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On congruence lattices on $QRAT$-rpp semigroups

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Abstract The $*$-congruences on $QRAT$-rpp semigroups are given by the $*$-congruence pairs abstractly which consists of congruences on the structure component parts T and Λ. The congruence lattices of these semigroups are also considered.

Keywords Abundant semigroup · Right adequate transversal · Congruence

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1 Introduction and preliminaries

Let S be a semigroup. By $a\mathcal{R}^*b$ we mean that $xa = ya$ if and only if $xb = yb$ for all x, $y \in S^1$. The relation $\mathcal{L}^*$ is defined dually. It is obvious that $\mathcal{R}^*$ is a left congruence and $\mathcal{L}^*$ is a right congruence. It is easy to show that if a and b are regular elements of S, then $a\mathcal{R}^*b$ if and only if $a\mathcal{R}b$. Call a semigroup S *right principal projective semigroup*, in short, *rpp semigroup* if and only if each $\mathcal{L}^*$-class of S contains an idempotent. Dually, we can define left principal projective (in short, lpp) semigroup. Following Fountain [4], a rpp semigroup S is called *right adequate* if $E(S)$ (the set of idempotents of S) is a semilattice. A semigroup S is called *abundant* if each $\mathcal{R}^*$-class and each $\mathcal{L}^*$-class contains an idempotent. Obviously, an abundant semigroup is just a semigroup which is both lpp and rpp. Moreover, an abundant semigroup in which idempotents commute is called *adequate*. It is clear that regular semigroups are abundant and that inverse semigroups are adequate.

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The multiplicative inverse transversals of regular semigroups were first introduced by Blyth and McFadden in 1982 (see [1]). As an analogue of an inverse transversal, an adequate transversal, was introduced by El-Qallali in [3]. Afterwards, Chen and Guo studied the subclass of abundant semigroups with adequate transversals in [4] and [5-7], respectively. Recently, Guo and Hu studied rpp semigroups having a quasi-ideal right adequate transversal (in short, *QRAT-rpp semigroup*) and gave the structure theory on this kind of semigroups in [6]. The congruences on regular semigroups with inverse transversals were studied by Wang and Tang in [10]. In this paper, we shall give $*$-congruences on $QRAT$-rpp semigroups by the $*$-congruence pairs and show that the set of all $*$-congruences on this kind of semigroups is a complete lattice.

Lemma 1.1 **([5])** *Let S be a semigroup and $a, b \in S$. Then the following statements are equivalent:*
(1) *$a\mathcal{L}^*b$;*
(2) *for all $x, y \in S$, $ax = ay$ if and only if $bx = by$.*

Lemma 1.2 **([5])** *Let $e \in E(S)$ and $a \in S$. Then $e\mathcal{L}^*a$ if and only if $a = ae$ and for all $x, y \in S^1$, $ax = ay$ implies $ex = ey$.*

Let S be rpp. For any $a \in S$, we denote a typical idempotent $\mathcal{L}^*$-related to a by a^*. It is easy to show that if S is right adequate and $a, b \in S$ then $a\mathcal{L}^*b$ if and only if $a^* = b^*$.

Lemma 1.3 **([4])** *If S is a right adequate semigroup, then for all $a, b \in S$, $(ab)^* = (a^*b)^*$.*

It is obvious that an adequate semigroup must be right adequate. But a right adequate semigroup need not be adequate as the following example demonstrates.

Example 1.1 Let $T = \{\alpha, \beta, 0, a, b, c\}$ with the following multiplication table

	α	β	0	a	b	c
α	α	0	0	c	0	c
β	0	β	0	0	0	0
0	0	0	0	0	0	0
a	0	a	0	0	0	0
b	0	b	0	0	0	0
c	0	c	0	0	0	0

From the above table, the associativity may be checked directly and $E = \{\alpha, \beta, 0\}$ is a semilattice. Also it is easy to see that $\{\beta, a, b, c\}$, $\{\alpha\}$, $\{0\}$ are the $\mathcal{L}^*$-classes of S and $\{a\}$, $\{b\}$, $\{\alpha, c\}$, $\{\beta\}$, $\{0\}$ are the $\mathcal{R}^*$-classes of S. Hence T is right adequate and not adequate.

