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On weakly n-absorbing ideals of commutative rings

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Abstract All rings are commutative with $1 \neq 0$. The purpose of this paper is to investigate the concept of weakly n-absorbing ideals generalizing weakly 2-absorbing ideals. We prove that over a u-ring R the Anderson-Badawi's conjectures about n-absorbing ideals and the Badawi-Yousefian's question about weakly 2-absorbing ideals hold.

Keywords Prime ideals · 2-absorbing ideals · n-absorbing ideals · Weakly n-absorbing ideals

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1 Introduction

Throughout this paper all rings are commutative with a nonzero identity. Recall from [2] that a proper ideal I of a commutative ring R is said to be a *weakly prime ideal of* R if whenever $a, b \in R$ and $0 \neq ab \in I$, then either $a \in I$ or $b \in I$. Badawi in [4] generalized the concept of prime ideals in a different way. He defined a nonzero proper ideal I of R to be a *2-absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Anderson and Badawi [3] generalized the concept of 2-absorbing ideals to n-absorbing ideals. According to their definition, a proper ideal I of R is called an *n-absorbing* (resp. *strongly n-absorbing*) ideal if whenever $a_1 \cdots a_{n+1} \in I$ for $a_1, \dots, a_{n+1} \in R$ (resp. $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \dots, I_{n+1}$ of R), then there are n of the a_i's (resp. n of the I_i's) whose product is in I. Thus a strongly 1-absorbing ideal is just a prime ideal. Clearly a strongly n-absorbing ideal of R is also

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an n-absorbing ideal of R. Anderson and Badawi conjectured that these two concepts are equivalent, e.g., they proved that an ideal I of a Prüfer domain R is strongly n-absorbing if and only if I is an n-absorbing ideal of R (see [3, Corollary 6.9]). They also gave several results relating strongly n-absorbing ideals. The concept 2-absorbing ideals has another generalization, called weakly 2-absorbing ideals, which has studied in [5]. A proper ideal I of R to be a *weakly 2-absorbing ideal of* R if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Generally, we say that a proper ideal I of R is called a *weakly n-absorbing* (resp. *strongly weakly n-absorbing*) ideal if whenever $0 \neq a_1 \cdots a_{n+1} \in I$ for $a_1, \ldots, a_{n+1} \in R$ (resp. $0 \neq I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \ldots, I_{n+1}$ of R), then there are n of the a_i's (resp. n of the I_i's) whose product is in I. Clearly a strongly weakly n-absorbing ideal of R is also a weakly n-absorbing ideal of R. In [13], Quartararo et al. said that a commutative ring R is a *u-ring* provided R has the property that an ideal contained in a finite union of ideals must be contained in one of those ideals. They show that every Bézout ring is a u-ring. Moreover, they proved that every Prüfer domain is a u-domain.

In section 2, we give some basic properties of weakly n-absorbing ideals. For example, we show that I is a weakly n-absorbing ideal of an integral domain R if and only if $\langle I, X\rangle$ is a weakly n-absorbing ideal of $R[X]$. If I is a secondary ideal of a ring R and J is a weakly n-absorbing ideal of R, then $I \cap J$ is secondary. Let I be a weakly n-absorbing ideal of R that is not an n-absorbing ideal. Then $\sqrt{I} = \mathrm{Nil}(R)$, and also if $w \in \mathrm{Nil}(R)$, then either $w^n \in I$ or $w^{n-i}I^{i+1} = \{0\}$ for every $0 \leq i \leq n-1$.

In section 3, we prove that if R is a ring and n is a positive integer such that every proper ideal of R is a weakly n-absorbing ideal of R, then $\dim(R) = 0$, R has at most $n+1$ prime ideals that are pairwise comaximal, and $Jac(R)^{n+1} = 0$. Let $(R_1, M_1), \ldots, (R_s, M_s)$ be quasi-local commutative rings and let $R = R_1 \times \cdots \times R_s$. If every proper ideal of R is a weakly n-absorbing ideal of R, then $M_1^n = M_2^n = \cdots = M_s^n = \{0\}$. Moreover we show that every proper ideal of a decomposable commutative ring $R = R_1 \times R_2 \times \cdots \times R_{n+1}$ is a weakly n-absorbing ideal of R if and only if all of R_i's are fields.

In section 4, we investigate the following conjectures of Anderson and Badawi [3]:

Conjecture 1 *Let n be a positive integer. Then a proper ideal I of a ring R is a strongly n-absorbing ideal of R if and only I is an n-absorbing ideal of R.*

Conjecture 2 *Let n be a positive integer, and let I be an n-absorbing ideal of a ring R. Then $(\sqrt{I})^n \subseteq I$.*

In [3], they proved that Conjecture 1 implies Conjecture 2. Also, they show that if R is a Bézout ring and I is an n-absorbing ideal of R such that $\sqrt{I}$ is a prime ideal of R, then $(\sqrt{I})^n \subseteq I$. For $n = 2$, Badawi [4] shows that these two conjectures hold. In the case where R is a u-ring, we show that these conjectures hold. In [5], Badawi and Yousefian offered a question as follows:

Question Let I be a weakly 2-absorbing ideal of R. Is I a strongly weakly 2-absorbing ideal of R?

Regarding this question we will prove that for an arbitrary positive integer n, a weakly n-absorbing ideal I of a u-ring R is a strongly weakly n-absorbing ideal of R.

2 Properties of weakly n-absorbing ideals

Let n be a positive integer. It is obvious that any n-absorbing ideal of a ring R is a weakly n-absorbing ideal of R, also the zero ideal is a weakly n-absorbing ideal of R, by definition. Therefore $I = \{0\}$ is a weakly n-absorbing ideal of the ring $\mathbb{Z}_{2^{n+1}}$, but it is easy to see that I is not an n-absorbing ideal of $\mathbb{Z}_{2^{n+1}}$.

Consider elements $a_1, \ldots, a_n$ and ideals $I_1, \ldots, I_n$ of a ring R. Throughout this paper we use the following notations:

$a_1 \cdots \widehat{a_i} \cdots a_n$: *i-th* term is excluded from $a_1 \cdots a_n$.

Similarly; $I_1 \cdots \widehat{I_i} \cdots I_n$: *i-th* term is excluded from $I_1 \cdots I_n$.

Moreover, $\mathrm{Nil}(R)$ denotes the ideal of nilpotent elements of R.

Theorem 2.1 *Let R be a ring and let m and n be positive integers.*

1. *A proper ideal I of R is a weakly n-absorbing ideal if and only if whenever $0 \neq x_1 \cdots x_m \in I$ for $x_1, \ldots, x_m \in R$ with $m > n$, then there are n of the x_i's whose product is in I.*
2. *If I is a weakly n-absorbing ideal of R, then I is a weakly m-absorbing ideal of R for all $m \geq n$.*
3. *If I_i is a weakly n_i-absorbing ideal of R for each $1 \leq i \leq k$, then $I_1 \cap \cdots \cap I_k$ is a weakly n-absorbing ideal of R for $n = n_1 + \cdots + n_k$. In particular, if $P_1, \ldots, P_n$ are weakly prime ideals of R, then $P_1 \cap \cdots \cap P_n$ is a weakly n-absorbing ideal of R.*
4. *If $P_1, \ldots, P_n$ are weakly prime ideals of R that are pairwise comaximal ideals, then $I = P_1 \cdots P_n$ is a weakly n-absorbing ideal of R.*

Proof. The proof of (1) and (2) are routine, so it is left out.

(3) Let $a_1, \ldots, a_{n+1} \in R$ such that $0 \neq a_1 \cdots a_{n+1} \in I_1 \cap \cdots \cap I_k$. Since I_i's are weakly n_i-absorbing, then, for each $1 \leq i \leq k$, there exist integers $1 \leq j_1, j_2, \ldots, j_{n_i} \leq n+1$ such that $a_{j_1} a_{j_2} \cdots a_{j_{n_i}} \in I_i$. So we have $a_{1_1} a_{1_2} \cdots a_{1_{n_1}} a_{2_1} a_{2_2} \cdots a_{2_{n_2}} \cdots a_{n_1} a_{n_2} \cdots a_{n_{n_k}} \in I_1 \cap \cdots \cap I_k$ which implies that $I_1 \cap \cdots \cap I_k$ is weakly n-absorbing.

(4) is a direct consequence of (3). □

Proposition 2.2 *Let I be a proper ideal of a ring R. Then the following conditions are equivalent:*

1. *I is strongly weakly n-absorbing;*
2. *For any ideals $I_1, \ldots, I_{n+1}$ of R such that $I \subseteq I_1$, $0 \neq I_1 \cdots I_{n+1} \subseteq I$ implies that there are n of I_i's whose product is in I.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) Let $J, I_2, \ldots, I_{n+1}$ be ideals of R such that $0 \neq J I_2 \cdots I_{n+1} \subseteq I$. Then we have that $0 \neq (J + I) I_2 \cdots I_{n+1} = (J I_2 \cdots I_{n+1}) + (I I_2 \cdots I_{n+1}) \subseteq I$. Set $I_1 := J + I$. Then, by hypothesis $I_2 \cdots I_{n+1} \subseteq I$ or there exists $2 \leq i \leq n+1$ such that $(J + I) I_2 \cdots \widehat{I_i} \cdots I_{n+1} \subseteq I$. Therefore, $I_2 \cdots I_{n+1} \subseteq I$ or there exists $2 \leq i \leq n+1$ such that $J I_2 \cdots \widehat{I_i} \cdots I_{n+1} \subseteq I$. So I is strongly weakly n-absorbing. □

Theorem 2.3 *Let R be a commutative ring and J be a weakly n-absorbing ideal of R.*

1. *If I is an ideal of R with $I \subseteq J$, then J/I is a weakly n-absorbing ideal of R/I.*
2. *If T is a subring of R, then $J \cap T$ is a weakly n-absorbing ideal of T.*

3. If S is a multiplicatively closed subset of R with $J \cap S = \emptyset$, then J_S is a weakly n-absorbing ideal of R_S.

