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Rickart modules determined by preradicals $\overline{Z}(\cdot)$ and $\delta(\cdot)$

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Abstract Let R be an arbitrary ring with identity and M a right R-module with the endomorphism ring $S = \mathrm{End}_R(M)$. Let F be a fully invariant submodule of M. The module M is called *F-inverse split* if for all $f \in S$, $f^{-1}(F)$ is a direct summand of M. This definition produces Rickart modules in the sense of Lee, Rizvi and Roman, namely, if M is F-inverse split, then M/F is Rickart. In this paper, we specialize the fully invariant submodule F of the module M as $\overline{Z}(M)$ and $\delta(M)$, and study various basic characterizations and properties of $\overline{Z}(\cdot)$-inverse split modules and $\delta(\cdot)$-inverse split modules.

Keywords $\overline{Z}(\cdot)$-inverse split module · $\delta(\cdot)$-inverse split module · Rickart module

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1 Introduction

Throughout this paper R denotes a ring with identity, modules are unital right R-modules and $S = \mathrm{End}_R(M)$ denotes the ring of all right R-module endomorphisms of a module M unless otherwise stated. Principally projective rings were introduced by Hattori [5] to study the torsion theory, that is, a ring is called *left (right) principally projective* if every principal left (right) ideal is projective. This is equivalent to the left (right) annihilator of any element of the ring is generated by an idempotent as a left (right) ideal, i.e., the ring is *left (right) Rickart*. The concept of Rickart rings was extended by Rizvi and Roman [13] to the general module theoretic setting, and investigated in [1], [7] and [8]. A module M is called *Rickart* if for any $f \in S$, $\mathrm{Ker}\, f$ is a direct summand of M. In [3], a concept of T-Rickart modules was defined, that is, a module M is called *T-Rickart* if $f^{-1}(Z_2(M))$ is a direct summand of M for

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every $f \in S$ where $Z_2(M)$ is the second singular (or Goldie torsion) submodule of M. Motivated by the works on Rickart modules and T-Rickart modules, in [15], the present authors concerned about the conditions under which fully invariant submodules produce Rickart modules. Recall that a submodule F of a module M is called *fully invariant* if $fF \subseteq F$ for every $f \in S$. Let F be a fully invariant submodule of a module M. Then M is said to be *F-inverse split* [15] if $f^{-1}(F)$ is a direct summand of M for every $f \in S$.

In [14], Talebi and Vanaja introduced *cosingular submodule* of a module M as $\overline{Z}(M) = \bigcap\{\mathrm{Ker}\, g \mid g \in \mathrm{Hom}(M, L), L \in \mathcal{S}\}$ where $\mathcal{S}$ is the class of all small modules. On the other hand, for a module M, the submodule $\delta(M) = \bigcap\{N \leq M \mid M/N \text{ is singular simple}\}$ was defined by Zhou in [18]. The submodules $\overline{Z}(M)$ and $\delta(M)$ are fully invariant in M. It is a natural question that what kind of additional properties do the fully invariant submodules $\overline{Z}(M)$ and $\delta(M)$ of a module M provide to M in terms of inverse splitness? Motivated by this question, our purpose is to define $\overline{Z}(M)$-inverse split modules and $\delta(M)$-inverse split modules, and investigate their properties in Section 3 and Section 4, respectively. A module M being $\overline{Z}(M)$-inverse split or $\delta(M)$-inverse split reveals out some properties of M. Let $\tau(\cdot)$ denote $\overline{Z}(\cdot)$ or $\delta(\cdot)$ for short in the following. It is shown that M is $\tau(M)$-inverse split if and only if $M = \tau(M) \oplus N$ where N is Rickart if and only if M has a decomposition $M = K \oplus N$ such that N is Rickart, $\tau(N) = 0$ and $\tau(K) = K$. We investigate some classes of rings over which every module is $\overline{Z}(\cdot)$-inverse split. We also determine δ-semiperfect $\delta(\cdot)$-inverse split modules, namely, they are exactly the modules M with $M = \delta(M)$. On the other hand, we consider the concepts of $\overline{Z}(\cdot)$-inverse splitness and $\delta(\cdot)$-inverse splitness for the rings. If R is right $\tau(R_R)$-inverse split, then for any $e^2 = e \in R$, we show that the R-module eR has a what kind of inverse splitting property. Also, any ring R which has an R-module decomposition $R = \tau(R_R) \oplus I$ with I a hereditary module is characterized in terms of $\tau(\cdot)$-inverse splitness.

In what follows, by $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{Z}_n$ we denote, respectively, integers, rational numbers and the ring of integers modulo n. For a module M, $E(M)$ and $\mathrm{Rad}(M)$ are the injective hull and the radical of M, also, $J(R)$ and $\mathrm{Soc}(R_R)$ stand for the Jacobson radical and the right socle of a ring R, respectively.

2 Preliminaries

Let *Mod-R* denote the category of unitary right R-modules. Recall that a functor $\tau :$ *Mod-R* $\to$ *Mod-R* is called a *preradical* on *Mod-R* if it satisfies the following properties:

(1) For every right R-module M, $\tau(M) \leq M$.
(2) For any homomorphism $f\colon M \to N$ in *Mod-R*, $f(\tau(M)) \leq \tau(N)$ and $\tau(f)$ is the restriction of f to $\tau(M)$.

The beforementioned $\overline{Z}$ and δ are examples for preradicals. A preradical τ is said to be *hereditary* if for any submodule $K \leq M$, $\tau(K) = K \cap \tau(M)$. It is known that any fully invariant submodule defines a preradical and so a torsion theory. Also it is clear that for any preradical τ and a module M, $\tau(M)$ is fully invariant in M.

In [15], $\tau(\cdot)$-inverse split modules for any preradical τ on *Mod-R* is introduced. By using the relations between preradicals and fully invariant submodules, a module M

is called *F-inverse split* [15] if $f^{-1}(F)$ is a direct summand of M for every $f \in S$ where F is a fully invariant submodule of M. In [15], a basic characterization of an F-inverse split module M is given as the following.

Theorem 2.1 ([15], Theorem 2.3) *The following are equivalent for a module M.*

(1) *M is F-inverse split.*
(2) *$M = F \oplus N$ where N is a Rickart module.*

In the next sections of this paper, we consider preradicals $\overline{Z}$ and δ, and so the fully invariant submodules $\overline{Z}(M)$ and $\delta(M)$ of a module M. Theorem 2.1 is a common approach for parts of Theorem 3.3 and Theorem 4.5.

3 $\overline{Z}(\cdot)$-Inverse split modules

A submodule N of a module M is said to be *small* (*or superfluous*) in M if $M = N+K$ for any submodule K of M implies $M = K$. In [9], a module M is called *small* if it is a small submodule of some module and it is shown that M is small if and only if it is a small submodule of $E(M)$. We note some known facts about small modules and small submodules to be used in the sequel.

(1) Submodules, quotient modules and finite direct sums of small modules are small.

(2) If M is a module with submodules N, K such that $N \subseteq K \subseteq M$ and N is small in K, then N is small in M.

(3) If N is a small submodule of a module M and N is contained in a direct summand K of M, then N is small in K.

(4) Let M be a module and $\mathcal{S}$ denote the class of all small modules. In [14], Talebi and Vanaja define $\overline{Z}(M)$ as follows:

$$\overline{Z}(M) = Rej_M(\mathcal{S}) = \bigcap\{\operatorname{Ker} g \mid g \in \operatorname{Hom}(M, L), L \in \mathcal{S}\}$$

It is shown that $\overline{Z}(M)$ is a submodule of M and it is called a *cosingular submodule* of M. Also, $\overline{Z}(M) = \bigcap\{N \leq M \mid M/N \text{ is small}\}$. The module M is called *cosingular* if $\overline{Z}(M) = 0$ and it is said to be *noncosingular* if $\overline{Z}(M) = M$. Clearly, every small module is cosingular. The class $\mathcal{F}_{\mathcal{S}}$ of all cosingular modules is closed under submodules, direct products and extensions (see [14, Corollary 2.2]). The class $\mathcal{T}_{\mathcal{S}}$ of all noncosingular modules is closed under taking factor modules, extensions, and arbitrary direct sums, therefore the pair $\tau_S = (\mathcal{T}_{\mathcal{S}}, \mathcal{F}_{\mathcal{S}})$ forms a torsion theory. It is also a torsion theory cogenerated by small modules. It is easily checked that $\mathcal{F}_{\mathcal{S}}$ consists of all modules for which every nonzero submodule is not noncosingular.

