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QUASI-VARIATIONAL CHARACTERIZATION OF SOME OPTIMAL POINTS IN BANACH SPACES

BY

ROBERT T. VESCAN

1. **Introduction**. The quasi-variational inequalities (QVI's), which were the object of our concern in [23] [24] [25], are a class of problems involving partial derivatives, but which will be used here (in their most general formulation) to characterize some optimal points in Banach space. In [19], a QVI is stated as follows: *let $C_0 \subset C_1 \subset C$ be subsets of a real linear space E, h and g real functions with*

$$h : C_1 \times C \to (-\infty, +\infty], \; h(u, .) \not\equiv +\infty,$$
$$g : C_1 \times C \times C \to R, \; g(u, w, w) \leq 0, \; \forall u \in C_1, \; \forall w \in C;$$

the problem is to find $u \in E$ such that

$$u \in C_0; \; h(u, w) \geq g(u, u, w) + h(u, u), \; \forall w \in C. \tag{1.1}$$

In the beginning of this paper, we prove an abstract existence result (Th. 1) for (1.1), based on a nonempty *intersection* property; just because of this method our theorem is different from all known existence results which construct the solution of a QVI as a fixed point of a selection map (see, besides [19], also [5] [6] [8] [10] [13] [14] [15] [20]). To illustrate, but without exhausting the possibilities of our general framework, we apply first Theorem 1 to the *generalized complementarity problem*.

In Theorem 2 we initiate the study of weak *Pareto optimality under quasi-variational form*; next, the existence of weak Pareto minima is deduced by means of the same Theorem 1.

Further, we show the *equivalence* between a QVI and *a problem with complementarity type constraints* and establish one of the possible interpretations for this problem in the case of the Kuhn-Tucker conditions (Th. 4). The existence of solutions for that QVI is deduced as a corollary of a second abstract result (Th. 5) based, as classical, on a fixed point property of the usual selection map, but assuming that the implicit obstacle operator is *order bounded from below*; let us note that other known existence theorems assume either monotonicity properties (e.g. [5] [8] [15] [19]) or norm boundedness for the obstacle operator (e.g. [6] [10] [13]).

2. **QVI's for monotone functions.** Let E be a real linear separated topological space and C_1, C_2, C_3 convex subsets with $C_1 \cap C_2 \cap C_3 \neq \emptyset$; suppose that C_1 and C_3 are closed in E.

Theorem 1. *Let us consider a function g defined on $C_1 \times C_2 \times C_3$ with values in $R \cup \{+\infty\}$, so that the following conditions hold:*

(2.1) *for fixed $u \in C_1$, $v \in C_2$, $g(u, v, .)$ is proper convex on C_3,*

(2.2) $\forall u \in C_1$, $g(u, v, v) = 0$ *and* $g(u, v, w) + g(u, w, v) \leqslant 0$, $\forall v, w \in C_2 \cap C_3$;

(2.3) $\forall u \in C_1$, $v_1, v_2 \in C_2$, $w \in C_3$, *one has*

$$\limsup_{t \downarrow 0+} g(u, (1-t)v_1 + tv_2, w) = g(u, v_1, w);$$

(2.4) *for fixed $v \in C_2$, $g(w, v, w)$ is l.s.c. in the variable $w \in C_1 \cap C_3$.*

(2.5) *there exist a non-void compact $K \subset C_1 \cap C_2 \cap C_3$ and an element $v_0 \in K$ such that $g(w, v_0, w) > 0$, $\forall w \in (C_1 \cap C_3) \setminus K$.*

Under the above hypotheses, the

(2.6) $QVI \qquad g(u, u, w) \geqslant 0, \qquad \forall w \in C_1 \cap C_2 \cap C_3$

has at least one solution $u \in K$.

Proof. If, for any $v \in C_1 \cap C_2 \cap C_3$, we set $G(v) = \{u \in C_1 \cap \overline{C}_2 \cap C_3 ;\ g(u, v, u) \leqslant 0\}$, where $\overline{C}_2$ is the closure of C_2, then the sets $G(v)$ are nonempty ($v \in G(v)$) and closed (by (2.4)). For a given finite family $v_1, \ldots, v_n \in C_1 \cap C_2 \cap C_3$ each convex combination $\sum_{i=1}^{n} a_i v_i$ belongs just to $\bigcup_{i=1}^{n} G(v_i)$: otherwise, the inequalities $g\left(\sum_{i=1}^{n} a_i v_i, v_i, \sum_{i=1}^{n} a_i v_i\right) > 0$, $i = 1, \ldots, n$, would imply by (2.2), $g\left(\sum_{i=1}^{n} a_i v_i, \sum_{i=1}^{n} a_i v_i, v_i\right) < 0$, $i = 1, \ldots, n$ and further, using (2.1),

$$g\left(\sum_{i=1}^{n} a_i v_i, \sum_{i=1}^{n} a_i v_i, \sum_{i=1}^{n} a_i v_i\right) < 0,$$

which contradicts (2.2). Next, let us note that by (2.5), one has $G(v_0) \subset K$, hence $G(v_0)$ is compact.