Proof. (1) Let $\bar{R} = R/I, \bar{J} = J/I$ and $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{n+1} \in \bar{R}$ such that $\bar{a}_1\bar{a}_2 \cdots \bar{a}_{n+1} \in \bar{J}\setminus\{0\}$. Since $\bar{a}_1\bar{a}_2 \cdots \bar{a}_{n+1} \neq 0$, so $a_1a_2 \cdots a_{n+1} \in R\setminus I$. Hence $a_1a_2 \cdots a_{n+1} \in J\setminus\{0\}$. As J is a weakly n-absorbing ideal of R, we have n of a_i's whose product is in J. Then there are n of $\bar{a}_i$'s whose product is in $\bar{J}$.

(2) It's obvious.

(3) Assume that $0 \neq (a_1/s_1)(a_2/s_2)\cdots(a_{n+1}/s_{n+1}) \in J_S$ such that a_1, a_2, ..., $a_{n+1} \in R$ and $s_1, s_2, \ldots, s_{n+1} \in S$ and

$$(a_1/s_1)(a_2/s_2)\cdots\widehat{(a_i/s_i)}\cdots(a_{n+1}/s_{n+1}) \notin J_S,$$

for any $2 \leq i \leq n+1$. Now let $(a_1a_2 \cdots a_{n+1})/(s_1s_2 \cdots s_{n+1}) = (x/u)$ for some $x \in J$ and $u \in S$. Then there exists $v \in S$ such that $vu(a_1a_2 \cdots a_{n+1}) = vx(s_1s_2 \cdots s_{n+1})$. So we have $(vua_1)a_2 \cdots a_{n+1} \in J\setminus\{0\}$ but the product of (vua_1) with $n-1$ of a_i's for $2 \leq i \leq n+1$ is not in J. So we conclude $a_2 \cdots a_{n+1} \in J$ and then $(a_2/s_2)\cdots(a_{n+1}/s_{n+1}) \in J_S$, that is, J_S is a weakly n-absorbing ideal of R_S. □

Theorem 2.4 *Let $I \subseteq J$ be proper ideals of a ring R. If I is a weakly n-absorbing ideal of R and J/I is a weakly n-absorbing ideal of R/I, then J is a weakly n-absorbing ideal of R.*

Proof. Suppose that I is a weakly n-absorbing ideal of R and J/I is a weakly n-absorbing ideal of R/I. Let $0 \neq a_1 \cdots a_{n+1} \in J$ where $a_1, \ldots, a_{n+1} \in R$, so $(a_1 + I)\cdots(a_{n+1}+I) \in J/I$. If $a_1 \cdots a_{n+1} \in I$, then for some $1 \leq i \leq n+1$, $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \in I \subseteq J$, because I is weakly n-absorbing. If $a_1 \cdots a_{n+1} \notin I$, then for some $1 \leq i \leq n+1$, $(a_1 + I)\cdots\widehat{(a_i + I)}\cdots(a_{n+1} + I) \in J/I$, since J/I is a weakly n-absorbing ideal of R/I. So $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \in J$. Consequently J is a weakly n-absorbing ideal of R. □

Theorem 2.5 *Let I be an ideal of an integral domain R. Then $\langle I, X\rangle$ is a weakly n-absorbing ideal of $R[X]$ if and only if I is a weakly n-absorbing ideal of R.*

Proof. By Theorem 2.3(1), Theorem 2.4 and regarding the isomorphism $\langle I, X\rangle/\langle X\rangle \simeq I$ in $R[X]/\langle X\rangle \simeq R$ we have the result. □

Proposition 2.6 *Let I be a weakly primary ideal of a ring R, and let $(\sqrt{I})^n \subseteq I$ for some positive integer n (for example, if $\sqrt{I}$ is a finitely generated ideal). Then I is a weakly n-absorbing ideal of R.*

Proof. Let $0 \neq a_1 \cdots a_{n+1} \in I$ for $a_1, \ldots, a_{n+1} \in R$. If one of the a_i's is not in $\sqrt{I}$, then the product of the other a_i's is in I, since I is weakly primary. Thus we may assume that every a_i is in $\sqrt{I}$. Since $(\sqrt{I})^n \subseteq I$, we have $a_1 \cdots a_n \in I$. Hence I is a weakly n-absorbing ideal of R. □

Remark 2.1 Let R be a ring such that its zero ideal is n-absorbing (e.g., let R be an integral domain). Then every weakly n-absorbing ideal of R is an n-absorbing ideal.

Let M be an R-module. We say that M is secondary precisely when $M \neq 0$ and, for each $r \in R$, either $rM = M$ or there exists $n \in \mathbb{N}$ such that $r^n M = 0$. When this is the case, $P := \sqrt{(0 :_R M)}$ is a prime ideal of R: in these circumstances, we say that M is a P-secondary R-module. A secondary ideal of R is just a secondary submodule of the R-module R (see [10]).

Theorem 2.7 *Let I be a secondary ideal of a ring R. If J is a weakly n-absorbing ideal of R, then $I \cap J$ is secondary.*

Proof. Let I be a P-secondary ideal of R, and let $a \in R$. If $a \in P = \sqrt{(0 :_R I)}$, then clearly $a \in \sqrt{(0 :_R I \cap J)}$. If $a \notin P$, then $a^n \notin P$, and so $a^n I = I$. We suppose that $a(I \cap J) = I \cap J$. Assume that $0 \neq x \in I \cap J$. There is an element $b \in I$ such that $x = a^n b \in J$. Since J is weakly n-absorbing we have either $a^n \in J$ or $a^{n-1}b \in J$. If $a^n \in J$, then $I = a^n I \subseteq J$ and so $a(I \cap J) = aI = I = I \cap J$. If $a^{n-1}b \in J$, then $x = a^n b \in a(I \cap J)$ and we are done. $\square$

A weakly prime ideal P of a ring R is said to be a divided weakly prime ideal if $P \subset xR$ for every $x \in R \backslash P$; thus a divided weakly prime ideal is comparable to every ideal of R.

Theorem 2.8 *Let P be a divided weakly prime ideal of a ring R, and let I be a weakly n-absorbing ideal of R with $\sqrt{I} = P$. Then I is a weakly primary ideal of R.*

Proof. Let $0 \neq xy \in I$ for $x, y \in R$ and $y \notin P$. Then $x \in P$. If $y^{n-1} = 0$, then $y \in \sqrt{I} = P$, which is a contradiction. Therefore $y^{n-1} \neq 0$, and so $y^{n-1} \notin P$. Thus $P \subset y^{n-1}R$, because P is a divided weakly prime ideal of R. Hence $x = y^{n-1}z$ for some $z \in R$. As $0 \neq y^n z = yx \in I$, $y^n \notin I$, and I is a weakly n-absorbing ideal of R, we have $x = y^{n-1}z \in I$. Hence I is a weakly primary ideal of R. $\square$

Let I be a weakly n-absorbing ideal of a ring R and $a_1, ..., a_{n+1} \in R$. We say $(a_1, \ldots, a_{n+1})$ is an $(n+1)$-tuple-zero of I if $a_1 \cdots a_{n+1} = 0$, and for each $1 \leq i \leq n+1$, $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \notin I$.

In the following Theorem $a_1 \cdots \widehat{a_i} \cdots \widehat{a_j} \cdots a_n$ denotes that a_i and a_j are eliminated from $a_1 \cdots a_n$.

Theorem 2.9 *Let I be a weakly n-absorbing ideal of a ring R and suppose that $(a_1, \ldots, a_{n+1})$ is an $(n+1)$-tuple-zero of I for some $a_1, ..., a_{n+1} \in R$. Then for every $1 \leq \alpha_1, \alpha_2, \ldots, \alpha_m \leq n+1$ which $1 \leq m \leq n$,*

$$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots a_{n+1} I^m = \{0\}.$$

Proof. We use induction on m. Let $m = 1$ and suppose that

$$a_1 \cdots \widehat{a_{\alpha_1}} \cdots a_{n+1} x \neq 0,$$

for some $x \in I$. Then $a_1 \cdots \widehat{a_{\alpha_1}} \cdots a_{n+1}(a_{\alpha_1} + x) \neq 0$. Since I is weakly n-absorbing and $a_1 \cdots \widehat{a_{\alpha_1}} \cdots a_{n+1} \notin I$, we conclude that

$$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots a_{n+1}(a_{\alpha_1} + x) \in I,$$

for some $1 \leq \alpha_2 \leq n+1$ distinct from α_1. Hence $a_1 \cdots \widehat{a_{\alpha_2}} \cdots a_{n+1} \in I$, a contradiction. Thus $a_1 \cdots \widehat{a_{\alpha_1}} \cdots a_{n+1} I = \{0\}$.