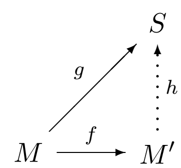
To each module M, there corresponds a submodule called noncosingular submodule denoted by $\tau_{\mathcal{S}}(M)$ which is the largest noncosingular submodule of M, that is, $\tau_{\mathcal{S}}(M)$ is the largest noncosingular submodule of M that belongs to $\mathcal{T}_{\mathcal{S}}$. Set $\overline{Z}^0(M) = M$, $\overline{Z}^1(M) = \overline{Z}(M)$ and define inductively $\overline{Z}^\alpha(M)$ for any ordinal α. Thus if α is not a limit ordinal, $\overline{Z}^\alpha(M) = \overline{Z}\big(\overline{Z}^{\alpha-1}(M)\big)$, while if α is a limit ordinal, $\overline{Z}^\alpha(M) = \bigcap_{\beta<\alpha} \overline{Z}^\beta(M)$. It is easy to see that if N is a noncosingular submodule of M, then $N \subseteq \overline{Z}^\alpha(M)$ for every ordinal α. Hence $\tau_{\mathcal{S}}(M) = \bigcap_\alpha \overline{Z}^\alpha(M)$. The following lemma summarize various properties of the preradical $\overline{Z}(\cdot)$ so that they may be easily referenced in the paper.

Lemma 3.1 *Let M and M' be R-modules. Then the following hold:*

1. *If $M \xrightarrow{f} M'$ is a homomorphism, then $f\big(\overline{Z}(M)\big) \subseteq \overline{Z}(M')$.*
2. *If $M \xrightarrow{f} M'$ is an epimorphism and Ker f is noncosingular, then*
 (a) $f^{-1}(\overline{Z}(M')) \subseteq \overline{Z}(M)$.
 (b) $f\big(\overline{Z}(M)\big) = \overline{Z}(M')$.
3. *If N is any submodule of M, then $\overline{Z}(N) \subseteq \overline{Z}(M)$.*
4. *Let $(M_i)_{i\in I}$ be any collection of R-modules. Then $\overline{Z}(\bigoplus_{i\in I} M_i) = \bigoplus_{i\in I} \overline{Z}(M_i)$.*
5. *$M/\overline{Z}(M)$ is cosingular.*

Proof. (1) Since $\overline{Z}$ is a preradical, (1) holds by the definition.

(2)(a) Let $x \in f^{-1}(\overline{Z}(M'))$. Then $f(x) \in \overline{Z}(M')$. Let $M \xrightarrow{g} S$ be a homomorphism



where S is a small module. Since Ker f is noncosingular, Ker $f \subseteq$ Ker g, for if Ker f is noncosingular, then $\overline{Z}\big(\text{Ker} f\big) = \text{Ker} f$. Ker $f \subseteq M$ implies $\overline{Z}(\text{Ker} f) \subseteq \overline{Z}(M)$ and so Ker $f \subseteq \overline{Z}(M) \subseteq$ Ker g. Then there exists a homomorphism $M' \xrightarrow{h} S$ such that $hf = g$. Since $f(x) \in \overline{Z}(M')$, $h(f(x)) = (hf)(x) = 0$. It implies $g(x) = (hf)(x) = 0$. Hence $x \in$ Ker g for every g from M to the small module S. Hence $x \in \overline{Z}(M)$.

(2)(b) Since f is an epimorphism, $f\big(f^{-1}(\overline{Z}(M')\big) = \overline{Z}(M')$. Then by (2)(a), $\overline{Z}(M') = f\big(f^{-1}(\overline{Z}(M')\big) \subseteq f\big(\overline{Z}(M)\big)$. We invoke (1) to conclude that $f\big(\overline{Z}(M)\big) = \overline{Z}(M')$.

(3) and (4) are clear.

(5) Let $x + \overline{Z}(M) \in \overline{Z}(M/\overline{Z}(M))$ and $f : M \to S$ be a homomorphism where S is any small module. Since $\overline{Z}(M) \subseteq$ Ker f, by the Factor Theorem, there exists a homomorphism $g : M/\overline{Z}(M) \to S$ such that $f = g\pi$ where $\pi : M \to M/\overline{Z}(M)$ is the natural projection. Then $f(x) = g\pi(x) = g(x + \overline{Z}(M)) = 0$ because of $x + \overline{Z}(M) \in \overline{Z}(M/\overline{Z}(M))$. Hence $x \in$ Ker f, and so $x \in \overline{Z}(M)$. Therefore $\overline{Z}(M/\overline{Z}(M)) = 0$, i.e., $M/\overline{Z}(M)$ is cosingular. □

Now we introduce one of the main definitions of this paper.

Definition 3.2 *A module M is called $\overline{Z}(M)$-inverse split if $f^{-1}(\overline{Z}(M))$ is a direct summand of M for every $f \in S$.*

Obviously, every noncosingular module is $\overline{Z}(\cdot)$-inverse split. It is easily deduced that Rickartness and $\overline{Z}(\cdot)$-inverse splitness are the same for cosingular modules.

Example 3.1 Let M be a semisimple injective module which is not projective. Since M is injective, any nonzero direct summand of M is not small. Let S be a small module and $f : M \to S$ a homomorphism. Then $M = \text{Ker} f \oplus K$. The submodule K is small as it is isomorphic to a submodule of small module S. It follows that $K = 0$. Hence Ker $f = M$ and so $\overline{Z}(M) = M$. Therefore M is $\overline{Z}(M)$-inverse split.

An idempotent e of a ring R is called *left semicentral* if $Re = eRe$, equivalently, for every $x \in R$, $xe = exe$. Now we characterize $\overline{Z}(\cdot)$-inverse split modules.

Theorem 3.3 *The following are equivalent for a module M.*

(1) *M is $\overline{Z}(M)$-inverse split.*
(2) *$M = \overline{Z}(M) \oplus N$ where N is a cosingular Rickart module and $\overline{Z}(M) = eM$ for some left semicentral $e^2 = e \in S$.*
(3) *$M = \overline{Z}(M) \oplus N$ where N is a Rickart module.*
(4) *M has a decomposition $M = K \oplus N$ where K is noncosingular and N is cosingular Rickart.*

Proof. (1) ⇒ (2) Let M be an $\overline{Z}(M)$-inverse split module and 1_M denote the identity endomorphism of M. Then $1_M^{-1}(\overline{Z}(M)) = \overline{Z}(M)$ is a direct summand of M. Let $M = \overline{Z}(M) \oplus N$ for some submodule N of M and $f \in \mathrm{End}_R(N)$. Hence $1_{\overline{Z}(M)} \oplus f \in S$, say $g = 1_{Z(M)} \oplus f$. This implies that $g^{-1}(\overline{Z}(M)) = \overline{Z}(M) \oplus \mathrm{Ker}\, f$. By assumption, $g^{-1}(\overline{Z}(M))$ is a direct summand of M. It follows that $\mathrm{Ker}\, f$ is a direct summand of N. Therefore N is Rickart. Also, by Lemma 3.1(5), N is cosingular. On the other hand, since $\overline{Z}(M)$ is a direct summand of M, $\overline{Z}(M) = eM$ for some $e^2 = e \in S$. For any $h \in S$, $he = ehe$ due to $heM \leq eM$. So e is left semicentral.