Let S be an *rpp* semigroup and let U be an *rpp* subsemigroup of S. We say that U is a $*$-*subsemigroup* of S if

$$\mathcal{L}^*(U) = \mathcal{L}^*(S) \cap (U \times U).$$

It can be shown that U is a $*$-subsemigroup of S if and only if for all $a \in U$ there exist $e \in E(U)$ such that $e \in L^*_a(S)$.

Suppose that S° is a right adequate $*$-subsemigroup of an *rpp* semigroup S. S° is called a *right adequate transversal* of S if for any element $x \in S$, there exists a unique element $x^\circ \in S^\circ$ and idempotent $f \in E$, such that $x = x^\circ f$, where $f\mathcal{R}^* x^{\circ *}$ for some $x^{\circ *} \in E^\circ$. It is straightforward to show [6], that such f is uniquely determined by x. After here we denote the f by f_x. We say that the right adequate transversal S° is *quasi-ideal* if for each $x \in S$ and $a \in S^\circ$, $f_x a \in S^\circ$.

As in [9], we call a nonempty set M a *partial groupoid* if there is a partial operation on M. A groupoid $(M, \circ)$ is said to be a *partial semigroup* if, for all $x, y, z \in M$, if xy, yz and one of $(xy)z$ and $x(yz)$ are defined, then the other one of $(xy)z$ and $x(yz)$ are defined and $(xy)z = x(yz)$. If, in addition, there exists a unipotent S (that is, a semigroup in which each $\mathcal{L}$-class and each $\mathcal{R}$-class contains at most one idempotent) such that $E(S) = M$, then M is called a *partial semilattice.* A partial semigroup M is called a *left* (*right*) *regular partial band* if M is the disjoint union of left (right) zero semigroups M_α with $\alpha \in Y$, where Y is a partial semilattice, satisfying the following conditions:

(PB1) for all $\alpha, \beta \in Y$ and $x \in M_\alpha, y \in M_\beta$, if $\alpha \preceq \beta$, then xy and yx are defined for all $x \in M_\alpha, y \in M_\beta$;

(PB2) for all $x \in M$ and $y \in M_\alpha$, if xy and yx are defined and such that $xy = x$ and $yx = y$ ($xy = y$ and $yx = x$), then $x \in M_\alpha$.

In this case, we call Y the *structure partial semilattice* of the left (right) regular partial band M. Moreover, a subset N of the left (right) regular partial band $(M, \circ) = \bigcup_{\alpha \in Y} M_\alpha$ with structure partial semilattice Y is called a *skeleton* if $(N, \circ)$ is a partial semilattice isomorphic to Y and $|N \bigcap M_\alpha| = 1$ for all Y. In what follows, by a left (right) regular partial band M with semilattice skeleton Y, we mean that M is a left (right) regular partial band in which Y is a skeleton of M and Y is a semilattice.

By the following example, we can see that right partial bands with semilattice skeleton are not right regular bands.

Example 1.2 Let $\Lambda = \{\alpha, \beta, x, 0\}$. Define a partial multiplication by

	α	β	x	0
α	α	0	0	0
β	0	β	x	0
x	$-$	β	x	0
0	0	0	0	0

It is clear that Λ is a partial semigroup. If we let $Y = \{\alpha, \beta, 0\}$, then Y is a semilattice. Obviously, $\Lambda_\alpha = \{\alpha\}$, $\Lambda_\beta = \{\beta\}$, $\Lambda_0 = \{0\}$, $\Lambda_\beta = \{\beta, x\}$ are all right zero rectangular bands in the partial semigroup Λ. By routine computing, Λ is a right regular partial band with semilattice skeletion Y.