Now suppose $m > 1$ and assume that for all integers less than m the claim holds. Let $a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots a_{n+1} x_1 x_2 \cdots x_m \neq 0$ for some $x_1, x_2, \ldots, x_m \in I$. By induction hypothesis, we conclude that

$$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots a_{n+1}(a_{\alpha_1} + x_1)(a_{\alpha_2} + x_2) \cdots (a_{\alpha_m} + x_m) \\ = a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots a_{n+1} x_1 x_2 \cdots x_m \neq 0.$$

Hence either

$$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots a_{n+1}(a_{\alpha_1} + x_1) \cdots (\widehat{a_{\alpha_i} + x_i}) \cdots (a_{\alpha_m} + x_m) \in I$$

for some $1 \leq i \leq m$; or

$$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots \widehat{a_j} \cdots a_{n+1}(a_{\alpha_1} + x_1)(a_{\alpha_2} + x_2) \cdots (a_{\alpha_m} + x_m) \in I,$$

for some j distinct from α_i's. Thus either $a_1 \cdots a_{\alpha_1} \cdots \widehat{a_{\alpha_i}} \cdots a_{\alpha_m} \cdots a_{n+1} \in I$ or $a_1 \cdots a_{\alpha_1} \cdots a_{\alpha_m} \cdots \widehat{a_j} \cdots a_{n+1} \in I$, a contradiction. Thus

$$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots a_{n+1} I^m = \{0\}.$$

□

Now we state a version of Nakayama's lemma.

Theorem 2.10 *Let I be a weakly n-absorbing ideal of R that is not an n-absorbing ideal. Then*

1. $I^{n+1} = \{0\}$.
2. $\sqrt{I} = Nil(R)$.
3. *If M is an R-module and $IM = M$, then $M = \{0\}$.*

Proof. (1) Since I is not an n-absorbing ideal of R, I has an $(n+1)$-truple-zero $(a_1, \ldots, a_{n+1})$ for some $a_1, \ldots, a_{n+1} \in R$. Suppose that $x_1 x_2 \cdots x_{n+1} \neq 0$ for some $x_1, x_2, \ldots, x_{n+1} \in I$. Then by Theorem 2.9 we have $(a_1 + x_1) \cdots (a_{n+1} + x_{n+1}) = x_1 x_2 \cdots x_{n+1} \neq 0$. Hence $(a_1 + x_1) \cdots (\widehat{a_i + x_i}) \cdots (a_{n+1} + x_{n+1}) \in I$ for some $1 \leq i \leq n+1$. Thus $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \in I$, a contradiction. Hence $I^{n+1} = \{0\}$.

(2) Clearly, $\mathrm{Nil}(R) \subseteq \sqrt{I}$. As $I^{n+1} = \{0\}$, we get $I \subseteq \mathrm{Nil}(R) = \sqrt{\{0\}}$; hence $\sqrt{I} \subseteq \mathrm{Nil}(R)$, as required.

(3) Since $IM = M$, we have $M = IM = I^{n+1}M = \{0\}$. □

The following example shows that a proper ideal I of a ring R with $I^{n+1} = \{0\}$ need not be a weakly n-absorbing ideal of R.

Example 2.1 Let $R = \mathbb{Z}_{2^{n+2}}$. Then $I = \{0, 2^{n+1}\}$ is an ideal of $\mathbb{Z}_{2^{n+2}}$ and $I^{n+1} = \{0\}$, but $2 \cdots 2 = 2^{n+1} \in I$ and $2^n \notin I$.

Corollary 2.11 *Let R be a ring such that $Nil(R)$ is an n-absorbing (resp. a weakly n-absorbing) ideal of R. If I is a weakly n-absorbing ideal of R, then $\sqrt{I}$ is an n-absorbing (resp. a weakly n-absorbing) ideal of R.*

Proof. Assume that $\mathrm{Nil}(R)$ is an n-absorbing (resp. a weakly n-absorbing) ideal of R and I is a weakly n-absorbing ideal of R. If I is an n-absorbing ideal of R, then $\sqrt{I}$ is an n-absorbing ideal, [3, Theorem 2.1(e)] and so $\sqrt{I}$ is a weakly n-absorbing ideal. If I is not an n-absorbing ideal of R, then by Theorem 2.10 and by our hypothesis, $\sqrt{I} = \mathrm{Nil}(R)$ which is an n-absorbing (resp. a weakly n-absorbing) ideal. □

Theorem 2.12 *Let I be a weakly n-absorbing ideal of a ring R that is not n-absorbing and let J be a weakly m-absorbing ideal of R that is not m-absorbing, and $n \geq m$. Then $I + J$ is a weakly n-absorbing ideal of R. In particular, $\sqrt{I+J} = Nil(R)$.*

Proof. By Theorem 2.10, we have $\sqrt{I} + \sqrt{J} = \sqrt{0} \neq R$, so $I + J$ is a proper ideal of R. Since $(I + J)/J \simeq I/(I \cap J)$ and I is weakly n-absorbing, we get that $(I + J)/J$ is a weakly n-absorbing ideal of R/J, by Theorem 2.3(1). On the other hand J is also weakly n-absorbing, by Theorem 2.1(2). Now, the assertion follows from Theorem 2.4. Finally, by [14, 2.25(i)] we have $\sqrt{I+J} = \sqrt{\sqrt{I} + \sqrt{J}} = \sqrt{\sqrt{0}} = \mathrm{Nil}(R)$. □

Let R be a ring and M an R-module. A submodule N of M is called a pure submodule if the sequence $0 \to N \otimes_R E \to M \otimes_R E$ is exact for every R-module E.

As another consequence of Theorem 2.10 we have the following corollary.

Corollary 2.13 *Let R be a ring. Then the following conditions hold:*

1. *Every nonzero weakly n-absorbing ideal of $R/Nil(R)$ is n-absorbing.*
2. *If I is a pure weakly n-absorbing ideal of R that is not n-absorbing, then $I = \{0\}$.*
3. *If R is von Neumann regular ring, then the only weakly n-absorbing ideal of R that is not n-absorbing can only be $\{0\}$.*

Proof. (1) Notice that $\mathrm{Nil}(\frac{R}{\mathrm{Nil}(R)}) = \{0\}$.

(2), (3) Note that every pure ideal is idempotent, and every ideal of a von Neumann regular ring is pure (see [9]). □

Theorem 2.14 *Let I be a weakly n-absorbing ideal of R that is not an n-absorbing ideal. Then*

1. *If $w \in Nil(R)$, then either $w^n \in I$ or $w^{n-i}I^{i+1} = \{0\}$ for every $0 \leq i \leq n-1$.*
2. *$\mathcal{N}^n I^n = \{0\}$, in which $\mathcal{N}$ denotes the ideal of R generated by all the $(n-1)$-th powers of elements of $Nil(R)$.*

Proof. (1) Suppose that $w \in \mathrm{Nil}(R)$ and $w^n \notin I$. We show that for every $0 \leq i \leq n-1$, $w^{n-i}I^{i+1} = \{0\}$. We use induction on i. For the first step, fix $i = 0$. Assume $w^n I \neq 0$. Let m be the least positive integer such that $w^m = 0$. Then $m \geq n+1$ ($w^n \notin I$) and for some $x \in I$ we have $w^n(x + w^{m-n}) = w^n x \neq 0$. Since $w^n \notin I$ and I is a weakly n-absorbing ideal, then $(w^{n-1}x + w^{m-1}) \in I$. Hence $w^{m-1} \in I$. On the other hand $w^{m-1} \neq 0$. Therefore $w^n \in I$, which is a contradiction. Now, assume that for every $0 \leq j < i$ the claim holds. We will show that $w^{n-i}I^{i+1} = \{0\}$. Assume that

$w^{n-i}x_1x_2\cdots x_{i+1}\neq 0$. By hypothesis we have

$$w^{n-i}(w+x_1)\cdots(w+x_i)(w^{m-n}+x_{i+1})=w^m+w^{m-1}(\sum_{1\leq r\leq i}x_r)$$
$$w^{m-2}(\sum_{\substack{1\leq r,s\leq i\\ r\neq s}}x_rx_s)+\cdots+w^{m-i}x_1\cdots x_i+w^nx_{i+1}+w^{n-1}(\sum_{1\leq r\leq i}x_r)x_{i+1}$$
$$+w^{n-2}(\sum_{\substack{1\leq r,s\leq i\\ r\neq s}}x_rx_s)x_{i+1}+\cdots+w^{n-i}x_1\cdots x_ix_{i+1}=w^{n-i}x_1\cdots x_{i+1}.$$

Therefore, either $w^{n-i}(w+x_1)\cdots(w+x_i)\in I$ or for some $1\leq t\leq i$, $w^{n-i}(w+x_1)\cdots\widehat{(w+x_t)}\cdots(w+x_i)(w^{m-n}+x_{i+1})\in I$ or $w^{n-i-1}(w+x_1)\cdots(w+x_i)(w^{m-n}+x_{i+1})\in I$ which the first case implies that $w^n\in I$, a contradiction, and two other cases imply that $w^{m-1}\in I$. Now $w^{m-1}\neq 0$ again shows that $w^n\in I$, a contradiction.