(3) ⇒ (1) Assume that $M = \overline{Z}(M) \oplus N$ where N is a Rickart module. Let $f \in S$ and π_N denote the projection on N along $\overline{Z}(M)$. Then $\pi_N f_{|_N} \in \mathrm{End}_R(N)$ and $\overline{Z}(M)$ being fully invariant implies that $f^{-1}(\overline{Z}(M)) = \overline{Z}(M) \oplus \mathrm{Ker}\,(\pi_N f_{|_N})$. Since N is Rickart, $\mathrm{Ker}\,(\pi_N f_{|_N})$ is a direct summand of N, and so $f^{-1}(\overline{Z}(M))$ is a direct summand of M as desired.

(2) ⇒ (4) obvious.

(4) ⇒ (3) Let $M = K \oplus N$ for a noncosingular submodule K and a cosingular Rickart submodule N. Due to Lemma 3.1(4), we have $\overline{Z}(M) = K$. □

It is known that $\mathbb{Z}$ is a Rickart $\mathbb{Z}$-module and $\overline{Z}(\mathbb{Z}) = 0$, and so $\mathbb{Z}$ is $\overline{Z}(\mathbb{Z})$-inverse split. On the other hand, if we consider $\mathbb{Q}$ as a $\mathbb{Z}$-module, then $\overline{Z}(\mathbb{Q}) = \mathbb{Q}$, thus $\mathbb{Q}$ is also $\overline{Z}(\mathbb{Q})$-inverse split. The next results provide another source of examples for $\overline{Z}(\cdot)$-inverse split modules.

Example 3.2 Let R be a ring such that $\mathcal{F}_{\mathcal{S}}$ is closed under taking homomorphic images and every module belonging to $\mathcal{F}_{\mathcal{S}}$ is projective, then every module M is $\overline{Z}(M)$-inverse split.

Proof. Let M be a module. By Lemma 3.1(5), $\overline{Z}(M/\overline{Z}(M)) = 0$ and so $M/\overline{Z}(M)$ is cosingular. Hence it belongs to $\mathcal{F}_{\mathcal{S}}$ and so it is projective. There exists a submodule K of M with $M = \overline{Z}(M) \oplus K$. Then $\overline{Z}(K) = 0$ and so K belongs to $\mathcal{F}_{\mathcal{S}}$. Let L be any submodule of K. By hypothesis K/L also belongs to $\mathcal{F}_{\mathcal{S}}$. Hence K/L is projective, and therefore L is a direct summand of K. It follows that K is semisimple. Since every semisimple module is Rickart, K is Rickart. By Theorem 3.3, M is $\overline{Z}(M)$-inverse split. □

Theorem 3.4 *Let R be a right hereditary ring. Then the following hold:*

1. *If every cosingular R-module is projective, then every R-module M is* $\overline{Z}(M)$*-inverse split.*
2. *Every injective R-module M is* $\overline{Z}(M)$*-inverse split.*

Proof. (1) Let M be a module. By Lemma 3.1(5), $M/\overline{Z}(M)$ is cosingular, and so it is projective. It follows that $M = \overline{Z}(M) \oplus N$ for some submodule N of M. Since N is also projective, R being right hereditary implies that N is Rickart due to [7, Theorem 2.26]. Therefore M is $\overline{Z}(M)$-inverse split by Theorem 3.3.

(2) Because of [14, Proposition 2.7], R being right hereditary implies that every injective R-module M is noncosingular, and so M is $\overline{Z}(M)$-inverse split. □

Let R be a ring. In the literature, R is called a *right V-ring* if every simple R-module is injective. By [6, Theorem 3.75], R is a right V-ring if and only if for every R-module M, $\mathrm{Rad}(M) = 0$.

Proposition 3.5 *Every module M over a right V-ring is* $\overline{Z}(M)$*-inverse split.*

Proof. Let R be a right V-ring and M an R-module. Consider any homomorphism $f : M \to S$ where S is a small module. Then $S \leq \mathrm{Rad}(T)$ for some R-module T. Since R is a right V-ring, we have $S = 0$. Hence $M = \overline{Z}(M)$. Therefore M is $\overline{Z}(M)$-inverse split. □

Proposition 3.6 *Every indecomposable* $\overline{Z}(M)$*-inverse split module M is either cosingular, Rickart, cogenerated by small modules and every endomorphism of M is a monomorphism or M is noncosingular.*

Proof. Let M be an indecomposable $\overline{Z}(M)$-inverse split module and 1_M denote the identity endomorphism of M. Since $1_M^{-1}(\overline{Z}(M)) = \overline{Z}(M)$ is a direct summand of M, $\overline{Z}(M) = 0$ or $\overline{Z}(M) = M$. Assume that $\overline{Z}(M) = 0$. Then M is Rickart. For if, $0 \neq f \in S$, then $f^{-1}(\overline{Z}(M)) = f^{-1}(0)$ is a direct summand. So $f^{-1}(0) = M$ or $f^{-1}(0) = 0$. $f^{-1}(0) = M$ implies $f = 0$, and $f^{-1}(0) = 0$ means that f is a monomorphism. Since $\bigcap\{\mathrm{Ker}\, g \mid g : M \to L, L \in \mathcal{S}\} = \overline{Z}(M) = 0$, M can be embedded in $\prod_g (M/\mathrm{Ker}\, g)$. This completes the proof. □

Proposition 3.7 *Let M be a module. If there exists a homomorphism* $f : M \to S$ *with* $\mathrm{Ker} f$ *small in M where S is a small module, then M is* $\overline{Z}(M)$*-inverse split if and only if M is cosingular and Rickart.*

Proof. Let M be a $\overline{Z}(M)$-inverse split module. By Theorem 3.3, $M = \overline{Z}(M) \oplus N$ for some Rickart module N. Since $\overline{Z}(M) \subseteq \mathrm{Ker}\, f$, we have $M = \mathrm{Ker}\, f + N$. Then $\mathrm{Ker}\, f$ being small in M implies that $M = N$. Thus M is Rickart and $\overline{Z}(M) = 0$. The converse is obvious. □

Recall that a module M is called $\overline{Z}(M)$*-lifting* ($\overline{Z}(M)$*-semiperfect*) if for any submodule N of M, there exists a (projective) direct summand A of M such that $A \subseteq N$ and $M = A \oplus B$ with $N \cap B \subseteq \overline{Z}(M)$. Obviously, every $\overline{Z}(M)$-semiperfect module M is $\overline{Z}(M)$-lifting.

Proposition 3.8 *Let R be a* $\overline{Z}(R_R)$*-lifting ring. Then the following hold:*

(1) *$R/\overline{Z}(R_R)$ is semisimple and R has a decomposition $R = A \oplus B$ with A semisimple and $\overline{Z}(R_R)$ is an essential right ideal in B and $B/\overline{Z}(R_R)$ is semisimple right R-module. Moreover, R is right $\overline{Z}(R_R)$-inverse split if and only if $R = \overline{Z}(R_R) \oplus A$ where A is semisimple.*
(2) *Every $R/\overline{Z}(R_R)$-module M has a decomposition $M = M_1 \oplus M_2$ with M_1 a semisimple $R/\overline{Z}(R_R)$-module and $\overline{Z}(M)$ essential in M_2 as an $R/\overline{Z}(R_R)$-module.*

Proof. (1) Assume that R is a $\overline{Z}(R_R)$-lifting ring. Let I be a right ideal of R. There exists $A \subseteq I$ such that $R = A \oplus B$ and $B \cap I \subseteq \overline{Z}(R_R)$. It is obvious that $(I + \overline{Z}(R_R))/\overline{Z}(R_R)$ is a direct summand of $R/\overline{Z}(R_R)$. So $R/\overline{Z}(R_R)$ is semisimple. By [10, Proposition 2.1], there exist right ideals A, B of R such that $R = A \oplus B$, A is semisimple, $\overline{Z}(R_R)$ is essential in B and $B/\overline{Z}(R_R)$ is semisimple. Assume further that R is right $\overline{Z}(R_R)$-inverse split. There exists a right ideal C such that $R = \overline{Z}(R_R) \oplus C$. Then $B = \overline{Z}(R_R) \oplus (B \cap C)$. Since $\overline{Z}(R_R)$ is essential in B, $B \cap C = 0$. Hence $R = \overline{Z}(R_R) \oplus A$ and A is semisimple. The converse is obvious by Theorem 3.3 since every semisimple module is Rickart.