Let T be a right adequate semigroup with semilattice Y and $\Lambda = \bigcup_{\alpha \in Y} \Lambda_\alpha$ a right regular partial band with semilattice skeleton Y. Define a mapping by $\langle , \rangle : \Lambda \times T \to T$, $(x, t) \to \langle x, t \rangle$ and put $Q = Q(T, \Lambda, \langle , \rangle) = \{(a, x) \in T \times \Lambda : x \in \Lambda_{a^*}\}$. The above triple $(T, \Lambda, \langle , \rangle)$ is called a *Q-system* provided

($Q1$) for any $x \in \Lambda, y \in Y$ and $s, t \in T$, if yx is defined in Λ, then $\langle yx, st\rangle = y\langle x, s\rangle t$.
($Q2$) for any $x \in \Lambda_\alpha, u \in T$, $\langle x, u\rangle = \alpha u$.
($Q3$) for any $(s, y) \in Q$, $\langle y, s^*\rangle = s^*$.
Given a Q-system $(T, \Lambda, \langle,\rangle)$. Define a multiplication on Q by

$$(s, x) \circ (t, y) = (s\langle x, t\rangle, (s\langle x, t\rangle)^* y).$$

Lemma 1.4 ([6]) *Let $Q = Q(T, \Lambda; \langle,\rangle)$ be a Q-system. Then Q is a $QRAT$-rpp semigroup and it has a quasi-ideal right adequate transversal $Q^\circ = \{(t, t^*) \in Q : t \in T\}$ is isomorphic to T.*

Conversely, every $QRAT$-rpp semigroup can be constructed in the above manner.

It is easy to see, if $Q = Q(T, \Lambda; \langle,\rangle)$ is a Q-system, then for any $(s, y) \in Q$, (s^*, y) is an idempotent.

Lemma 1.5 ([6]) *Let $Q = Q(T, \Lambda; \langle,\rangle)$ be a Q-system and $(s, y) \in Q$. Then:*
(1) *$(s, y) \in E(Q)$ if and only if $s\langle y, s\rangle = s$;*
(2) *$(s, y)\mathcal{L}^*(s^*, y)$.*

2 ∗-congruences on $QRAT$-rpp semigroups

In this section, we consider the ∗-congruences on $QRAT$-rpp semigroups.

Definition 2.1 *An equivalence ρ on a partial semigroup S is called normal, if, for all $a, b, c, d \in S$, if $a\ \rho\ b$, $c\ \rho\ d$ and ac, bd are defined, then $ac\ \rho\ bd$.*

Definition 2.2 *A congruence ρ on a right adequate semigroup S is a ∗-congruence, if for all $a, b \in S$, $a\rho b$ implies $a^*\ \rho\ b^*$.*

At first, we present certain examples of ∗-congruences on rpp semigroups.

Example 2.1 ([4], Proposition 2.3) Let S be a right adequate semigroup with semilattice Y. For each $a \in S$, define $\alpha_a : Y^1 \to Y^1$ by $x\alpha_a = (xa)^*$. Put $\mu_L = \{(a, b) \in S \times S : \alpha_a = \alpha_b\} = \{(a, b) \in S \times S : (xa)^* = (xb)^*$ for all $x \in Y^1\}$.

Then μ_L is the largest congruence contained in $\mathcal{L}^*$. It is easy to check that μ_L is a ∗-congruence on S.

Example 2.2 Let S be a right adequate semigroup as in Example 1.1. Then, by a routine verification, the equivalences α, 1_S on S with classes as follows $\alpha : \{\alpha\}, \{\beta\}, \{0\}, \{b\}, \{a, c\}$, $1_S : \{\alpha\}, \{\beta\}, \{0\}, \{a\}, \{b\}, \{c\}$ are ∗-congruences.