(2) Let $a_1,\ldots,a_n\in \mathrm{Nil}(R)$. If at least one of the a_i^n's does not belong to I, then $a_1\cdots a_nI^n=0$, by part (1). Therefore, $a_1^{n-1}\cdots a_n^{n-1}I^n=0$. Hence suppose that for every $1\leq i\leq n$, $a_i^n\in I$. Then $a_1\cdots a_n(a_1^{n-1}+\cdots+a_n^{n-1})\in I$. If $(a_1,\ldots,a_n,a_1^{n-1}+\cdots+a_n^{n-1})$ is an $(n+1)$-tuple-zero of I, then $a_1\cdots a_nI=0$, by Theorem 2.9, and hence $a_1^{n-1}\cdots a_n^{n-1}I^n=0$. If $(a_1,\ldots,a_n,a_1^{n-1}+\cdots+a_n^{n-1})$ is not an $(n+1)$-tuple-zero of I, then we can easily see that there is an $1\leq i\leq n$ such that $a_1\cdots a_i^{n-1}\cdots a_n\in I$ or $a_1\cdots a_n\in I$. Hence $a_1^{n-1}\cdots a_n^{n-1}\in I$, and so $a_1^{n-1}\cdots a_n^{n-1}I^n=0$, by Theorem 2.10(1). Consequently $\mathcal{N}^nI^n=\{0\}$. □

Theorem 2.15 *Let $R=R_1\times\cdots\times R_s$ be a decomposable commutative ring and let $L=I_1\times\cdots\times I_{\alpha_1-1}\times R_{\alpha_1}\times I_{\alpha_1+1}\times\cdots\times I_{\alpha_j-1}\times R_{\alpha_j}\times I_{\alpha_j+1}\times\cdots\times I_s$ be an ideal of R in which $\{\alpha_1,\ldots,\alpha_j\}\subset\{1,\ldots,s\}$. The following conditions are equivalent:*

1. *L is a weakly n-absorbing ideal of R;*
2. *L is an n-absorbing ideal of R;*
3. *$L':=I_1\times\cdots\times I_{\alpha_1-1}\times I_{\alpha_1+1}\times\cdots\times I_{\alpha_j-1}\times I_{\alpha_j+1}\times\cdots\times I_s$ is an n-absorbing ideal of $R':=R_1\times\cdots\times R_{\alpha_1-1}\times R_{\alpha_1+1}\times\cdots\times R_{\alpha_j-1}\times R_{\alpha_j+1}\times\cdots\times R_s$.*

Proof. (1) ⇒ (2) Clearly $L\nsubseteq Nil(R)$, so by Theorem 2.10(2), L is an n-absorbing ideal of R.

(2) ⇒ (3) Assume that L is an n-absorbing ideal of R and

$$(a_1^{(1)},\ldots,a_{\alpha_1-1}^{(1)},a_{\alpha_1+1}^{(1)},\ldots,a_{\alpha_j-1}^{(1)},a_{\alpha_j+1}^{(1)},\ldots,a_s^{(1)})\cdots$$
$$(a_1^{(n+1)},\ldots,a_{\alpha_1-1}^{(n+1)},a_{\alpha_1+1}^{(n+1)},\ldots,a_{\alpha_j-1}^{(n+1)},a_{\alpha_j+1}^{(n+1)},\ldots,a_s^{(n+1)})\in L',$$

in which for every $1\leq t\leq n+1$, $a_i^{(t)}$'s are in R_i, respectively. Then

$$(a_1^{(1)},\ldots,a_{\alpha_1-1}^{(1)},1,a_{\alpha_1+1}^{(1)},\ldots,a_{\alpha_j-1}^{(1)},1,a_{\alpha_j+1}^{(1)},\ldots,a_s^{(1)})\cdots$$
$$(a_1^{(n+1)},\ldots,a_{\alpha_1-1}^{(n+1)},1,a_{\alpha_1+1}^{(n+1)},\ldots,a_{\alpha_j-1}^{(n+1)},1,a_{\alpha_j+1}^{(n+1)},\ldots,a_s^{(n+1)})\in L.$$

So there are n of $(a_1^{(t)},\ldots,a_{\alpha_1-1}^{(t)},1,a_{\alpha_1+1}^{(t)},\ldots,a_{\alpha_j-1}^{(t)},1,a_{\alpha_j+1}^{(t)},\ldots,a_n^{(t)})$'s whose product is in L, because L is an n-absorbing ideal of R. Thus the product of n of $(a_1^{(t)},\dots,a_{\alpha_1-1}^{(t)},a_{\alpha_1+1}^{(t)},\dots,a_{\alpha_j-1}^{(t)},a_{\alpha_j+1}^{(t)},\dots,a_n^{(t)})$'s is in L', and so L' is an n-absorbing ideal of R'.

(3) $\Rightarrow$ (1) Let L' is an n-absorbing ideal of R'. It is routine to see that L is an n-absorbing ideal of R. Consequently, L is a weakly n-absorbing ideal of R. $\square$

Theorem 2.16 *Let $R = R_1 \times \cdots \times R_n$ where $R_1,\dots,R_n$ are commutative rings with identity. Suppose that $I_1 \times I_2 \times \cdots \times I_n$ is an ideal of R which $I_1 \neq 0$ and for each $1 \leq i \leq n-1$, I_i is a proper ideal of R_i, and for some $2 \leq i \leq n$, I_i is a nonzero ideal of R_i. The following conditions are equivalent:*

1. *$I_1 \times I_2 \times \cdots \times I_n$ is a weakly n-absorbing ideal of R;*
2. *$I_n = R_n$ and $I_1 \times I_2 \times \cdots \times I_{n-1}$ is an n-absorbing ideal of $R_1 \times \cdots \times R_{n-1}$ or I_n is a prime ideal of R_n and for each $1 \leq i \leq n-1$, I_i is a prime ideal of R_i, respectively;*
3. *$I_1 \times I_2 \times \cdots \times I_n$ is an n-absorbing ideal of R.*

Proof. (1)$\Rightarrow$(2) Suppose that $I_1 \times I_2 \times \cdots \times I_n$ is a weakly n-absorbing ideal of R. If $I_n = R_n$, then $I_1 \times I_2 \times \cdots \times I_{n-1}$ is an n-absorbing ideal of $R_1 \times \cdots \times R_{n-1}$, by Theorem 2.15. Assume that $I_n \neq R_n$. Fix $2 \leq i \leq n$. We show that I_i is a prime ideal of R_i. Suppose that $ab \in I_i$ for some $a, b \in R_i$. Let $0 \neq x \in I_1$. Then

$$\begin{aligned}
&(x,1,\dots,1)(1,\dots,1,\overbrace{a}^{i-th},1,\dots,1)(1,\dots,1,\overbrace{b}^{i-th},1,\dots,1)(1,0,1,\dots,1,\dots,1)\\
&(1,1,0,1,\dots,1,\dots,1)\cdots(1,\dots,1,0,\overbrace{1}^{i-th},\dots,1)(1,\dots,\overbrace{1}^{i-th},0,1,\dots,1)\cdots\\
&(1,\dots,1,0) = (x,0,\dots,0,\overbrace{ab}^{i-th},0,\dots,0) \in I_1 \times \cdots \times I_n \backslash \{(0,\dots,0)\},
\end{aligned}$$

Since $I_1 \times I_2 \times \cdots \times I_n$ is weakly n-absorbing and I_i's are proper, then either

$$\begin{aligned}
&(x,1,\dots,1)(1,\dots,1,\overbrace{a}^{i-th},1,\dots,1)(1,0,1,\dots,1,\dots,1)(1,1,0,1,\dots,1,\dots,1)\\
&\cdots(1,\dots,1,0,\overbrace{1}^{i-th},\dots,1)(1,\dots,\overbrace{1}^{i-th},0,1,\dots,1)\cdots(1,\dots,1,0)\\
&= (x,0,\dots,0,\overbrace{a}^{i-th},0,\dots,0) \in I_1 \times \cdots \times I_n,
\end{aligned}$$

or

$$\begin{aligned}
&(x,1,\dots,1)(1,\dots,1,\overbrace{b}^{i-th},1,\dots,1)(1,0,1,\dots,1,\dots,1)(1,1,0,1,\dots,1,\dots,1)\\
&\cdots(1,\dots,1,0,\overbrace{1}^{i-th},\dots,1)(1,\dots,\overbrace{1}^{i-th},0,1,\dots,1)\cdots(1,\dots,1,0)\\
&= (x,0,\dots,0,\overbrace{b}^{i-th},0,\dots,0) \in I_1 \times \cdots \times I_n,
\end{aligned}$$

and thus either $a \in I_i$ or $b \in I_i$. Consequently I_i is a prime ideal of R_i. Since for some $2 \leq i \leq n$, I_i is a nonzero ideal of R_i, similarly we can show that I_1 is a prime ideal of R_1.