Thus, according to Ky Fan's Lemma (see e.g. [21] – VII §2), the meet

$$\bigcap_{v \in C_1 \cap C_2 \cap C_3} G(v) \neq \emptyset, \quad \text{i.e.}$$

(2.7) $\qquad \exists u \in G(v_0) : g(u, v, u) \leqslant 0, \ \forall v \in C_1 \cap C_2 \cap C_3.$

It remains to show that (2.7) implies (2.6): if not, there would exist $\overline{w} \in C_1 \cap C_2 \cap C_3$ with $g(u, u, \overline{w}) < 0$; because $\limsup_{t \downarrow 0} g(u, (1-t)u + t\overline{w}, \overline{w}) = g(u, u, \overline{w})$, one would have

$$g(u, (1-t)u + t\overline{w}, \overline{w}) < 0 \quad \text{for} \quad \forall t \in (0, t_0];$$

as simultaneously (2.7) gives $g(u, (1-t)u + t\overline{w}, u) \leqslant 0$, we obtain by (2.1), $g(u, (1-t)u + t\overline{w}, t\overline{w} + (1-t)u) < 0$ contradictory to (2.2).

Remark 1. Non-void intersection properties have been previously used in solving (pure) variational inequalities, (see for instance [11] [19]); existence results for QVI's were established by this method only in our works [24] [25]. Theorem 1. differs from known existence theorems, not only because of the method, which permits not to ask the linear topology to be locally convex, but also for other reasons: the coercivity and continuity hypotheses have a *new adequate formulation*, (compare (2.5) (2.4) to (6.7) (6.4) from [19] – Ch. 1).

To illustrate the applicability of Theorem 1 to classical variational problems, we shall refer to the generalized complementarity problem ([17] [18]). Let E be a real reflexive Banach space with dual E' and P a closed convex cone in E with $P^+ = \{v' \in E' ;\ <v, v'> \geqslant 0, \ \forall v \in P\}$, where $< . , . >$ is the duality between E and E'. For a given map $F : P \to E'$ one looks for an element $u \in E$ which satisfies:

(2.8) $\qquad u \in P, \quad Fu \in P^+, \quad <u, Fu> = 0.$

Corollary 1. *If F is monotone hemicontinuous, then (2.8) has at least a solution, provided there exist $v_0 \in P$, $\alpha > 0$ such that*

(2.9) $\qquad <w - v_0, Fv_0> > 0, \ \forall w \in P$ *with* $\|w\| > \alpha$.

Proof. We use Th. 1, taking E with its weak topology, $C_1 = C_2 = C_3 = P$ and $g(u, v, w) = <w - v, Fv>$, $\forall u, v, w \in P$. Let us observe that (2.2) is fulfilled by the monotony of F and (2.3) by its hemicontinuity. The hypothesis (2.9) assures that (2.5) is satisfied for a ball $K \subset E$ with radius larger than α and $\|v_0\|$. Consequently, there is an $u \in P$ such that $<w - u, Fu> \geqslant 0$, $\forall w \in P$; it is known ([17]) that these last inequalities are equivalent to (2.8).

Remark 2. Corollary 1 improves a result due to M.S. B a z a r a a J.J. G o o d e, M.Z. N a s h e d [4], who assume that F is *additionally* bounded and has a *strong* coerciveness property (see also [3] – II 2.5).

3. **Weak Pareto minima.** Let $\mathcal{F}$ be a family of real functions on an arbitrary set X; $x_0 \in X$ is a weak Pareto minimum ($w.P.m.$) associated with the pair $(X, \mathcal{F})$ if

(3.1) for any $x \in X$, there exists $j_x \in \mathcal{F}$ such that $j_x(x) \geqslant j_x(x_0)$.

Denote by $\mathcal{A}(X, R)$ the linear space of all real functions on X, by conv $\mathcal{F}$ the convex hull and by co $\mathcal{F}$ the conical hull in $\mathcal{A}(X, R)$.