(2) Let M be any module. Then $M\overline{Z}(R_R) \subseteq \overline{Z}(M)$. Hence $M/\overline{Z}(M)$ is a semisimple $R/\overline{Z}(R_R)$-module. Again by [10, Proposition 2.1], there exist submodules M_1 and M_2 of M such that $M = M_1 \oplus M_2$ where M_1 is semisimple right $R/\overline{Z}(R_R)$-module and $\overline{Z}(M)$ is essential in M_2 as an $R/\overline{Z}(R_R)$-module and $M_2/\overline{Z}(M)$ is semisimple as an $R/\overline{Z}(R_R)$-module. □

Theorem 3.9 *Let M be a $\overline{Z}(M)$-semiperfect module with ascending chain condition on direct summands. Then M has a decomposition $M = \overline{Z}(M) \oplus N$ where $\overline{Z}(M)$ is projective and N is semisimple. In this case, M is $\overline{Z}(M)$-inverse split.*

Proof. Let M be a $\overline{Z}(M)$-semiperfect module. There exists a projective direct summand $A_1 \subseteq \overline{Z}(M)$ such that $M = A_1 \oplus B_1$. Then $\overline{Z}(M) = A_1 \oplus \big(B_1 \cap \overline{Z}(M)\big)$. Assume that $B_1 \cap \overline{Z}(M) \neq 0$, by the same reasoning, there exists a projective direct summand $A_2 \subseteq B_1 \cap \overline{Z}(M)$ such that $M = A_2 \oplus B_2$. Then $B_1 = A_2 \oplus \big(B_2 \cap B_1\big)$ and $\overline{Z}(M) = A_2 \oplus A_1 \oplus \big(B_2 \cap B_1 \cap \overline{Z}(M)\big)$ and $M = A_1 \oplus A_2 \oplus \big(B_2 \cap B_1\big)$. If $B_2 \cap B_1 \cap \overline{Z}(M) \neq 0$, then there exists a projective direct summand $A_3 \subseteq B_2 \cap B_1 \cap \overline{Z}(M)$ such that $M = A_3 \oplus B_3$. Then $B_2 \cap B_1 = A_3 \oplus (B_3 \cap B_2 \cap B_1)$, $M = A_3 \oplus A_2 \oplus A_1 \oplus \big(B_3 \cap B_2 \cap B_1\big)$ and $\overline{Z}(M) = A_3 \oplus A_2 \oplus A_1 \oplus \big(B_3 \cap B_2 \cap B_1 \cap \overline{Z}(M)\big)$. Continuing in this way, for each positive integer n, if $B_{n-1} \cap B_{n-2} \cap \cdots \cap B_1 \cap \overline{Z}(M) \neq 0$, then we may find a projective direct summand A_n of M such that $M = A_n \oplus B_n$ and $A_n \subseteq B_{n-1} \cap B_{n-2} \cap \cdots \cap B_1 \cap \overline{Z}(M) \neq 0$. By hypothesis, there must be a positive integer t such that the sequence $A_1 \subseteq A_1 \oplus A_2 \subseteq A_1 \oplus A_2 \oplus A_3 \subseteq \cdots$ of direct summands of M terminates at the step t and then $A_1 \oplus A_2 \oplus \cdots \oplus A_t = A_1 \oplus A_2 \oplus \cdots \oplus A_t \oplus A_{t+1} = \cdots$. At this step $B_1 \cap B_2 \cap \cdots \cap B_t \cap \overline{Z}(M) = 0$ must be held. It follows that $M = A_t \oplus A_{t-1} \oplus \cdots \oplus A_1 \oplus (B_t \cap B_{t-1} \cap \cdots \cap B_1)$ and $\overline{Z}(M) = A_1 \oplus A_2 \oplus \cdots \oplus A_t$ is projective. Let $N = B_t \cap B_{t-1} \cap \cdots \cap B_1$ and L be a submodule of N. By hypothesis, there exists a projective direct summand L_1 of M such that $M = L_1 \oplus L_2$ and $L_1 \subseteq L$ and $L_2 \cap L \subseteq \overline{Z}(M)$. Then $L_2 \cap L = 0$. Hence $L = L_1$ is a direct summand of N. Thus N is semisimple. By Theorem 3.3, M is $\overline{Z}(M)$-inverse split. □

Proposition 3.10 *Let M be a $\overline{Z}(M)$-lifting module. Then M is $\overline{Z}(M)$-inverse split if and only if $M = \overline{Z}(M) \oplus N$ where N is semisimple.*

Proof. Assume that M is a $\overline{Z}(M)$-lifting module. Let K be a submodule of M with $\overline{Z}(M) \subseteq K$. There exists a direct summand A of M such that $A \subseteq K$, $M = A \oplus B$ and $K \cap B \subseteq \overline{Z}(M)$. Then $M/\overline{Z}(M) = K/\overline{Z}(M) \oplus (B + \overline{Z}(M))/\overline{Z}(M)$. Hence $M/\overline{Z}(M)$ is semisimple. Now suppose that M is a $\overline{Z}(M)$-inverse split module. Then there exists a submodule N of M such that $M = \overline{Z}(M) \oplus N$. Hence N is semisimple. The converse is clear due to Theorem 3.3. □

Proposition 3.11 *Let R be a ring and consider the following conditions:*

(1) *R is $\overline{Z}(R_R)$-semiperfect.*
(2) *R is semiperfect and $J(R) \subseteq \overline{Z}(R_R)$.*

If $\overline{Z}(R_R) \subseteq J(R)$, then (1) $\Rightarrow$ (2). Also, (2) $\Rightarrow$ (1) if R is right $\overline{Z}(R_R)$-inverse split.

Proof. Assume that R is $\overline{Z}(R_R)$-semiperfect and $\overline{Z}(R_R) \subseteq J(R)$. Then $R/\overline{Z}(R_R)$ is semisimple and idempotents lift strongly modulo $\overline{Z}(R_R)$. Hence idempotents lift modulo $\overline{Z}(R_R)$. On the other hand, $R/\overline{Z}(R_R)$ being semisimple implies $J(R) \subseteq \overline{Z}(R_R)$. It follows that R is semiperfect. Conversely, let R be a semiperfect right $\overline{Z}(R_R)$-inverse split ring and $J(R) \subseteq \overline{Z}(R_R)$. Then $R/J(R)$ is semisimple. Since $R/\overline{Z}(R_R)$ is a homomorphic image of $R/J(R)$, $R/\overline{Z}(R_R)$ is semisimple. By Theorem 3.3, $R = \overline{Z}(R_R) \oplus I$ for some right ideal I of R. Hence idempotents lift strongly modulo $\overline{Z}(R_R)$ due to [12, Example 3]. Therefore R is $\overline{Z}(R_R)$-semiperfect. □

Remark 3.1 Let R be a ring. If $J(R) \subseteq \overline{Z}(R_R)$, then $J(R)$ is a nilpotent ideal of R, and moreover, nilpotency index of $J(R)$ is 2. To see this, let $r \in J(R)$ and $f : R \to rR$ denote the homomorphism defined by $f(a) = ra$ for any $a \in R$. Since $rR \subseteq J(R)$, rR is a small right ideal of R, and so $\overline{Z}(R_R) \subseteq \operatorname{Ker} f$. Hence $r\overline{Z}(R_R) = 0$. It follows that $J(R)\overline{Z}(R_R) = 0$. Therefore $J(R) \subseteq \overline{Z}(R_R)$ implies $J(R)^2 = 0$.

The next result shows when the homomorphic images of $\overline{Z}(\cdot)$-inverse split modules have the same property.