(2)⇒(3) If $I_n = R_n$ and $I_1 \times I_2 \times \cdots \times I_{n-1}$ is an n-absorbing ideal of $R_1 \times \cdots \times R_{n-1}$, then $I_1 \times I_2 \times \cdots \times I_n$ is an n-absorbing ideal of R, by Theorem 2.15. Now, assume that I_n is a prime ideal of R_n and for each $1 \leq i \leq n-1$, I_i is a prime ideal of R_i. Suppose that

$$(a_1^{(1)}, \ldots, a_n^{(1)})(a_1^{(2)}, \ldots, a_n^{(2)}) \cdots (a_1^{(n+1)}, \ldots, a_n^{(n+1)}) \in I_1 \times I_2 \times \cdots \times I_n,$$

in which $a_i^{(j)}$'s are in R_i. Then for any $1 \leq i \leq n$ at least one of the $a_i^{(j)}$'s is in I_i, say $a_i^{(i)}$. Thus $(a_1^{(1)}, \ldots, a_n^{(1)})(a_1^{(2)}, \ldots, a_n^{(2)}) \cdots (a_1^{(n)}, \ldots, a_n^{(n)}) \in I_1 \times I_2 \times \cdots \times I_n$. Consequently $I_1 \times I_2 \times \cdots \times I_n$ is an n-absorbing ideal of R.

(3)⇒(1) is obvious. □

Theorem 2.17 *Let $R = R_1 \times \cdots \times R_n$ be a commutative ring, and let for every $1 \leq i \leq n-1$, I_i be a proper ideal of R_i such that $I_1 \neq 0$ and I_n be an ideal of R_n. The following conditions are equivalent:*

1. *$I_1 \times \cdots \times I_n$ is a weakly n-absorbing ideal of R that is not an n-absorbing ideal of R.*
2. *I_1 is a weakly prime ideal of R_1 that is not a prime ideal and for every $2 \leq i \leq n$, $I_i = \{0\}$ is a prime ideal of R_i, respectively.*

Proof. (1)⇒(2) Assume that $I_1 \times \cdots \times I_n$ is a weakly n-absorbing ideal of R that is not an n-absorbing ideal. If for some $2 \leq i \leq n$ we have $I_i \neq \{0\}$, then $I_1 \times \cdots \times I_n$ is an n-absorbing ideal of R by Theorem 2.16, which contradicts our assumption. Thus for every $2 \leq i \leq n$, $I_i = \{0\}$. A proof similar to part (1)⇒(2) of Theorem 2.16 shows that for every $2 \leq i \leq n$, $I_i = \{0\}$ is a prime ideal of R_i. Now, we show that I_1 is a weakly prime ideal of R_1. Consider $a, b \in R_1$ such that $0 \neq ab \in I_1$. Note that

$$\begin{aligned}(a, 1, \ldots, 1)(b, 1, \ldots, 1)(1, 0, 1..., 1)(1, 1, 0, 1, \ldots, 1) \cdots (1, \ldots, 1, 0)\\ = (ab, 0, \ldots, 0) \in (I_1 \times \{0\} \times \cdots \times \{0\}) \backslash \{(0, \ldots, 0)\}.\end{aligned}$$

Since $I_1 \times \{0\} \times \cdots \times \{0\}$ is a weakly n-absorbing ideal of R, we have either $(a, 0, \ldots, 0) \in I_1 \times \{0\} \times \cdots \times \{0\}$ or $(b, 0, \ldots, 0) \in I_1 \times \{0\} \times \cdots \times \{0\}$. So either $a \in I_1$ or $b \in I_1$. Thus I_1 is a weakly prime ideal of R_1. Assume I_1 is a prime ideal of R_1, since for every $2 \leq i \leq n$, I_i is a prime ideal of R_i, it is easy to see that $I_1 \times \cdots \times I_n$ is an n-absorbing ideal of R, which is a contradiction.

(2)⇒(1) Suppose that I_1 is a weakly prime ideal of R_1 that is not a prime ideal and for every $2 \leq i \leq n$, $I_i = \{0\}$ is a prime ideal of R_i. Assume that

$$\begin{aligned}(a_1^{(1)}, \ldots, a_n^{(1)})(a_1^{(2)}, \ldots, a_n^{(2)}) \cdots (a_1^{(n+1)}, \ldots, a_n^{(n+1)})\\ \in I_1 \times \{0\} \times \cdots \times \{0\} \backslash \{(0, \ldots, 0)\}\end{aligned}$$

in which $a_i^{(j)}$'s are in R_i. Then at least one of the $a_1^{(j)}$'s is in I_1, say $a_1^{(1)}$, and for any $2 \leq i \leq n$ at least one of the $a_i^{(j)}$'s is zero, say $a_i^{(i)}$. Thus $(a_1^{(1)}, \ldots, a_n^{(1)})(a_1^{(2)}, \ldots, a_n^{(2)}) \cdots (a_1^{(n)}, \ldots, a_n^{(n)}) \in I_1 \times \{0\} \times \cdots \times \{0\}$. Consequently $I_1 \times \{0\} \times \cdots \times \{0\}$ is an weakly n-absorbing ideal of R. Since I_1 is not a prime ideal of R_1, there exist elements $a, b \in R_1$ such that $ab = 0$, but $a \notin I_1$ and $b \notin I_1$. Hence

$$\begin{aligned}&(a, 1, \ldots, 1)(b, 1, \ldots, 1)(1, 0, 1, \ldots, 1)(1, 1, 0, 1, \ldots, 1) \cdots\\ &\cdots (1, \ldots, 1, 0) = (0, \ldots, 0),\end{aligned}$$

but neither

$$(a,1,\dots,1)(1,0,1,\dots,1)(1,1,0,1,\dots,1)\cdots(1,\dots,1,0)\in I_1\times\{0\}\times\cdots\times\{0\},$$

nor

$$(b,1,\dots,1)(1,0,1,\dots,1)(1,1,0,1,\dots,1)\cdots(1,\dots,1,0)\in I_1\times\{0\}\times\cdots\times\{0\},$$

also the product of $(a,1,\dots,1)(b,1,\dots,1)$ with any $n-2$ of elements (1, 0, 1, …, 1), (1, 1, 0, 1, …, 1), …, (1, …, 1, 0) is not in $I_1\times\{0\}\times\cdots\times\{0\}$. Consequently $I_1\times\{0\}\times\cdots\times\{0\}$ is not an n-absorbing ideal of R. □

Theorem 2.18 *Let $R=R_1\times\cdots\times R_{n+1}$ where R_i's are commutative rings with identity. If I is a weakly n-absorbing ideal of R, then either $I=\{(0,\dots,0)\}$, or I is an n-absorbing ideal of R.*

Proof. We know that the ideal I is of the form $I_1\times\cdots\times I_{n+1}$ where I_i's are ideals of R_i's, respectively. Since $\{(0,\dots,0)\}$ is a weakly n-absorbing ideal of R, we may assume that $I=I_1\times\cdots\times I_{n+1}\neq\{(0,\dots,0)\}$. So, there is an element $(0,\dots,0)\neq(a_1,\dots,a_{n+1})\in I$. Then

$$(a_1,1,\dots,1)(1,a_2,1,\dots,1)\cdots(1,\dots,1,a_{n+1})\in I.$$

Since I is a weakly n-absorbing ideal of R, for some $1\leq i\leq n+1$

$$(a_1,1,\dots,1)\cdots(1,...1,a_{i-1},1,\dots,1)(1,...1,a_{i+1},1,\dots,1)\cdots(1,\dots,1,a_{n+1})$$
$$=(a_1,\dots,a_{i-1},1,a_{i+1},\dots,a_{n+1})\in I.$$

Then $I_i=R_i$, for some $1\leq i\leq n+1$. Hence $I\nsubseteq \mathrm{Nil}(R)$. Therefore, by Theorem 2.10, I must be an n-absorbing ideal of R. □

Theorem 2.19 *Let $R=R_1\times\cdots\times R_{n+1}$ where R_i's are commutative rings with identity. Let $L=I_1\times\cdots\times I_{n+1}$ be a nonzero proper ideal of R. The following conditions are equivalent:*

1. *$L=I_1\times\cdots\times I_{n+1}$ is a weakly n-absorbing ideal of R;*
2. *$L=I_1\times\cdots\times I_{n+1}$ is an n-absorbing ideal of R;*
3. *$L=I_1\times\cdots\times I_{i-1}\times R_i\times I_{i+1}\times\cdots\times I_{n+1}$ for some $1\leq i\leq n+1$ such that for each $1\leq t\leq n+1$ different from i, I_t is a prime ideal of R_t or $L=I_1\times\cdots\times I_{\alpha_1-1}\times R_{\alpha_1}\times I_{\alpha_1+1}\times\cdots\times I_{\alpha_j-1}\times R_{\alpha_j}\times I_{\alpha_j+1}\cdots\times I_{n+1}$ in which $\{\alpha_1,\dots,\alpha_j\}\subsetneq\{1,\dots,n+1\}$ and*

$$I_1\times\cdots\times I_{\alpha_1-1}\times I_{\alpha_1+1}\times\cdots\times I_{\alpha_j-1}\times I_{\alpha_j+1}\cdots\times I_{n+1}$$

is an n-absorbing ideal of

$$R_1\times\cdots\times R_{\alpha_1-1}\times R_{\alpha_1+1}\times\cdots\times R_{\alpha_j-1}\times R_{\alpha_j+1}\times\cdots\times R_{n+1}.$$

Proof. (1)⇒(2) Since L is a nonzero weakly n-absorbing ideal, L is an n-absorbing ideal of R by Theorem 2.18.