Lemma 1. *$x_0 \in X$ is a $w.P.m.$ associated with $(X, \mathcal{F})$ iff it is a $w.P.m.$ associated with X, conv $\mathcal{F}$) and iff it is a $w.P.m.$ for $(X$, co $\mathcal{F} \setminus \{0\})$.*

Proof. The assertions are obvious if one takes into account that (3.1) can be equally formulated as

(3.2) *there exists no $x \in X$ with $j(x) < j(x_0)$, $\forall j \in \mathcal{F}$.*

Remark 3. If an identically constant function belongs to $\mathcal{F}$ or conv $\mathcal{F}$, then every $x_0 \in X$ is a w.P.m.; hence, the preceding lemma allows to consider w.P.m. only for families $\mathcal{J} \setminus \{0\}$, where $\mathcal{J}$ is a convex cone generated in $\mathcal{A}(X, R)$ by a convex subfamily $\mathcal{F}$ which does not contain the function vanishing identically on X (as in Th. 3).

Remark 4. w.P. minima, for a subset X of a linear space and a convex cone $\mathcal{J}$, have been defined by a condition similar to (3.1) in [1]; a classical „efficient" point ([2] [7] [9] [12]) is particular w.P.m. not so much for it is associated with a finite family of functions, as for it involves a strengthening of the condition (3.1)⇔(3.2) (see [22]).

When the family $\mathcal{F}$ is infinite, the product topology on $\prod_{j \in \mathcal{F}} R_j$ with $R_j = R$, $\forall j$, can not be normed, therefore one can not proceed to study w.P. minima as optimal points for a Banach space valued map (like in the case of classical efficient points, (see e.g. [7]). This is the reason to consider the following convenient framework.

Let Y, Y', Z, Z' be linear spaces and consider the bilinear dualities $<.,.>: Y \times Y' \to R$, $<...>: Z \times Z' \to R$. For a subset C of another space S, we consider the maps $J: C \to Y$ and $L: C \to Z$. Given the convex (weakly) closed cones $P \subset Y$, $Q \subset Z$ and $D \subset C$ we denote

$$\hat{P} = \{p \in Y;\ <p, p'> > 0,\ \forall p' \in P^+,\ p' \neq 0\}$$

and

$$X = \{x \in D;\ Lx \in -Q\}.$$

Definition 1. *$x_0 \in X$ is called a w.P.m. of J on the set X if $J(x_0)$ is minimal element of $J(X)$ with respect to the partial transitive relation defined in Y by the cone $\hat{P}$.*

Remark 5. The w.P. minima of $J: X \to Y$ are just the w.P. minima in the sense (3.2) associated with $\mathcal{F} = \{j: X \to R;\ j = p' \circ J,\ p' \in P^+,\ p' \neq 0\}$. As a matter of fact, the situation from Def. 1. includes that from (3.2) by noteworthy constructions, (see the proof of Th. 3).

In view of the theorem which follows, let us suppose:

(3.3) *S is Banach space, C is closed convex, J is P-convex and L is Q-convex;*

(3.4) *$D = \operatorname{int} C \neq \emptyset$, the constraints $q' \circ L$, $q' \in Q^+$ and the cost criteria $p' \circ J$, $p' \in P^+$ are Gâteaux differentiable on D;*

(3.5) *P^+ and Q^+ are generated by convex (weakly) compact subsets, which do not contain $O_{Y'}$, respectively $O_{Z'}$;*

(3.6) *$\forall q' \in Q^+$, $q' \neq 0$, there exists $x \in D$ with $(q' \circ L)(x) < 0$.* We denote by „d" the differential and define for fixed $p' \in P^+$

(3.7) $g_{p'}(u, v, w) = d(p' \circ J)(x_2; x_3 - x_2) + d(q'_1 \circ L)(x_2; x_3 - x_2) + (q'_2 \circ L)(x_1) - (q'_3 \circ L)(x_1)$

with

$$u = (x_1, q'_1) \in C \times Q^+,\ v = (x_2, q'_2) \in D \times Q^+,\ w = (x_3, q'_3) \in S \times Q^+.$$

Theorem 2. *Let us suppose that (3.3) → (3.6) are valid. Then x_0 is a w.P.m. of J on X, iff there is a $p' \in P^+$, $p' \neq 0$ such that the*

(3.8) $QVI \quad g_{p'}(u, u, w) \geqslant 0, \qquad \forall w = (x, q') \in D \times Q^+$

has a solution of the form $u = (x_0, q'_0) \in D \times Q^+$.

Proof. We rewrite (3.8) in accordance with (3.7):

$$(3.9)\quad d(p' \circ J)(x_0; x - x_0) + d(q'_0 \circ L)(x_0; x - x_0) - (q' \circ L)(x_0) + (q'_0 \circ L)(x_0) \geqslant 0, \quad \forall x \in D,\ \forall q' \in Q^+.$$

If one takes $x = x_0$ in (3.9) and substitutes successively $2q'_0$, respectively 0 and $q' + q'_0$ for q', it follows $(q'_0 \circ L)(x_0) = 0$ as well as $(q' \circ L)(x_0) \leqslant 0$, $\forall q' \in Q^+$ the last inequalities show that $Lx_0 \in -Q^{++} = -Q$, hence $x_0 \in X$.

