)\operatorname{grad}\lambda$, for any $Z, W \in \mathcal{D}^\perp$, i.e., $N_\perp$ is totally umbilical in M with mean curvature $H = -\operatorname{grad}\lambda$. Now, we prove that $\operatorname{grad}\lambda$ is parallel corresponding to the normal connection D of $N_\perp$ in M. Consider $X \in \mathcal{D}^\theta \oplus \langle\xi\rangle$ and $W \in \mathcal{D}^\perp$, we derive $g(D_W \operatorname{grad}\lambda, X) = g(\nabla_W \operatorname{grad}\lambda, X) = W g(\operatorname{grad}\lambda, X) - g(\operatorname{grad}\lambda, \nabla_W X) = W(X(\lambda)) - g(\operatorname{grad}\lambda, [W,X]) - g(\operatorname{grad}\lambda, \nabla_X W) = X(W\lambda) + g(\nabla_X \operatorname{grad}\lambda, W) = 0$, since $W(\lambda) = 0$, $\forall\ W \in \mathcal{D}^\perp$ and thus $\nabla_X \operatorname{grad}\lambda \in \mathcal{D}^\theta \oplus \langle\xi\rangle$ and $\mathcal{D}^\theta \oplus \langle\xi\rangle$ being totally geodesic. This means that the mean curvature of $N_\perp$ is parallel. Thus the leaves of $\mathcal{D}^\perp$ are totally umbilical with parallel mean curvature $H = -\operatorname{grad}\lambda$. Thus, $N_\perp$ is an extrinsic sphere in M. Hence, by a result of Hiepko [8] we conclude that M is a warped product submanifold and the proof is complete. □

5 Inequality for warped product pseudo-slant submanifolds

In this section, we obtain a relation for the squared norm of the second fundamental form of a mixed geodesic warped product pseudo-slant submanifold with the warping function. First, we consider the following orthonormal frame for later use.

Let $M = N_\theta \times_f N_\perp$ be an n-dimensional warped product pseudo-slant submanifold of a $(2m+1)$-dimensional cosymplectic manifold $\widetilde{M}$ with fiber $N_\perp$ of dimension d_1 and base N_θ of dimension $d_2 = 2p+1$ such that ξ is tangent to N_θ. Let us consider the tangent spaces of $N_\perp$ and N_θ by $\mathcal{D}^\perp$ and $\mathcal{D}^\theta \oplus \langle\xi\rangle$. We set the orthonormal frames of $\mathcal{D}^\perp$ and $\mathcal{D}^\theta \oplus \langle\xi\rangle$, respectively as $\{e_1, e_2 \cdots, e_{d_1}\}$ and $\{e_{d_1+1} = e_1^*, \ldots, e_{d_1+p} = e_p^*, e_{d_1+p+1} = e_{p+1}^* = \sec\theta T e_1^*, \ldots, e_{d_1+2p} = e_{2p}^* = \sec\theta T e_p^*, e_{d_1+2p+1} = e_{2p+1}^* =$

$\xi\}$, where θ is the slant angle. Then the orthonormal frames of the normal sub-bundles of $\varphi\mathcal{D}^\perp$, $F\mathcal{D}^\theta$ and μ, respectively are $\{e_{n+1} = \tilde{e}_1 = \varphi e_1, e_{n+2} = \tilde{e}_2 = \varphi e_2, \ldots, e_{n+d_1} = \tilde{e}_{d_1} = \varphi e_{d_1}\}$, $\{e_{n+d_1+1} = \tilde{e}_{d_1+1} = \csc\theta F e_1^*, e_{n+d_1+2} = \tilde{e}_{d_1+2} = \csc\theta F e_2^*, \ldots, e_{n+d_1+p} = \tilde{e}_{d_1+p} = \csc\theta F e_p^*, e_{n+d_1+p+1} = \tilde{e}_{d_1+p+1} = \csc\theta\sec\theta F T e_1^*, \ldots, e_{n+d_1+2p} = \tilde{e}_{d_1+2p} = \csc\theta\sec\theta F T e_p^*\}$ and $\{e_{2n} = \tilde{e}_n, \ldots, e_{2m+1} = \tilde{e}_{2(m-n+1)}\}$.