(2)⇒(3) Suppose that L is an n-absorbing ideal of R, then for some $1\leq i\leq n+1$, $I_i=R_i$ by the proof of Theorem 2.18. Assume that $L=I_1\times\cdots\times I_{i-1}\times R_i\times I_{i+1}\times\cdots\times I_{n+1}$ for an $1\leq i\leq n+1$ such that for each $1\leq t\leq n+1$ different from i, I_t

is a proper ideal of R_t. Fix an I_t different from I_i with $t > i$. Let $ab \in I_t$ for some $a, b \in R_t$. In this case

$$
\begin{aligned}
&(1,\dots,1,\overbrace{a}^{t-th},1,\dots,1)(1,\dots,1,\overbrace{b}^{t-th},1,\dots,1)(0,1,\dots,1)(1,0,1,\dots,1)\cdots\\
&(1,\dots,1,0,\overbrace{1}^{i-th},\dots,1)(1,\dots,\overbrace{1}^{i-th},0,1,\dots,1)\cdots(1,\dots,1,0,\overbrace{1}^{t-th},\dots,1)\\
&(1,\dots,\overbrace{1}^{t-th},0,1,\dots,1)\cdots(1,\dots,1,0)\\
&=(0,\dots,0,\overbrace{1}^{i-th},0,\dots,0,\overbrace{ab}^{t-th},0,\dots,0)\in L.
\end{aligned}
$$

Since $I_1 \times \cdots \times I_{n+1}$ is weakly n-absorbing and I_j's different from I_i are proper, then either

$$
\begin{aligned}
&(1,\dots,1,\overbrace{a}^{t-th},1,\dots,1)(0,1,\dots,1)(1,0,1,\dots,1)\cdots(1,\dots,1,0,\overbrace{1}^{i-th},\dots,1)\\
&(1,\dots,\overbrace{1}^{i-th},0,1,\dots,1)\cdots(1,\dots,1,0,\overbrace{1}^{t-th},\dots,1)(1,\dots,\overbrace{1}^{t-th},0,1,\dots,1)\\
&\cdots(1,\dots,1,0)=(0,\dots,0,\overbrace{1}^{i-th},0,\dots,0,\overbrace{a}^{t-th},0,\dots,0)\in L,
\end{aligned}
$$

or

$$
\begin{aligned}
&(1,\dots,1,\overbrace{b}^{t-th},1,\dots,1)(0,1,\dots,1)(1,0,1,\dots,1)\cdots(1,\dots,1,0,\overbrace{1}^{i-th},\dots,1)\\
&(1,\dots,\overbrace{1}^{i-th},0,1,\dots,1)\cdots(1,\dots,1,0,\overbrace{1}^{t-th},\dots,1)(1,\dots,\overbrace{1}^{t-th},0,1,\dots,1)\\
&\cdots(1,\dots,1,0)=(0,\dots,0,\overbrace{1}^{i-th},0,\dots,0,\overbrace{b}^{t-th},0,\dots,0)\in L,
\end{aligned}
$$

and thus either $a \in I_t$ or $b \in I_t$. Consequently I_t is a prime ideal of R_t.

Now, assume that $L = I_1 \times \cdots \times I_{\alpha_1-1} \times R_{\alpha_1} \times I_{\alpha_1+1} \times \cdots \times I_{\alpha_j-1} \times R_{\alpha_j} \times I_{\alpha_j+1} \times \cdots \times I_{n+1}$ in which $\{\alpha_1, \dots, \alpha_j\} \subset \{1, \dots, n+1\}$. Since L is n-absorbing, then $I_1 \times \cdots \times I_{\alpha_1-1} \times I_{\alpha_1+1} \times \cdots \times I_{\alpha_j-1} \times I_{\alpha_j+1} \cdots \times I_{n+1}$ is an n-absorbing ideal of $R_1 \times \cdots \times R_{\alpha_1-1} \times R_{\alpha_1+1} \times \cdots \times R_{\alpha_j-1} \times R_{\alpha_j+1} \times \cdots \times R_{n+1}$, by Theorem 2.15.

(3)⇒(1) If L is one of the given two forms, then it is easily verified that L is an n-absorbing ideal of R, and hence L is a weakly n-absorbing ideal of R. □

3 Rings with property that all proper ideals are weakly n-absorbing

Theorem 3.1 *Let R be a ring and n a positive integer such that every proper ideal of R is a weakly n-absorbing ideal of R. Then*

1. *$dim(R) = 0$.*

2. *R has at most $n+1$ prime ideals that are pairwise comaximal, in particular, R has at most $n+1$ maximal ideals.*

Proof. (1) Suppose that $dim(R) \geq 1$; so R has prime ideals $P \subset Q$. Choose $x \in Q \backslash P$, and let $I = x^{n+1}R$. Then $x^n \in I$, since I is a weakly n-absorbing ideal of R and $0 \neq x^{n+1} \in I$. The reminder is similar to the proof of [3, Theorem 5.9].

(2) Suppose that $P_1, \ldots, P_{n+2}$ are prime ideals of R that are pairwise comaximal. Let $I = P_1 \cdots P_{n+1}$. By [3, Theorem 2.6], I is not an n-absorbing ideal of R. Hence I is a weakly n-absorbing ideal of R that is not an n-absorbing ideal of R. Thus $I^{n+1} = \{0\}$ by Theorem 2.10. Hence $I^{n+1} = P_1^{n+1} \cdots P_{n+1}^{n+1} = \{0\} \subseteq P_{n+2}$, and thus one of the P_i's, $1 \leq i \leq n+1$, is contained in P_{n+2}, which is a contradiction. Hence R has at most $n+1$ prime ideals that are pairwise comaximal. □

For a commutative ring R, we denote by $J(R)$ the intersection of all maximal ideals of R.

Lemma 3.2 *Let R be a commutative ring and $x_1, \ldots, x_{n+1} \in J(R)$. Then the ideal $x_1 \cdots x_{n+1}R$ is a weakly n-absorbing ideal of R if and only if $x_1 \cdots x_{n+1} = 0$.*

Proof. Set $I = x_1 \cdots x_{n+1}R$. If $x_1 \cdots x_{n+1} = 0$, then I is a weakly n-absorbing ideal of R. For the converse, assume that I is a weakly n-absorbing ideal of R and $x_1 \cdots x_{n+1} \neq 0$. Since $x_1 \cdots x_{n+1} \in I \setminus \{0\}$, then there are n of x_i' whose product is in I. We may assume that $y = x_1 \cdots x_n \in I$. Hence $y = yx_{n+1}b$ for some $b \in R$ and so $y(1 - x_{n+1}b) = 0$. Since $x_{n+1}b \in J(R)$, $1 - x_{n+1}b$ is a unit of R. Therefore $y = 0$ and then $x_1 \cdots x_{n+1} = 0$, which is a contradiction. Hence $x_1 \cdots x_{n+1} = 0$. □

Corollary 3.3 *Let R be a ring. If every proper ideal of R is weakly n-absorbing, then $Jac(R)^{n+1} = 0$, and so $Jac(R) = Nil(R)$.*

Theorem 3.4 *Let R be a semi-local ring with maximal ideals $M_1, \ldots, M_t$. If for every nonnegative integers $\alpha_1, \ldots, \alpha_t$ with $\alpha_1 + \cdots + \alpha_t = n+1$ we have $M_1^{\alpha_1} \cdots M_t^{\alpha_t} = \{0\}$, then every proper ideal of R is weakly n-absorbing.*

Proof. Let I be a proper ideal of R and suppose that $0 \neq x_1 \cdots x_{n+1} \in I$ for some $x_1, \ldots, x_{n+1} \in R$. If for some $1 \leq i \leq n+1$, x_i is invertible, then $x_1 \cdots \widehat{x_i} \cdots x_{n+1} \in I$. If for every $1 \leq i \leq n+1$, x_i is noninvertible, then there are nonnegative integers $\alpha_1, \ldots, \alpha_t$ with $\alpha_1 + \cdots + \alpha_t = n+1$ such that $x_1 \cdots x_{n+1} \in M_1^{\alpha_1} \cdots M_t^{\alpha_t} = \{0\}$, a contradiction. Consequently I is weakly n-absorbing. □

As an immediate consequence of Corollary 3.3 and Theorem 3.4 we have the next corollary.

Corollary 3.5 *Let (R, M) be a quasi-local ring. Then every proper ideal of R is weakly n-absorbing if and only if $M^{n+1} = 0$.*

Theorem 3.6 *Let R be a semi-local ring with maximal ideals $M_1, \ldots, M_t$. If for every nonnegative integers $\alpha_1, \ldots, \alpha_t$ with $\alpha_1 + \cdots + \alpha_t = n$ we have $M_1^{\alpha_1} \cdots M_t^{\alpha_t} = \{0\}$, then every proper ideal of R is n-absorbing.*

Proof. Let I be a proper ideal of R and suppose that $x_1 \cdots x_{n+1} \in I$ for some $x_1, \dots, x_{n+1} \in R$. By Theorem 3.4, I is a weakly n-absorbing ideal of R. Hence if $x_1 \cdots x_{n+1} \neq 0$, then we are done. Thus assume that $x_1 \cdots x_{n+1} = 0$.