Theorem 3.12 *Let M and M' be R-modules and $M \stackrel{f}{\to} M'$ an epimorphism. Suppose that M is $\overline{Z}(M)$-inverse split and $Ker f$ is noncosingular. Then M' is $\overline{Z}(M')$-inverse split.*

Proof. Assume that M is $\overline{Z}(M)$-inverse split. There exists a submodule K of M such that $M = \overline{Z}(M) \oplus K$. By Theorem 3.3, K is Rickart. Then $M' = f\big(\overline{Z}(M)\big) + f(K)$. Let $y \in f\big(\overline{Z}(M)\big) \cap f(K)$. There exist $x \in \overline{Z}(M)$ and $k \in K$ such that $y = f(x) = f(k)$. Then $x - k \in \operatorname{Ker} f$. Since $\operatorname{Ker} f$ is noncosingular, $\operatorname{Ker} f \subseteq \overline{Z}(M)$. So $x \in \overline{Z}(M)$ and $x - k \in \operatorname{Ker} f$ implies $k \in \overline{Z}(M)$. Hence $k = 0$ and so $y = 0$. It follows that $M' = f\big(\overline{Z}(M)\big) \oplus f(K)$. $\operatorname{Ker} f \subseteq \overline{Z}(M)$ implies that the restriction of f to K is an isomorphism from K onto $f(K)$. Since K is Rickart, so is $f(K)$. This and Theorem 3.3 complete the proof. □

In the following, we prove that every direct summand of a $\overline{Z}(\cdot)$-inverse split module inherits this property.

Lemma 3.13 *Every direct summand N of a $\overline{Z}(M)$-inverse split module M is also $\overline{Z}(N)$-inverse split.*

Proof. Let $M = N \oplus K$ be a $\overline{Z}(M)$-inverse split module with N and K submodules of M. By Lemma 3.1(4), $\overline{Z}(M) = \overline{Z}(N) \oplus \overline{Z}(K)$. There exists a submodule L of N such that $N = \overline{Z}(N) \oplus L$. Then $\overline{Z}(L) = 0$. Let $f \in \mathrm{End}(L)$. Consider $g = 1_{(\overline{Z}(N)\oplus K)} \oplus f$ where $1_{\overline{Z}(N)\oplus K}$ denotes the identity endomorphism of $\overline{Z}(N) \oplus K$. Then $g^{-1}(\overline{Z}(M)) = \overline{Z}(N) \oplus \overline{Z}(K) \oplus \operatorname{Ker} f$ is a direct summand of M. It follows that $\operatorname{Ker} f$ is a direct summand of L, and so L is Rickart. By Theorem 3.3, N is $\overline{Z}(N)$-inverse split. □

Corollary 3.14 *If R is a right $\overline{Z}(R_R)$-inverse split ring, then eR is a $\overline{Z}(eR)$-inverse split R-module for every $e^2 = e \in R$.*

Now we end this section by characterizing a ring R which has a decomposition $R = \overline{Z}(R_R) \oplus I$ with I a hereditary module in terms of $\overline{Z}(\cdot)$-inverse splitting property.

Theorem 3.15 *The following are equivalent for a ring R.*

(1) *Every free R-module M is $\overline{Z}(M)$-inverse split.*
(2) *Every projective R-module M is $\overline{Z}(M)$-inverse split.*
(3) *$R = \overline{Z}(R_R) \oplus I$ where I is a hereditary Rickart R-module.*
(4) *$R = \overline{Z}(R_R) \oplus I$ where I is a hereditary R-module.*

Proof. (1) ⇒ (2) Let P be a projective module. There exists a free module F such that $F = P \oplus K$. (1) implies that F is $\overline{Z}(F)$-inverse split. By Lemma 3.13, P is $\overline{Z}(P)$-inverse split.

(2) ⇒ (3) By (2), R is $\overline{Z}(R_R)$-inverse split. Then $R = \overline{Z}(R_R) \oplus I$, and by Theorem 3.3, I is a Rickart R-module. Let N be a submodule of I. There exists a short exact sequence $F \xrightarrow{f} N \to 0$ with F a free R-module. Consider the sequence of right R-module maps $F \xrightarrow{f} N \xrightarrow{\iota_1} I \xrightarrow{\iota_2} R \xrightarrow{\iota_3} F$ where ι_1, ι_2 and ι_3 are inclusions. So $\operatorname{Ker}(\iota_3\iota_2\iota_1 f) = \operatorname{Ker} f$. We identify f with $\iota_3\iota_2\iota_1 f$ and we may assume $f \in \mathrm{End}_R(F)$. Clearly, $\operatorname{Ker} f \subseteq f^{-1}(\overline{Z}(F))$. Let $x \in f^{-1}(\overline{Z}(F))$. Hence $f(x) \in \overline{Z}(F)$. So $\iota_1 f(x) \in \iota_1\overline{Z}(F) \subseteq \overline{Z}(I) = 0$. Hence $\iota_1 f(x) = 0$ and so $f(x) = 0$ or $x \in \operatorname{Ker} f$. Accordingly, $\operatorname{Ker} f = f^{-1}(\overline{Z}(F))$. By (2), F being $\overline{Z}(F)$-inverse split implies that $\operatorname{Ker} f$ is a direct summand of F. Since $F/\operatorname{Ker} f \cong N$, N is projective. Therefore I is hereditary.

(3) ⇒ (4) Obvious.

(4) ⇒ (1) Let $M = \bigoplus_{\mathcal{J}} R$ be a free R-module with an index set $\mathcal{J}$ and $f \in S$. By (4), there exists a hereditary right ideal I such that $R = \overline{Z}(R_R) \oplus I$ and $M = \overline{Z}(M) \oplus (\bigoplus_{\mathcal{J}} I)$. Then $\bigoplus_{\mathcal{J}} I = eM$ for some $e^2 = e \in S$ and $\mathrm{End}_R(\bigoplus_{\mathcal{J}} I) = eSe$. Since $\overline{Z}(M)$ is a fully invariant submodule of M, $f(\overline{Z}(M)) \subseteq \overline{Z}(M)$ and so $\overline{Z}(M) \subseteq f^{-1}(\overline{Z}(M))$. Then

$$f^{-1}(\overline{Z}(M)) = \overline{Z}(M) \oplus \big((\bigoplus_{\mathcal{J}} I) \cap f^{-1}(\overline{Z}(M))\big).$$

We claim that $(\bigoplus_{\mathcal{J}} I) \cap f^{-1}(\overline{Z}(M)) = \operatorname{Ker}(efe)$. Let $x \in (\bigoplus_{\mathcal{J}} I) \cap f^{-1}(\overline{Z}(M))$. Since $eM = \bigoplus_{\mathcal{J}} I$, $efe(x) = ef(x) \in \bigoplus_{\mathcal{J}} I \cap \overline{Z}(M) = 0$, and so $x \in \operatorname{Ker}(efe)$. Now let $y \in \operatorname{Ker}(efe)$. So $y \in \bigoplus_{\mathcal{J}} I$ and $efe(y) = ef(y) = 0$. Also, $f(y) = a + b$ for some $a \in \overline{Z}(M)$ and $b \in \bigoplus_{\mathcal{J}} I$. Then $0 = ef(y) = ea + eb = b$. Hence $f(y) = a \in \overline{Z}(M)$, and so $y \in (\bigoplus_{\mathcal{J}} I) \cap f^{-1}(\overline{Z}(M))$. Thus $(\bigoplus_{\mathcal{J}} I) \cap f^{-1}(\overline{Z}(M)) = \operatorname{Ker}(efe)$. On the other hand, by (4) and [17, 39.7], $\bigoplus_{\mathcal{J}} I$ is hereditary. It follows that $(\bigoplus_{\mathcal{J}} I)/\operatorname{Ker}(efe)$ is projective, so $\operatorname{Ker}(efe)$ is a direct summand of $\bigoplus_{\mathcal{J}} I$. Thus $f^{-1}(\overline{Z}(M))$ is a direct summand of M. Therefore M is $\overline{Z}(M)$-inverse split. □

4 $\delta(\cdot)$-inverse split modules

Zhou, in [18], studied δ-semiperfect modules by introducing a version of small submodules, namely, a submodule N of a module M is called *δ-small* in M and denoted by $N \ll_\delta M$, if $N + K \neq M$ for any proper submodule K of M with M/K singular. Every small submodule is δ-small. For the reader's convenience, we first record here some of the known results on δ-small submodules which will be used repeatedly in the sequel.