Now, we establish the ∗-congruence on $QRAT$-rpp semigroups by the Q-systems.

Let $Q = Q(T, \Lambda; \langle,\rangle)$ be a Q-system. Suppose ρ^T is a ∗-congruence on T and ρ^Λ is a normal equivalence on Λ. Then (ρ^T, ρ^Λ) is called a *∗-congruence pair* on Q if the following conditions hold:
(C.1) $\rho^T|_Y = \rho^\Lambda|_Y$;
(C.2) $(\forall\ z \in \Lambda)(\forall\ s, t \in T)\ s\ \rho^T\ t \Rightarrow \langle z, s\rangle\ \rho^T\ \langle z, t\rangle$;
(C.3) $(\forall\ x, y \in \Lambda)(\forall\ l \in T)\ x\ \rho^\Lambda\ y \Rightarrow \langle x, l\rangle\ \rho^T\ \langle y, l\rangle$.
Define $\rho^{(\rho^T, \rho^\Lambda)}$ on Q by

$$(s, x)\ \rho^{(\rho^T, \rho^\Lambda)}\ (t, y) \Leftrightarrow s\ \rho^T\ t, x\ \rho^\Lambda\ y.$$

Lemma 2.3 *Let ρ be a $*$-congruence on Q. We define the following equivalences on T and Λ, respectively,*
$(\forall\ s,t \in T)\ s\ \rho_T\ t \Leftrightarrow (s,s^*)\ \rho\ (t,t^*)$;
$(\forall\ x \in \Lambda_{s^*}, y \in \Lambda_{t^*})\ x\ \rho_\Lambda\ y \Leftrightarrow (s^*,x)\ \rho\ (t^*,y)$.
Then ρ_T is a $*$-congruence on T and ρ_Λ is normal on Λ.

Proof. Since ρ is a $*$-congruence on Q, we have that ρ_Λ is an equivalence on Λ and that ρ_T is a $*$-congruence on T.

Let $(s,x),(t,y),(s_1,x_1),(t_1,y_1) \in Q$. If $x\ \rho_\Lambda\ y$ and $x_1\ \rho_\Lambda\ y_1$, then

$$(s^*,x)\ \rho\ (t^*,y) \text{ and } (s_1^*,x_1)\ \rho\ (t_1^*,y_1).$$

Now we immediately get $(s^*,x)(s_1^*,x_1)\ \rho\ (t^*,y)(t_1^*,y_1)$. It follows that

$$(s^*\langle x,s_1^*\rangle,(s^*\langle x,s_1^*\rangle)^*x_1)\ \rho\ (t^*\langle y,t_1^*\rangle,(t^*\langle y,t_1^*\rangle)^*y_1).$$

Thus, by $(Q2)$, $(s^*s_1^*,xx_1)\ \rho\ (t^*t_1^*,yy_1)$. Therefore $xx_1\rho_\Lambda yy_1$ and ρ_Λ is normal. □

Now we can prove our main theorem.

Theorem 2.4 *Let Q be a QRAT-rpp semigroup as in Lemma* 1.4*, and (ρ^T,ρ^Λ) be a $*$-congruence pair on Q. Then $\rho^{(\rho^T,\rho^\Lambda)}$ is a $*$-congruence on Q.*

Conversely, every $*$-congruence on Q can be constructed in the above manner.