If for some $1 \leq i \leq n$, x_i is invertible, then $x_1 \cdots \widehat{x_i} \cdots x_{n+1} = 0 \in I$. If for every $1 \leq i \leq n$, x_i is noninvertible, then there are nonnegative integers $\alpha_1, \dots, \alpha_t$ with $\alpha_1 + \cdots + \alpha_t = n$ such that $x_1 \cdots x_n \in M_1^{\alpha_1} \cdots M_t^{\alpha_t} = \{0\}$. Consequently I is n-absorbing. □

Corollary 3.7 *Let (R, M) be a quasi-local ring such that $M^n = \{0\}$. Then every proper ideal of R is n-absorbing.*

Theorem 3.8 *Let $s > 1$ be an integer, $(R_1, M_1), \dots, (R_s, M_s)$ be quasi-local commutative rings and let $R = R_1 \times \cdots \times R_s$. If every proper ideal of R is a weakly n-absorbing ideal of R, then $M_1^n = M_2^n = \cdots = M_s^n = \{0\}$.*

Proof. Assume that every proper ideal of R is a weakly n-absorbing ideal. Take an arbitrary integer $1 \leq i \leq s$ and let $a_1, ..., a_n \in M_i$ such that $a_1 \cdots a_n \neq 0$. Then

$$I = \{0\} \times \cdots \times \{0\} \times (a_1 \cdots a_n R_i) \times \{0\} \times \cdots \times \{0\}$$

is a weakly n-absorbing ideal of R. So we have

$$\begin{aligned}(1, \dots, 1, a_1, 1, \dots, 1) \cdots (1, \dots, 1, a_n, 1, \dots, 1)(0, \dots, 0, 1, 0, \dots, 0)\\ = (0, \dots, 0, a_1 \cdots a_n, 0, \dots, 0) \in I \backslash \{(0, \dots, 0)\}.\end{aligned}$$

Since I is weakly n-absorbing, there exists $1 \leq j \leq n$ such that

$$\begin{aligned}(1, \dots, 1, a_1, 1, \dots, 1) \cdots (1, \dots, 1, a_{j-1}, 1, \dots, 1)(1, \dots, 1, a_{j+1}, 1, \dots, 1)\\ \cdots (1, \dots, 1, a_n, 1, \dots, 1)(0, \dots, 0, 1, 0, \dots, 0) \in I.\end{aligned}$$

Then $a_1 \cdots a_{j-1} a_{j+1} \cdots a_n = a_1 \cdots a_n b$ for some $b \in R_i$.

So $a_1 \cdots a_{j-1} a_{j+1} \cdots a_n (1 - a_j b) = 0$. As $1 - a_j b$ is a unit of R_i, we can conclude $a_1 \cdots a_{j-1} a_{j+1} \cdots a_n = 0$, a contradiction. Thus for every $1 \leq i \leq s$, $M_i^n = \{0\}$. □

Theorem 3.9 *Let (R_1, M_1) and (R_2, M_2) be quasi-local commutative rings with $M_1^n = M_2^n = \{0\}$ and let $R = R_1 \times R_2$. If either R_1 or R_2 is a field, then every proper ideal of R is a weakly n-absorbing ideal of R.*

Proof. Let R_2 be a field. Since $M_1^n = \{0\}$, so every proper ideal of R_1 is an n-absorbing ideal, by Corollary 3.7. Thus, by Theorem 2.15 the ideal $\{0\} \times R_2$ is a weakly n-absorbing ideal of R. Since R_2 is a field, the ideal $R_1 \times \{0\}$ is a weakly n-absorbing ideal of R. Now let J be a proper ideal of R_1 such that $J \neq \{0\}$. Then J is an n-absorbing ideal of R_1 and so $J \times R_2$ is a weakly n-absorbing ideal of R, by Theorem 2.15. At last, we show that $I = J \times \{0\}$ is a weakly n-absorbing ideal of R. Assume that $(a_1, b_1) \cdots (a_{n+1}, b_{n+1}) \in I \setminus \{(0, 0)\}$ such that $a_1, \dots, a_{n+1} \in R_1$ and $b_1, \dots, b_{n+1} \in R_2$. Since $0 \neq a_1 \cdots a_{n+1} \in J$ and $M_1^n = \{0\}$, then at least two of the a_i's are not in M_1, say a_n and a_{n+1}. Since a_n and a_{n+1} are unites of R_1 and $a_1 \cdots a_{n+1} \in J$ we conclude that $a_1 \cdots a_{n-1} \in J$. On the other hand, R_2 is a field and $b_1 \cdots b_{n+1} = 0$, at least one of the b_i's is equal to 0, say $b_{n+1} = 0$. Hence $(a_1, b_1) \cdots (a_{n-1}, b_{n-1})(a_{n+1}, 0) \in I$. Therefore I is a weakly n-absorbing ideal of R. □

Theorem 3.10 *Let $R_1, R_2, \ldots, R_{n+1}$ be commutative rings and let $R = R_1 \times R_2 \times \cdots \times R_{n+1}$. Then every proper ideal of R is a weakly n-absorbing ideal of R if and only if all of R_i's are fields.*

Proof. Assume that every proper ideal of R is a weakly n-absorbing ideal of R and one of the R_i's is not a field. Now we may assume that R_1 is not a field. Hence R_1 has a proper ideal. Say J such that $J \neq \{0\}$. So the ideal $I = J \times \{0\} \times \cdots \times \{0\}$ of R is a weakly n-absorbing ideal. Let $a \in J$ such that $a \neq 0$. Then

$$\begin{aligned}(a, 1, \ldots, 1)(1, 0, 1, \ldots, 1)(1, 1, 0, 1, \ldots, 1) \cdots (1, \ldots, 1, 0)\\ = (a, 0, \ldots, 0) \in I \setminus \{(0, \ldots, 0)\}.\end{aligned}$$

Since J is proper, there exists $2 \leq i \leq n+1$ such that

$$(a, 0, \ldots, 0, \overbrace{1}^{i-th}, 0, \ldots, 0) \in I,$$

which is a contradiction. Hence all of the R_i's are fields. Conversely, suppose that R is ring-isomorphic to $D = F_1 \oplus F_2 \oplus \cdots \oplus F_{n+1}$ where F_i's are fields. Since every nonzero proper ideal of D is a product (intersection) of m distinct maximal ideals of D, for some $1 \leq m \leq n$, we conclude that every nonzero proper ideal of D is an n-absorbing ideal of D, [3, Theorem 2.1]. Hence every nonzero proper ideal of R is a weakly n-absorbing ideal of R. □

4 On Anderson-Badawi's conjecture and Badawi-Yousefian's question

Theorem 4.1 *Let I be a proper ideal of a u-ring R. Then the following conditions are equivalent:*

(*a*) *I is strongly n-absorbing;*

(*b*) *I is n-absorbing;*

(*c*) *For every t ideals $I_1, \ldots, I_t$, $0 \leq t \leq n$, and for every elements $x_1, \ldots, x_{n-t} \in R$ such that $x_1 \cdots x_{n-t} I_1 \cdots I_t \nsubseteq I$,*

$$\begin{aligned}(I :_R x_1 \cdots x_{n-t} I_1 \cdots I_t) = [\bigcup_{i=1}^{n} (I :_R x_1 \cdots \widehat{x_i} \cdots x_{n-t} I_1 \cdots I_t)]\\ \cup [\bigcup_{i=1}^{n} (I :_R x_1 \cdots x_{n-t} I_1 \cdots \widehat{I_i} \cdots I_t)];\end{aligned}$$

(*d*) *For every t ideals $I_1, \ldots, I_t$, $0 \leq t \leq n$, and for every elements $x_1, \ldots, x_{n-t} \in R$ such that $x_1 \cdots x_{n-t} I_1 \cdots I_t \nsubseteq I$,*

$$(I :_R x_1 \cdots x_{n-t} I_1 \cdots I_t) = (I :_R x_1 \cdots \widehat{x_i} \cdots x_{n-t} I_1 \cdots I_t),$$

for some $1 \leq i \leq n-t$ or

$$(I :_R x_1 \cdots x_{n-t} I_1 \cdots I_t) = (I :_R x_1 \cdots x_{n-t} I_1 \cdots \widehat{I_j} \cdots I_t),$$

for some $1 \leq j \leq t$.

Proof. (a)⇒(b) It is clear.