Lemma 4.1 *Let M be a module. Then the following hold:*

1. *The following are equivalent for a submodule N of M.*
 (a) *N is δ-small in M.*
 (b) *If $X + N = M$, then $M = X \oplus Y$ for a projective semisimple submodule Y with $Y \subseteq N$.*
2. *For submodules N, K, L of M with $K \subseteq N$, we have*
 (a) *$N \ll_\delta M$ if and only if $K \ll_\delta M$ and $N/K \ll_\delta M/K$.*
 (b) *$N + L \ll_\delta M$ if and only if $N \ll_\delta M$ and $L \ll_\delta M$.*
3. *If $K \ll_\delta M$ and $f : M \to N$ is a homomorphism, then $f(K) \ll_\delta N$. In particular, if $K \ll_\delta M \subseteq N$, then $K \ll_\delta N$.*
4. *Let $K_1 \subseteq M_1 \subseteq M$, $K_2 \subseteq M_2 \subseteq M$ and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \ll_\delta M_1 \oplus M_2$ if and only if $K_1 \ll_\delta M_1$ and $K_2 \ll_\delta M_2$.*

Proof. See [18, Lemma 1.2 and Lemma 1.3]. □

As a consequence of Lemma 4.1(1), a module M is a δ-small submodule of itself if and only if M is semisimple projective. δ-small submodules of a module M produce a fully invariant submodule of M denoted by $\delta(M)$. Let $\mathcal{P}$ be the class of all singular simple modules. For a module M, let $\delta(M) = \sum\{L \subseteq M \mid L \text{ is a } \delta\text{-small submodule of } M\}$. By definition, $\operatorname{Rad}(M) \subseteq \delta(M)$, and for any ring R, we have $J(R) \subseteq \delta(R_R)$. Consider the integers $\mathbb{Z}$ as a $\mathbb{Z}$-module. Then $\delta(\mathbb{Z}) = 0$ and for any prime integer p, $\delta(\mathbb{Z}/p\mathbb{Z}) = 0$. As for the rational numbers $\mathbb{Q}$ as a $\mathbb{Z}$-module, $\delta(\mathbb{Q}) = \mathbb{Q}$ since every cyclic $\mathbb{Z}$-submodule of $\mathbb{Q}$ is small.

Lemma 4.2 *Let $\mathcal{P}$ be the class of all singular simple R-modules and M an R-module. Then the following hold:*

1. *$\delta(M) = Rej_M(\mathcal{P}) = \bigcap\{N \leq M \mid M/N \in \mathcal{P}\}$.*
2. *If $f : M \to N$ is an R-homomorphism, then $f(\delta(M)) \subseteq \delta(N)$.*

3. *If $M = \bigoplus_{i\in I} M_i$, then $\delta(M) = \bigoplus_{i\in I} \delta(M_i)$.*
4. *If M is a coatomic module, that is, every proper submodule of M is contained in a maximal submodule of M, then $\delta(M)$ is the unique largest δ-small submodule of M.*
5. $\delta(M/\delta(M)) = 0$.

Proof. See [18, Lemma 1.5] for (1)-(4). In order to prove (5), let $0+\delta(M) \neq m+\delta(M) \in \delta(M/\delta(M))$. Since $m \notin \delta(M)$, there exists a submodule K of M with M/K singular simple and $m \notin K$. Hence $\delta(M) \subseteq K$ and K is maximal in M. This implies $\delta(M/\delta(M)) \subseteq K/\delta(M)$, and so $m \in K$. This is a contradiction. Therefore $\delta(M/\delta(M)) = 0$. □

Corollary 4.3 *$\delta(R_R)$ is δ-small in the ring R.*

Proof. Let K be a right ideal of R such that $R = \delta(R_R) + K$ and R/K singular. We may assume that K is maximal. Then R/K is a simple singular R-module. Hence $\delta(R_R) \subseteq K$. Thus $K = R$. A contradiction. □

Example 4.1 (1) There exists a module M such that $\overline{Z}(M) = M$ and $\delta(M) \neq M$.
(2) There exists a module M such that $\overline{Z}(M) \neq M$ and $\delta(M) = M$.

Proof. (1) Let M be an injective semisimple module but not projective and $M = \bigoplus_{i\in I} M_i$. By Example 3.1, $\overline{Z}(M) = M$. Since M is not projective, at least one of the direct summands, say M_i is singular. Let π_i denote the projection $M \xrightarrow{\pi_i} M_i$ with $\operatorname{Ker}\pi_i = \bigoplus_{j\neq i} M_j$. Then $\delta(M) \subseteq \bigoplus_{j\neq i} M_j \neq M$.

(2) Let M be a semisimple module. Assume that M is projective but not injective. Let T be a simple singular module and $M \xrightarrow{f} T$ be an R-homomorphism. Then $M = \operatorname{Ker} f \oplus K$ for some submodule K of M. Then K is a projective singular module as it is singular since it is isomorphic to a submodule of the singular module T, and it is projective since it is a direct summand of the projective module M. Accordingly, $K = 0$. Thus $M = \delta(M)$. Now we may assume $M = M_1 \oplus M_2$. Since M is not injective, one of M_1 and M_2 should not be injective, say M_1 is not injective. Therefore M_1 is small. Consider the natural projection $M \xrightarrow{\pi_1} M_1$ with kernel M_2. Then $\overline{Z}(M) \subseteq M_2 \neq M$. □

Now we introduce the other main definition of the paper.

Definition 4.4 *A module M is called $\delta(M)$-inverse split if $f^{-1}(\delta(M))$ is a direct summand of M for every $f \in S$.*

Example 4.2 The module M considered in Example 4.1(2) is $\delta(M)$-inverse split because of $\delta(M) = M$.

The next result gives some characterizations of the class of $\delta(\cdot)$-inverse split modules.

Theorem 4.5 *The following are equivalent for a module M.*

(1) *M is $\delta(M)$-inverse split.*
(2) *$M = \delta(M) \oplus N$ where N is a Rickart module with $\delta(N) = 0$ and $\delta(M) = eM$ for some left semicentral $e^2 = e \in S$.*
(3) *$M = \delta(M) \oplus N$ where N is a Rickart module.*

(4) *M has a decomposition $M = K \oplus N$ for some submodules K, N with N Rickart, $\delta(N) = 0$ and $\delta(K) = K$.*

Proof. The proof of Theorem 3.3 works verbatim by replacing $\overline{Z}(M)$ by $\delta(M)$. For (1) $\Rightarrow$ (2), by Lemma 4.2(5), we have $\delta(N) = 0$. □

It is known that $\delta(\mathbb{Z}) = 0$ and $\mathbb{Z}$ is a Rickart $\mathbb{Z}$-module. Hence $\mathbb{Z}$ is $\delta(\mathbb{Z})$-inverse split. On the other hand, $\delta(\mathbb{Q}) = \mathbb{Q}$ implies that $\mathbb{Q}$ is a $\delta(\mathbb{Q})$-inverse split $\mathbb{Z}$-module.

Proposition 4.6 *Let M be a module with $\delta(M)$ maximal in M. If every cyclic submodule of M is δ-small, then M is $\delta(M)$-inverse split.*

Proof. Let $m \in M \setminus \delta(M)$. Since $\delta(M)$ is a maximal submodule of M, we have $M = \delta(M) + mR$. By hypothesis and Lemma 4.1(1), there exists a semisimple projective $Y \subseteq mR$ such that $M = \delta(M) \oplus Y$. Y being semisimple implies that it is Rickart. Therefore Theorem 4.5 completes the proof. □

The next result shows that the $\delta(\cdot)$-inverse splitting property for modules is inherited by the direct summands.