Proof. Let (ρ^T,ρ^Λ) be a $*$-congruence pair on Q. Obviously, $\rho^{(\rho^T,\rho^\Lambda)}$ is an equivalence on Q. Now assume $(s,x),(t,y),(l,z) \in Q$, and $(s,x)\ \rho^{(\rho^T,\rho^\Lambda)}\ (t,y)$. Then $s\ \rho^T\ t,\ x\ \rho^\Lambda\ y$. From C.2, we have $\langle z,s\rangle\ \rho^T\ \langle z,t\rangle$ and so $l\langle z,s\rangle\ \rho^T\ l\langle z,t\rangle$. Since ρ^T is a $*$-congruence, we have $(l\langle z,s\rangle)^*\ \rho^T\ (l\langle z,t\rangle)^*$. Together with the fact $\rho^T|_Y = \rho^\Lambda|_Y$, we have $(l\langle z,s\rangle)^*\ \rho^\Lambda\ (l\langle z,t\rangle)^*$. By (Q1), we have $\langle z,s\rangle s^* = \langle z,s\rangle$ and so $l\langle z,s\rangle s^* = l\langle z,s\rangle$. It follows that $(l\langle z,s\rangle)^*s^* = (l\langle z,s\rangle)^*$ and $(l\langle z,s\rangle)^*\omega s^*$. So, $(l\langle z,s\rangle)^*x$ is defined and $(l\langle z,s\rangle)^*x \in \Lambda_{(l\langle z,s\rangle)^*}$.

Similarly, $(l\langle z,t\rangle)^*y \in \Lambda_{(l\langle z,t\rangle)^*}$. Thus $(l\langle z,s\rangle)^*x\rho^\Lambda(l\langle z,t\rangle)^*y$ and so

$$(l\langle z,s\rangle,(l\langle z,s\rangle)^*x)\ \rho^{(\rho^T,\rho^\Lambda)}\ (l\langle z,t\rangle,(l\langle z,t\rangle)^*y).$$

That is, $(l,z)(s,x)\ \rho^{(\rho^T,\rho^\Lambda)}\ (l,z)(t,y)$. Similarly, $(s,x)(l,z)\ \rho^{(\rho^T,\rho^\Lambda)}\ (t,y)(l,z)$. So we have proved that $\rho^{(\rho^T,\rho^\Lambda)}$ is a congruence.

If $(s,x)\ \rho^{(\rho^T,\rho^\Lambda)}\ (t,y)$, then $s\ \rho^T\ t,\ x\ \rho^\Lambda\ y$. Since ρ^T is a $*$-congruence, we have $s^*\ \rho^T\ t^*$. Hence, together with $x\ \rho^\Lambda\ y$, $(s^*,x)\ \rho^{(\rho^T,\rho^\Lambda)}\ (t^*,y)$. Thus, by Lemma 1.5, $\rho^{(\rho^T,\rho^\Lambda)}$ is a $*$-congruence.

Conversely, assume that ρ is a $*$-congruence on Q. We define the following equivalences on T and Λ, respectively,
$(\forall\ s,t \in T)\ s\ \rho_T\ t \Leftrightarrow (s,s^*)\ \rho\ (t,t^*)$;
$(\forall\ x \in \Lambda_{s^*}, y \in \Lambda_{t^*})\ x\ \rho_\Lambda\ y \Leftrightarrow (s^*,x)\ \rho\ (t^*,y)$.
By Lemma 2.3, we have ρ_T is a $*$-congruence and ρ_Λ is normal. And we have the following cases:
(1) $\rho_T|_Y = \rho_\Lambda|_Y$ is obvious. So (C.1) holds.
(2) Let $s,t \in T$ be such that $s\rho_T t$ and $(l^*,z) \in Q$. Then

$$(s,s^*)\ \rho\ (t,t^*) \text{ and so } (l^*,z)(s,s^*)\ \rho\ (l^*,z)(t,t^*).$$

That is, $(l^*\langle z,s\rangle,(l^*\langle z,s\rangle)^*s^*)\ \rho\ (l^*\langle z,t\rangle,(l^*\langle z,t\rangle)^*t^*)$. From Lemma 1.3, we have $(l^*\langle z,s\rangle)^* = (l^*\langle z,s\rangle s^*)^* = ((l^*\langle z,s\rangle)^*s^*)^* = (l^*\langle z,s\rangle)^*s^*$. Similarly, $(l^*\langle z,t\rangle)^* = (l^*\langle z,t\rangle)^*t^*$. Thus $l^*\langle z,s\rangle\ \rho_T\ l^*\langle z,t\rangle$. By (Q1), we have $l^*\langle z,s\rangle = \langle z,s\rangle$ and $l^*\langle z,t\rangle = \langle z,t\rangle$. Thus $\langle z,s\rangle\ \rho_T\ \langle z,t\rangle$. Thus (C.2) holds.