(b)⇒(c) We use induction on t. For $t = 0$, consider elements $x_1, \ldots, x_n \in R$ such that $x_1 \cdots x_n \notin I$. We show that

$$(I :_R x_1 \cdots x_n) = \bigcup_{i=1}^{n} (I :_R x_1 \cdots \widehat{x_i} \cdots x_n).$$

Let $a \in (I :_R x_1 \cdots x_n)$, so $x_1 \cdots x_n a \in I$. Since $x_1 \cdots x_n \notin I$, then for some $1 \leq i \leq n$ we have $x_1 \cdots \widehat{x_i} \cdots x_n a \in I$, i.e., $a \in (I :_R x_1 \cdots \widehat{x_i} \cdots x_n)$. Therefore

$$(I :_R x_1 \cdots x_n) \subseteq \bigcup_{i=1}^{n} (I :_R x_1 \cdots \widehat{x_i} \cdots x_n).$$

Now suppose $t > 0$ and assume that for integer $t-1$ the claim holds. Let $x_1, \ldots, x_{n-t}$ be elements of R and let $I_1, \ldots, I_t$ be ideals of R such that $x_1 \cdots x_{n-t} I_1 \cdots I_t \nsubseteq I$. Consider element $a \in (I :_R x_1 \cdots x_{n-t} I_1 \cdots I_t)$. Thus $I_t \subseteq (I :_R x_1 \cdots x_{n-t} a I_1 \cdots I_{t-1})$. By hypothesis

$$(I :_R x_1 \cdots x_{n-t} a I_1 \cdots I_{t-1}) = (I :_R x_1 \cdots \widehat{x_i} \cdots x_{n-t} a I_1 \cdots I_{t-1})$$

for some $1 \leq i \leq n-t$ or

$$(I :_R x_1 \cdots x_{n-t} a I_1 \cdots I_{t-1}) = (I :_R x_1 \cdots x_{n-t} a I_1 \cdots \widehat{I_j} \cdots I_{t-1}),$$

for some $1 \leq j \leq t-1$.

Consequently either $a \in (I :_R x_1 \cdots \widehat{x_i} \cdots x_{n-t} I_1 \cdots I_{t-1} I_t)$ for some $1 \leq i \leq n-t$ or $a \in (I :_R x_1 \cdots x_{n-t} I_1 \cdots \widehat{I_j} \cdots I_{t-1} I_t)$ for some $1 \leq j \leq t-1$. Hence

$$\begin{aligned}(I :_R x_1 \cdots x_{n-t} I_1 \cdots I_t) = {} & [\bigcup_{i=1}^{n} (I :_R x_1 \cdots \widehat{x_i} \cdots x_{n-t} I_1 \cdots I_t)] \\ & \cup [\bigcup_{i=1}^{n} (I :_R x_1 \cdots x_{n-t} I_1 \cdots \widehat{I_i} \cdots I_t)].\end{aligned}$$

(c)⇒(d) Since R is a u-ring, we are done.

(d)⇒(a) In special case of part (d), for every ideals $I_1, \ldots, I_n$ of R such that $I_1 \cdots I_n \nsubseteq I$ we have

$$(I :_R I_1 \cdots I_n) = (I :_R I_1 \cdots \widehat{I_i} \cdots I_n),$$

for some $1 \leq i \leq n$. Now, easily we can see that I is strongly n-absorbing. □

Remark 4.1 Note that in Theorem 4.2, for the case $n = 2$ we can omit the condition u-ring, by the fact that if an ideal (a subgroup) is the union of two ideals (two subgroups), then it is equal to one of them.

In the next theorem we investigate weakly n-absorbing ideals over u-rings. Notice that any Bézout ring is a u-ring, [13, Corollary 1.2].

Theorem 4.2 *Let I be a proper ideal of a u-ring R. Then the following conditions are equivalent:*
(*a*) *I is strongly weakly n-absorbing;*
(*b*) *I is weakly n-absorbing;*
(*c*) *For every t ideals $I_1, \ldots, I_t$, $0 \le t \le n$, and for every elements $x_1, \ldots, x_{n-t} \in R$ such that $x_1 \cdots x_{n-t} I_1 \cdots I_t \nsubseteq I$,*

$$\begin{aligned}(I :_R x_1 \cdots x_{n-t} I_1 \cdots I_t) = &[\bigcup_{i=1}^{n} (I :_R x_1 \cdots \widehat{x_i} \cdots x_{n-t} I_1 \cdots I_t)] \\ &\cup [\bigcup_{i=1}^{n} (I :_R x_1 \cdots x_{n-t} I_1 \cdots \widehat{I_i} \cdots I_t)] \\ &\cup (0 :_R x_1 \cdots x_{n-t} I_1 \cdots I_t);\end{aligned}$$

(*d*) *For every t ideals $I_1, \ldots, I_t$, $0 \le t \le n$, and for every elements $x_1, \ldots, x_{n-t} \in R$ such that $x_1 \cdots x_{n-t} I_1 \cdots I_t \nsubseteq I$,*

$$(I :_R x_1 \cdots x_{n-t} I_1 \cdots I_t) = (I :_R x_1 \cdots \widehat{x_i} \cdots x_{n-t} I_1 \cdots I_t),$$

for some $1 \le i \le n - t$ or

$$(I :_R x_1 \cdots x_{n-t} I_1 \cdots I_t) = (I :_R x_1 \cdots x_{n-t} I_1 \cdots \widehat{I_j} \cdots I_t),$$

for some $1 \le j \le t$ or

$$(I :_R x_1 \cdots x_{n-t} I_1 \cdots I_t) = (0 :_R x_1 \cdots x_{n-t} I_1 \cdots I_t).$$

Proof. (a)⇒(b) It is clear.
(b)⇒(c) We use induction on t. For $t = 0$, consider elements $x_1, \ldots, x_n \in R$ such that $x_1 \cdots x_n \notin I$. We show that

$$(I :_R x_1 \cdots x_n) = \cup_{i=1}^{n} (I :_R x_1 \cdots \widehat{x_i} \cdots x_n) \cup (0 :_R x_1 \cdots x_n).$$

Let $a \in (I :_R x_1 \cdots x_n)$, so $x_1 \cdots x_n a \in I$. Assume that $x_1 \cdots x_n a \neq 0$. Since $x_1 \cdots x_n \notin I$, then for some $1 \le i \le n$ we have $x_1 \cdots \widehat{x_i} \cdots x_n a \in I$, i.e., $a \in (I :_R x_1 \cdots \widehat{x_i} \cdots x_n)$. Consequently

$$(I :_R x_1 \cdots x_n) = [\bigcup_{i=1}^{n} (I :_R x_1 \cdots \widehat{x_i} \cdots x_n)] \cup (0 :_R x_1 \cdots x_n).$$

Now suppose $t > 0$ and assume that for integer $t - 1$ the claim holds. Let $x_1, \ldots, x_{n-t}$ be elements of R and let $I_1, \ldots, I_t$ be ideals of R such that $x_1 \cdots x_{n-t} I_1 \cdots I_t \nsubseteq I$. Consider element $a \in (I :_R x_1 \cdots x_{n-t} I_1 \cdots I_t)$. Thus $I_t \subseteq (I :_R x_1 \cdots x_{n-t} a I_1 \cdots I_{t-1})$. By hypothesis

$$(I :_R x_1 \cdots x_{n-t} a I_1 \cdots I_{t-1}) = (I :_R x_1 \cdots \widehat{x_i} \cdots x_{n-t} a I_1 \cdots I_{t-1}),$$

for some $1 \le i \le n - t$ or

$$(I :_R x_1 \cdots x_{n-t} a I_1 \cdots I_{t-1}) = (I :_R x_1 \cdots x_{n-t} a I_1 \cdots \widehat{I_j} \cdots I_{t-1}),$$

for some $1 \leq j \leq t-1$ or

$$(I :_R x_1 \cdots x_{n-t} a I_1 \cdots I_{t-1}) = (0 :_R x_1 \cdots x_{n-t} a I_1 \cdots I_{t-1}).$$

Consequently either $a \in (I :_R x_1 \cdots \widehat{x_i} \cdots x_{n-t} I_1 \cdots I_{t-1} I_t)$ for some $1 \leq i \leq n-t$ or $a \in (I :_R x_1 \cdots x_{n-t} I_1 \cdots \widehat{I_j} \cdots I_{t-1} I_t)$ for some $1 \leq j \leq t-1$ or $a \in (0 :_R x_1 \cdots x_{n-t} I_1 \cdots I_t)$. Hence

$$\begin{aligned}(I :_R x_1 \cdots x_{n-t} I_1 \cdots I_t) = & [\bigcup_{i=1}^{n} (I :_R x_1 \cdots \widehat{x_i} \cdots x_{n-t} I_1 \cdots I_t)] \\ & \cup [\bigcup_{i=1}^{n} (I :_R x_1 \cdots x_{n-t} I_1 \cdots \widehat{I_i} \cdots I_t)] \\ & \cup (0 :_R x_1 \cdots x_{n-t} I_1 \cdots I_t).\end{aligned}$$

(c)⇒(d) Since R is a u-ring, we are done.

(d)⇒(a) In special case of part (d), for every ideals $I_1, \ldots, I_n$ of R such that $I_1 \cdots I_n \nsubseteq I$ we have $(I :_R I_1 \cdots I_n) = (I :_R I_1 \cdots \widehat{I_i} \cdots I_n)$ for some $1 \leq i \leq n$ or $(I :_R I_1 \cdots I_n) = (0 :_R I_1 \cdots I_n)$. Now, easily we can see that I is strongly weakly n-absorbing. □

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