Lemma 4.7 *Let M be a $\delta(M)$-inverse split module. Then every direct summand L of M is also $\delta(L)$-inverse split.*

Proof. Let $M = L \oplus K$ for some submodules K and L of M. Then there exists a submodule N of M with $M = \delta(M) \oplus N$. By Lemma 4.2, $\delta(M) = \delta(L) \oplus \delta(K)$. Then $L = \delta(L) \oplus \big(L \cap (\delta(K) \oplus N)\big)$ and $M = \delta(L) \oplus K \oplus \big(L \cap (\delta(K) \oplus N)\big)$. To prove that L is $\delta(L)$-inverse split, we show that $L \cap \big(\delta(K) \oplus N\big)$ is Rickart. Let $f \in \mathrm{End}_R(L \cap (\delta(K) \oplus N))$ and α denote the identity map of $\delta(L) \oplus K$. Consider $g = \alpha \oplus f$. Then $g \in S$ and $g^{-1}(\delta(M)) = \delta(L) \oplus \delta(K) \oplus f^{-1}(0)$ is a direct summand of M. It follows that $f^{-1}(0)$ is a direct summand of $L \cap (\delta(K) \oplus N)$. Hence $L \cap (\delta(K) \oplus N)$ is Rickart. By Theorem 4.5, L is $\delta(L)$-inverse split. □

Recall that a ring is called *abelian* if every idempotent is central. A module M is called *abelian* if $fem = efm$ for any $f \in S, e^2 = e \in S, m \in M$. Note that M is an abelian module if and only if S is an abelian ring.

Corollary 4.8 *The following hold:*

(1) *If R is a right $\delta(R_R)$-inverse split ring, then eR is a $\delta(eR)$-inverse split R-module for every $e^2 = e \in R$.*
(2) *If R is an abelian right $\delta(R_R)$-inverse split ring, then eRe is a right $\delta(eRe_{eRe})$-inverse split ring for every $e^2 = e \in R$.*
(3) *Let M be an abelian module with S a right $\delta(S_S)$-inverse split ring. Then every direct summand of M has a right $\delta(\cdot)$-inverse split endomorphism ring.*

Proposition 4.9 *Let R be a ring. If every finitely generated projective R-module has a right $\delta(\cdot)$-inverse split endomorphism ring, then $M_n(R)$ is right $\delta(M_n(R)_{M_n(R)})$-inverse split for every positive integer n.*

Proof. $M_n(R)$ can be viewed as the endomorphism ring of a finitely generated projective R-module R^n for a positive integer n. Hence $M_n(R)$ is right $\delta(M_n(R)_{M_n(R)})$-inverse split. □

Proposition 4.10 *Let M be an indecomposable module. If M is $\delta(M)$-inverse split, then either M has no maximal essential submodule or M is cogenerated by simple singular modules and every endomorphism of M is a monomorphism.*

Proof. There exists a direct summand N of M such that $M = \delta(M) \oplus N$. By Theorem 4.5, N is Rickart. Assume that $M = \delta(M)$. Otherwise, M is Rickart and $\delta(M) = 0$. Hence M is cogenerated by singular simple modules. □

In the following result, it can be seen that under what conditions a homomorphic image of a $\delta(\cdot)$-inverse split module is also $\delta(\cdot)$-inverse split.

Proposition 4.11 *Let M, N be R-modules and $f : M \to N$ an epimorphism with $\mathrm{Ker} f \subseteq \delta(M)$. If M is $\delta(M)$-inverse split, then N is also $\delta(N)$-inverse split.*

Proof. Let M be a $\delta(M)$-inverse split module. Then $M = \delta(M) \oplus K$ for some Rickart module K. Hence $N = f(\delta(M)) + f(K)$. By [2, Corollary 8.17(2)], $f(\delta(M)) = \delta(N)$. Also being $\mathrm{Ker}\, f \subseteq \delta(M)$ implies that $N = \delta(N) \oplus f(K)$. Since $K \cong f(K)$, $f(K)$ is Rickart. Thus Theorem 4.5 completes the proof. □

A δ-cover of a module M is an epimorphism from a module F onto M with a δ-small kernel. By the next result, if a δ-cover of a module is $\delta(\cdot)$-inverse split, then the module also has the same property.

Corollary 4.12 *Let N be a module and M a δ-cover of N. If M is $\delta(M)$-inverse split, then N is $\delta(N)$-inverse split.*

Recall that a projective module P is called a *projective δ-cover* of M if there exists an epimorphism $P \xrightarrow{f} M$ with $\mathrm{Ker}\, f$ is δ-small in P. Every projective cover of a module is also projective δ-cover. Let M be a projective module. It is called *δ-semiperfect* if every homomorphic image of M has a projective δ-cover. Every semiperfect module defined in [11] is δ-semiperfect. In [18], a ring R is called *δ-semiperfect* if every simple R-module has a projective δ-cover. Also, in [16], a ring R is said to be *right δ-semiperfect* if for every right ideal K, there exists $e^2 = e \in K$ with $(1-e)K \subseteq \delta(R_R)$, equivalently, for every right ideal K, there exists $e^2 = e \in R$ such that $K = eR \oplus L$ with $L \subseteq \delta(R_R)$. By [18, Theorem 3.6], these two notions for rings coincide. Now we determine δ-semiperfect $\delta(\cdot)$-inverse split modules.

Theorem 4.13 *Let M be a δ-semiperfect module. Then M is $\delta(M)$-inverse split if and only if $M = \delta(M)$.*

Proof. Let M be a $\delta(M)$-inverse split module. There exists a submodule N such that $M = \delta(M) \oplus N$. We first prove that N is semisimple. Let K be a submodule of N. Since M/K has a projective δ-cover, by [18, Lemma 2.4] there exists a direct summand $L \subseteq K$ of M such that $M = L \oplus U$ with $K \cap U$ δ-small in U and it is also δ-small in M. So $K \cap U \subseteq \delta(M)$. Since $K \cap U \subseteq N$ and $N \cap \delta(M) = 0$, $K \cap U = 0$. It follows that $K = L$ and $N = K \oplus (N \cap U)$. Hence N is semisimple. Therefore N is a semisimple projective module with $\delta(N) = 0$. Let $0 \neq x \in N$. Then there exist a singular simple module T and an R-homomorphism $N \xrightarrow{f} T$ such that $f(x) \neq 0$. Then $N = \mathrm{Ker}\, f \oplus V$ and V is a projective singular module. Hence $V = 0$ and so $f = 0$. This is a contradiction. Thus $N = 0$ and $M = \delta(M)$. The converse is clear. This completes the proof. □

A module M is called *$\delta(M)$-semiperfect* if for any submodule N of M, there exists a projective direct summand A of M such that $A \subseteq N$ and $M = A \oplus B$ with $N \cap B \subseteq \delta(M)$. By [18, Lemma 2.4], a projective module M is $\delta(M)$-semiperfect if and only if it is δ-semiperfect. The ring R is said to be *$\delta(R_R)$-semiperfect* provided that the module R_R is $\delta(R_R)$-semiperfect. Then a ring R being $\delta(R_R)$-semiperfect, right δ-semiperfect and δ-semiperfect coincide.

Theorem 4.14 *A ring R is right $\delta(R_R)$-inverse split and $\delta(R_R)$-semiperfect if and only if it is semisimple.*

Proof. R being $\delta(R_R)$-inverse split implies that there exists a right ideal I of R such that $R = \delta(R_R) \oplus I$. Then $\delta(I) = 0$. By Corollary 4.3, $\delta(R_R)$ is δ-small in R, so there exists a semisimple right ideal Y of R such that $R = Y \oplus I$. Let K be any right ideal of I. There exists a right ideal L of I such that $I = L \oplus L'$, $L \subseteq K$ and $K \cap L' \subseteq \delta(I)$. Since $\delta(R_R) = 0$, we have $K \cap L' = 0$. By the modularity condition, $K = L$ and so I is semisimple. Thus $R = Y \oplus I$ is semisimple. The converse statement is obvious. □

The next result gives some information about the Jacobson radical and the socle of right $\delta(\cdot)$-inverse split rings.