Theorem 5.1 *Let $M = N_\theta \times_f N_\perp$ be a n-dimensional mixed geodesic warped product pseudo-slant submanifold of a $(2m+1)$-dimensional cosymplectic manifold $\widetilde{M}$ such that $\xi \in TN_\theta$, where N_θ is a proper slant submanifold of dimension $2p+1$ and $N_\perp$ is an anti-invariant submanifold of dimension d_1 of $\widetilde{M}$. Then*

(i) *The squared norm of the second fundamental form of M satisfies*

$$\|h\|^2 \geq d_1 \cot^2\theta \|\operatorname{grad}\ln f\|^2. \tag{5.1}$$

(ii) *If the equality sign of (5.1) holds identically, then N_θ is totally geodesic and $N_\perp$ is totally umbilical in $\widetilde{M}$.*

Proof. From definition $\|h\|^2 = \|h(\mathcal{D},\mathcal{D}\|^2 + 2\|h(\mathcal{D},\mathcal{D}^\perp)\|^2 + \|h(\mathcal{D}^\perp,\mathcal{D}^\perp\|^2$, where $\mathcal{D} = \mathcal{D}^\theta \oplus \langle\xi\rangle$. Since M is mixed geodesic, hence the middle term of right hand side should be identically zero. Then we have

$$\|h\|^2 = \sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{2p+1} g(h(e_i^*, e_j^*), e_r)^2 + \sum_{r=n+1}^{2m+1}\sum_{l,k=1}^{d_1} g(h(e_l, e_k), e_r)^2.$$

The above equation can be separated for the $\phi\mathcal{D}^\perp$, $F\mathcal{D}^\theta$ and μ components as follows

$$\begin{aligned}\|h\|^2 = &\sum_{r=1}^{d_1}\sum_{i,j=1}^{2p+1} g(h(e_i^*, e_j^*), \tilde{e}_r)^2 + \sum_{r=d_1+1}^{2p+d_1}\sum_{i,j=1}^{2p+1} g(h(e_i^*, e_j^*), \tilde{e}_r)^2 \\ &+ \sum_{r=n}^{2(m-n+1)}\sum_{i,j=1}^{2p+1} g(h(e_i^*, e_j^*), \tilde{e}_r)^2 + \sum_{r=1}^{d_1}\sum_{l,k=1}^{d_1} g(h(e_l, e_k), \tilde{e}_r)^2 \\ &+ \sum_{r=d_1+1}^{2p+d_1}\sum_{l,k=1}^{d_1} g(h(e_l, e_k), \tilde{e}_r)^2 + \sum_{r=n}^{2(m-n+1)}\sum_{l,k=1}^{d_1} g(h(e_l, e_k), \tilde{e}_r)^2.\end{aligned} \tag{5.2}$$

Using (4.3)-(4.6), the first term of right hand side in (5.2) vanishes identically and we shall leave all the terms except the fifth term in (5.2), then we get

$$\|h\|^2 \geq \sum_{r=d_1+1}^{2p+d_1}\sum_{l,k=1}^{d_1} g(h(e_l, e_k), \tilde{e}_r)^2.$$

Using the frame of $F\mathcal{D}^\theta$, we arrive at

$$\begin{aligned}\|h\|^2 &\geq \sum_{j=1}^{p}\sum_{l,k=1}^{d_1} g(h(e_l,e_k), \csc\theta F e_j^*)^2\\ &+\sum_{j=1}^{p}\sum_{l,k=1}^{d_1} g(h(e_l,e_k), \csc\theta\sec\theta FTe_j^*)^2.\end{aligned}$$

Hence, by Lemma 4.3, we obtain

$$\begin{aligned}\|h\|^2 &\geq \csc^2\theta\sum_{j=1}^{p}\sum_{l,k=1}^{d_1}(Te_j^*\ln f)^2 g(e_l,e_k)^2+\cot^2\theta\sum_{j=1}^{p}\sum_{l,k=1}^{d_1}(e_j^*\ln f)^2 g(e_l,e_k)^2\\ &= d_1(\csc^2\theta\sum_{j=1}^{p}(Te_j^*\ln f)^2+\cot^2\theta\sum_{j=1}^{p}(e_j^*\ln f)^2)\\ &= d_1(\csc^2\theta\sum_{j=1}^{p}g(e_j^*, T\operatorname{grad}\ln f)^2+\cot^2\theta\sum_{j=1}^{p}(e_j^*\ln f)^2).\end{aligned}$$