(3) Similarly, we can show that (C.3) also holds.

Now from the above proof, (ρ_T,ρ_Λ) is a $*$-congruence pair on Q.

By the direct part, $\rho^{(\rho_T,\rho_\Lambda)}$ is a $*$-congruence. If $(s,x)\ \rho^{(\rho_T,\rho_\Lambda)}\ (t,y)$, then $s\ \rho_T\ t, x\ \rho_\Lambda\ y$. Thus $(s,s^*)\ \rho\ (t,t^*), (s^*,x)\ \rho\ (t^*,y)$. It follows that $(s,s^*)(s^*,x)\ \rho\ (t,t^*)(t^*,y)$. That is, $(s,x)\ \rho\ (t,y)$. Thus, $\rho^{(\rho_T,\rho_\Lambda)} \subseteq \rho$. Since $\rho \subseteq \rho^{(\rho_T,\rho_\Lambda)}$ is obvious, $\rho^{(\rho_T,\rho_\Lambda)} = \rho$. □

We close this section with the following example.

Example 2.3 Let $S = \{a,b,c,d,e,f,g,h,i,j\}$ be a set with the following multiplication table

	a	b	c	d	e	f	g	h	i	j
a	i	i	i	i	i	i	a	b	i	i
b	i	i	i	i	i	i	a	b	i	i
c	i	i	i	i	i	i	c	d	i	i
d	i	i	i	i	i	i	c	d	i	i
e	i	i	i	i	i	i	e	f	i	i
f	i	i	i	i	i	i	e	f	i	i
g	i	i	i	i	i	i	g	h	i	i
h	i	i	i	i	i	i	g	h	i	i
i	i	i	i	i	i	i	i	i	i	i
j	e	f	i	i	e	f	i	i	i	j

By a routine verification, we have S is an rpp semigroup with a right adequate transversal S°, where $S^\circ = \{a,c,e,g,i,j\}$. The equivalences α, 1_S on S with classes as follows:

$$\alpha : \{i\},\{j\},\{g\},\{h\},\{c\},\{d\},\{a,e\},\{b,f\},$$
$$1_S : \{a\},\{b\},\{c\},\{d\},\{e\},\{f\},\{g\},\{h\},\{i\},\{j\}$$

are $*$-congruences.

3 $*$-congruence lattices

We denote the set of all $*$-congruences on Q and the set of all $*$-congruence pairs on Q constructed as in Theorem 2.4 by $C^*(Q)$ and $CP^*(Q)$, respectively.

Lemma 3.1 *If $(\rho_1^T,\rho_1^\Lambda),(\rho_2^T,\rho_2^\Lambda) \in CP^*(Q)$, then*

$$\rho^{(\rho_1^T,\rho_1^\Lambda)} \subseteq \rho^{(\rho_2^T,\rho_2^\Lambda)} \Leftrightarrow \rho_1^T \subseteq \rho_2^T, \rho_1^\Lambda \subseteq \rho_2^\Lambda.$$

Proof. Suppose $\rho^{(\rho_1^T,\rho_1^\Lambda)} \subseteq \rho^{(\rho_2^T,\rho_2^\Lambda)}$. Let $s\ \rho_1^T\ t$. By the proof of Theorem 2.4, $((s,s^*),(t,t^*)) \in \rho^{(\rho_1^T,\rho_1^\Lambda)} \subseteq \rho^{(\rho_2^T,\rho_2^\Lambda)}$. Hence $s\ \rho_2^T\ t$, and immediately we get $\rho_1^T \subseteq \rho_2^T$. Similarly, we have $\rho_1^\Lambda \subseteq \rho_2^\Lambda$.