Theorem 4.15 *Let R be a right $\delta(R_R)$-inverse split ring. Then $R = \delta(R_R) \oplus I$, $J(R) = 0$, $Soc(R_R) = \delta(R_R)$ and I is a Rickart R-module.*

Proof. By Theorem 4.5, $R = \delta(R_R) \oplus I$ where I is a Rickart R-module. By Corollary 4.3, $\delta(R_R)$ is δ-small, therefore $R = Y \oplus I$ for some projective semisimple submodule $Y \subseteq \delta(R_R)$ from [18, Corollary 1.2]. Note that Y is a right ideal and a direct sum of minimal right ideals. By the modularity condition, we have $Y = \delta(R_R)$. Also $\mathrm{Soc}(R_R) \subseteq \delta(R_R)$ as $\mathrm{Soc}(R_R)$ is the intersection of all essential right ideals. Hence $\mathrm{Soc}(R_R) = \delta(R_R)$. Since $\delta(R_R)/\mathrm{Soc}(R_R) = J(R/\mathrm{Soc}(R_R)) = J(I)$, we have $J(I) = 0$. Hence $J(R) = 0$. □

We end this paper by observing some results about the notion of $\delta(\cdot)$-inverse splitness for the classes of free, projective and flat modules.

Theorem 4.16 *The following are equivalent for a ring R.*

(1) *Every free R-module M is $\delta(M)$-inverse split.*
(2) *Every projective R-module M is $\delta(M)$-inverse split.*
(3) *$R = \delta(R_R) \oplus I$ where I is a hereditary Rickart R-module.*
(4) *$R = \delta(R_R) \oplus I$ where I is a hereditary R-module.*

Proof. Similar to the proof of Theorem 3.15. □

Corollary 4.17 *Let R be a right $\delta(R_R)$-inverse split ring with $R = \delta(R_R) \oplus I$ for some hereditary R-module I. Then R is right nonsingular.*

Proof. By Theorem 4.15, R being right $\delta(R_R)$-inverse split implies that R has a decomposition $R = \delta(R_R) \oplus I$ with $\delta(R_R)$ semisimple and I hereditary right R-module. Let $x \in Z(I)$. Then xR is a projective module. Then $xR \cong R/r_R(x)$ implies $r_R(x)$ is a direct summand of R. Hence $x = 0$ since $r_R(x)$ is an essential right ideal of R. Let $x \in Z(\delta(R_R))$ and $x = x_1 + x_2 + \cdots + x_t \in I_1 \oplus I_2 \oplus \cdots \oplus I_t$, $I_i = x_i R$. Obviously, $r_R(x) = \bigcap_{i=1}^{t} r_R(x_i)$. Hence for each $1 \leq i \leq t$, $r_R(x_i)$ is an essential right ideal of R. Then $I_i \cong R/r_R(x_i)$ is a projective singular module. It follows $I_i = 0$ for each $1 \leq i \leq t$. This contradiction implies $Z(R_R) = 0$. □

Proposition 4.18 *If every flat module M is $\delta(M)$-inverse split, then every projective module F is $\delta(F)$-inverse split. The converse holds if the pure submodules of projective modules are δ-small.*

Proof. The first assertion is obvious. For the converse, let M be a flat module. Then M is a homomorphic image of a free module F. Hence there exists an exact sequence $0 \to K \to F \xrightarrow{f} M \to 0$ where $K = \operatorname{Ker} f$. By hypothesis, F is $\delta(F)$-inverse split. Since M is flat, K is a pure submodule of F. Also $K \subseteq \delta(F)$. Therefore M is also $\delta(M)$-inverse split by Proposition 4.11. □

Remark 4.1 Due to [4, Theorem 3], if P is a projective module, then 0 is the only small pure submodule of P. But this is not the case for δ-small notion. Let R be a semisimple ring. By Lemma 4.1(1), R is δ-small in itself. On the other hand, R is a pure submodule of itself as an R-module.

Theorem 4.19 *Let R be a ring. Consider the following statements.*

(1) *R is right $\delta(R_R)$-inverse split.*
(2) *$R/\delta(R_R)$ is a projective Rickart R-module.*
(3) *Every cyclic projective R-module M is $\delta(M)$-inverse split.*
(4) *Every finitely generated projective R-module M is $\delta(M)$-inverse split.*
(5) *Every projective R-module M is $\delta(M)$-inverse split.*

Then (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Leftrightarrow$ (2) $\Leftrightarrow$ (1). If R is right semihereditary, then (1) $\Rightarrow$ (4). Moreover, if R is right hereditary, then all of them are equivalent.

Proof. (1) $\Rightarrow$ (2) Clear by Theorem 4.5. (2) $\Rightarrow$ (1) and (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (1) are obvious.

(1) $\Rightarrow$ (3) Let M be a cyclic projective R-module. Then $M \cong I$ for some direct summand right ideal I of R. By (1), R is $\delta(R_R)$-inverse split as an R-module. Hence I is a $\delta(I)$-inverse split module from Lemma 4.7, so M is $\delta(M)$-inverse split.

(1) $\Rightarrow$ (4) Assume that R is right semihereditary. Let M be a finitely generated projective R-module. Then M is a direct summand of $\bigoplus_{i=1}^{n} R_i$ for some positive integer n where $R_i = R$ for all i. Let F denote the module $\bigoplus_{i=1}^{n} R_i$. By (1), $R = \delta(R_R) \oplus I$ with I a Rickart right ideal of R. Due to Lemma 4.2(3), $F = \delta(F) \oplus (\bigoplus_{i=1}^{n} I_i)$ where $I_i = I$ for all i. By [8, Theorem 3.6], $\bigoplus_{i=1}^{n} I_i$ is Rickart. Thus F is a $\delta(F)$-inverse split module. Therefore M is $\delta(M)$-inverse split by Lemma 4.7.

(1) $\Rightarrow$ (5) Assume that R is right hereditary. Let M be a projective R-module. So there exists an index set $\mathcal{J}$ such that M is a direct summand of $R^{(\mathcal{J})}$. Since $R = \delta(R_R) \oplus I$ for some Rickart right ideal I of R, we have $R^{(\mathcal{J})} = \delta(R^{(\mathcal{J})}) \oplus I^{(\mathcal{J})}$. The ring R being hereditary implies that $I^{(\mathcal{J})}$ is Rickart by [7, Theorem 2.26]. It follows that $R^{(\mathcal{J})}$ is $\delta(R^{(\mathcal{J})})$-inverse split, and so is M by Lemma 4.7. □

Corollary 4.20 *Let R be a right hereditary right $\delta(R_R)$-inverse split ring. Then $R^{(\mathbb{N})} \cong R[x]$ and $R^{(\mathbb{R})}$ are $\delta(\cdot)$-inverse split R-modules.*

Proposition 4.21 *Let R be a ring. Consider the following conditions:*

(1) *Every finitely generated projective R-module M is $\delta(M)$-inverse split.*
(2) *R is right semihereditary.*

If $\delta(R_R) = 0$, then (1) $\Rightarrow$ (2)*. If $\delta(R_R)$ is injective as a right ideal, then* (2) $\Rightarrow$ (1).

Proof. Let $\delta(R_R) = 0$ and assume (1). Let M be a finitely generated projective R-module. Then $M = \delta(M) \oplus N$ for some Rickart module N. Since $\delta(R_R) = 0$, we have $\delta(M) = 0$, and so $M = N$. It follows that R is right semihereditary due to [8, Theorem 3.6]. Now assume (2), and let $\delta(R_R)$ be injective as a right ideal, M a finitely generated projective R-module. Hence M is a direct summand of a finitely generated free R-module. $\delta(R_R)$ being injective implies that $\delta(M)$ is also injective. Thus $M = \delta(M) \oplus K$ for some submodule K of M. Since $K \cong M/\delta(M)$, K is finitely generated and projective. By [8, Theorem 3.6], K is Rickart. Therefore Theorem 4.5 completes the proof. $\square$

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