To satisfy (4.2), the above expression can be simplified as

$$\begin{aligned}\|h\|^2 &\geq d_1[\csc^2\theta(\|T\operatorname{grad}\ln f\|^2-\sum_{j=1}^{p}g(e_{p+j}^*, T\operatorname{grad}\ln f)^2)\\ &+\cot^2\theta\sum_{j=1}^{p}(e_j^*\ln f)^2], \quad (\text{since } T\operatorname{grad}\ln f\in\mathcal{D} \text{ and } T\xi=0)\\ &= d_1[\csc^2\theta(\|T\operatorname{grad}\ln f\|^2-\cos^2\theta\sum_{j=1}^{p}g(e_j^*,\operatorname{grad}\ln f)^2)\\ &+\cot^2\theta\sum_{j=1}^{p}(e_j^*\ln f)^2]\\ &= d_1[\cot^2\theta\|\operatorname{grad}\ln f\|^2-\cot^2\theta\sum_{j=1}^{p}(e_j^*\ln f)^2+\cot^2\theta\sum_{j=1}^{p}(e_j^*\ln f)^2].\end{aligned}$$

Thus, the inequality (5.1) follows from the last expression. If the equality holds in (5.1), then from the leaving terms in (5.2), we get the following relations from the second and the third leaving terms of (5.2) $g(h(\mathcal{D},\mathcal{D}),F\mathcal{D}^\theta)=0$, $g(h(\mathcal{D},\mathcal{D}),\mu)=0$ which implies

$$h(\mathcal{D},\mathcal{D})\perp F\mathcal{D}^\theta,\quad h(\mathcal{D},\mathcal{D})\perp\mu\ \Rightarrow h(\mathcal{D},\mathcal{D})\in\varphi\mathcal{D}^\perp. \tag{5.3}$$

Also, for a mixed geodesic warped product submanifold we have from Lemma 4.2 that $g(h(\mathcal{D},\mathcal{D}),\varphi\mathcal{D}^\perp)=0$, which means that

$$h(\mathcal{D},\mathcal{D})\perp\varphi\mathcal{D}^\perp. \tag{5.4}$$

Then (5.3) and (5.4) imply that $h(\mathcal{D},\mathcal{D})=0$, using this relation with the fact that N_θ is totally geodesic in M [1], we conclude that N_θ is totally geodesic in $\widetilde{M}$.

Also, from the leaving forth and the sixth terms of (5.2) in right hand side, we conclude that $g(h(\mathcal{D}^\perp,\mathcal{D}^\perp),\varphi\mathcal{D}^\perp)=0$, $g(h(\mathcal{D}^\perp,\mathcal{D}^\perp),\mu)=0$ which means that

$$h(\mathcal{D}^\perp,\mathcal{D}^\perp)\perp\varphi\mathcal{D}^\perp,\quad h(\mathcal{D}^\perp,\mathcal{D}^\perp)\perp\mu\ \Rightarrow h(\mathcal{D}^\perp,\mathcal{D}^\perp)\in F\mathcal{D}^\theta. \tag{5.5}$$

Also, for a mixed geodesic warped product submanifold, from Lemma 4.3 (i), we have

$$g(h(Z,W),FX)=(TX\ln f)g(Z,W), \tag{5.6}$$

for any $X\in TN_\theta$ and $Z,W\in TN_\perp$. Hence, by the relations (5.5), (5.6) and the fact that $N_\perp$ is totally umbilical in M [1], we find that $N_\perp$ is totally umbilical in $\widetilde{M}$. This completes the proof. □

Example 5.1 ([3]) Consider the product of a compact Kaehler manifold (V,J,g_1) with a circle S^1. Then the cosymplectic structure (φ,ξ,η,g) on a product manifold $\widetilde{M}=V\times S^1$ is defined by $\varphi=J\circ(pr_1)_*$, $\xi=\frac{E}{c}$, $\eta=c(pr_2)_*(\theta)$, $g=(pr_1)_*(g_1)+(pr_2)_*c^2(\theta\otimes\theta)$ where pr_1 and pr_2 are the projections of $\widetilde{M}$ to V and S^1, respectively and $*$ is the symbol for the tangent maps, θ is the length element of S^1, E is its dual vector field and c is real number, $c\neq 0$. If $c>0$, then $\widetilde{M}$ is a warped product manifold of base V and fiber S^1 with the warping function c.

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