The reverse implication is obvious. □

Define $\leq$ on $CP^*(Q)$ by $(\rho_1^T, \rho_1^\Lambda) \leq (\rho_2^T, \rho_2^\Lambda) \Leftrightarrow \rho_1^T \subseteq \rho_2^T, \rho_1^\Lambda \subseteq \rho_2^\Lambda$. Then $CP^*(Q)$ is a partial ordered set with respect to $\leq$. By Theorem 2.4 and Lemma 3.1, we can easily see that $C^*(Q)$ and $CP^*(Q)$ are isomorphic as partial ordered sets.

Proposition 3.2 *Let $\Omega \subseteq C^*(Q)$ and $T_\rho = (\rho^T, \rho^\Lambda)$ where $\rho \in \Omega$. Then*

$$T_{(\cap_{\rho\in\Omega}\rho)} = (\bigcap_{\rho\in\Omega} \rho^T, \bigcap_{\rho\in\Omega} \rho^\Lambda)$$

and

$$T_{(\bigvee_{\rho\in\Omega}\rho)} = (\bigvee_{\rho\in\Omega} \rho^T, \bigvee_{\rho\in\Omega} \rho^\Lambda).$$

Proof. The first equality is obvious, we only need to prove the second equality. Let $x \in \Lambda_{s^*}$ and $y \in \Lambda_{t^*}$ be such that $x\ (\bigvee_{\rho\in\Omega}\rho)^\Lambda\ y$. Then

$$i = (s^*, x) \bigvee_{\rho\in\Omega} \rho\ (t^*, y) = j.$$

Hence, there exist $\rho_i \in \Omega$ and $a_i = (s_i, x_i) \in Q$ such that $i\ \rho_1\ a_1\ \rho_2\ a_2 \cdots a_{n-1}\ \rho_n\ j$. This implies that $i^*\ \rho_1\ a_1^*\ \rho_2\ a_2^* \cdots a_{n-1}^*\ \rho_n\ j^*$. And then $x\ \rho_1^\Lambda\ x_1\ \rho_2^\Lambda\ x_2 \cdots x_{n-1}\ \rho_n^\Lambda\ y$. We have proved that

$$(\bigvee_{\rho\in\Omega} \rho)^\Lambda \subseteq \bigvee_{\rho\in\Omega} \rho^\Lambda.$$

The converse is obvious.

Next, we assume that

$$s\ (\bigvee_{\rho\in\Omega} \rho)^T\ t.$$

Then

$$s = (s, s^*) \bigvee_{\rho\in\Omega} \rho\ (t, t^*) = t.$$

Hence, there exist $s_i = (s_i, s_i^*) \in Q$, and $\rho_i \in \Omega$ such that $s\ \rho_1\ s_1\ \rho_2\ s_2 \cdots \rho_{n-1}\ s_{n-1}\ \rho_n\ t$. Moreover, $s\ \rho_1^T\ s_1\ \rho_2^T\ s_2 \cdots \rho_{n-1}^T\ s_{n-1}\ \rho_n^T\ t$. Therefore,

$$(\bigvee_{\rho\in\Omega} \rho)^T \subseteq \bigvee_{\rho\in\Omega} \rho^T.$$

$\bigvee_{\rho\in\Omega} \rho^T \subseteq (\bigvee_{\rho\in\Omega}\rho)^T$ is clear. □

Now, by summing up the above results, we obtain the following theorem.

Theorem 3.3 *Let Q be constructed in Lemma 1.4. Then $CP^*(Q)$ forms a complete lattice with respect to $\leq$ and $C^*(Q)$ is isomorphic to $CP^*(Q)$ as complete lattices.*

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