Besides, with $q' = q_0$ in (3.9), one obtains

$$(3.10)\qquad d(p' \circ J)(x_0; x - x_0) + d(q'_0 \circ L)(x_0; x - x_0) \geqslant 0,\ \forall x \in D.$$

Since $p' \circ J$ and $q'_0 \circ L$ can be extended outside C preserving their convexity (3.3) (assigning the value $+\infty$), the differentials in x_0 satisfy the characteristic property of the Moreau-Rockafellar subgradient (see [3] — II 2.1), therefore

$$(p' \circ J)(x) - (p' \circ J)(x_0) + (q'_0 \circ L)(x) - (q'_0 \circ L)(x_0) \geqslant d(p' \circ J)(x_0; x - x_0) + d(q'_0 \circ L)(x_0; x - x_0),\ \forall x \in C.$$

Consequently, recalling (3.10) and $(q'_0 \circ L)(x_0) = 0$, one has $(p'_0 \circ J)(x) - (p' \circ J)(x_0) \geqslant 0$ for $\forall x \in X$. Thus x_0 must be a w.P.m. of J on X (see Rem. 5).

The proof of the *necessity*. Conversely, let $x_0 \in X$ be a w.P.m. of J on X; if x_0 is a w.P.m. even on D we can use Th. 1.2.1 from [1] to infer that x_0 minimizes on D at least one function $p' \circ J$ with $p' \in P^+$ $p' \neq 0$; thus, in this situation, for that p' we have $d(p' \circ J)(x_0) = 0 \in S'$ and taking also $u = (x_0, 0) \in X \times Q^+$ we are in position to satisfy (3.9)⇔(3.8).

If $x_0 \in X$ is not a w.P.m. on D, there exists $\bar{x} \in D$ such that $(p' \circ J)(\bar{x}) < (p' \circ J)(x_0)$, $\forall p' \in P^+$, $p' \neq 0$. Then, if $\mathcal{P}$ denotes the cone of all positive functions on D and $\mathcal{J} = \{u: D \to R;\ u(\cdot) = <J(\cdot) - J, p'>(x_0),\ p' \in P^+\}$ the set $\mathcal{P} - \mathcal{J} \subset \mathcal{A}(D, R)$ is closed in the pointwise convergence topology (see Prop. 2.2.2 in [1]). Consider $\mathcal{L} = \{h: D \to R;\ h = q' \circ L,\ q' \in$ the weakly compact subset that generates $Q^+\}$; as the mapping $q' \in Q^+ \to q' \circ L$ is continuous from the weak topology of Z' to the pointwise convergence topology, $\mathcal{L}$ is compact in $\mathcal{A}(D, R)$ endowed with the mentioned topology. Suppose that $(\mathcal{P} - \mathcal{J}) \cap \mathcal{L} = \emptyset$; then, a hyperplane defined by a function $\bar{w}: D \to R$ with finite support, separates $\mathcal{P} - \mathcal{J}$ and $\mathcal{L}$ in $\mathcal{A}(D, R)$:

$$\sum_{x \in D} \bar{w}(x) h(x) < \alpha,\ \sum_{x \in D} \bar{w}(x)(p(x) - u(x)) > \alpha,\ \forall h \in \mathcal{L},\ p \in \mathcal{P},\ u \in \mathcal{J}.$$

It is easy to see that $\bar{x} = \sum_{x \in D} (\bar{w}(x) / \sum_{x \in D} \bar{w}(x)) x$ is a convex combination in D with $h(\bar{x}) < 0$, $\forall h \in \mathcal{L}$; hence

$$(q' \circ L)(\bar{x}) < 0,\ \forall q' \in Q^+,\ q' \neq 0$$

and we can use Th. 2.1.1 from [1] to deduce that x_0 minimizes on D a function $p' \circ J + q'_0 \circ L$ with $p' \in P^+$, $p' \neq 0$, $(q'_0 \circ L)(x_0) = 0$; clearly this means $(\mathcal{J} + \mathcal{L}) \cap \mathcal{P} \neq \emptyset$ in contradiction to $(\mathcal{P} - \mathcal{J}) \cap \mathcal{L} = \emptyset$. We conclude that there must exist $p' \in P^+$ and $q'_0 \in Q^+$, $q'_0 \neq 0$ with

$$<J(x) - J(x_0),\ p'> + (q_0 \circ L)(x) \geqslant 0,\ \forall x \in D.$$

$p' = 0$ is not possible because of (3.6). Taking $x = x_0$, one finds that $(q'_0 \circ L)(x_0) = 0$. Thus in this case x_0 is a minimum point in D of $p' \circ J + q'_0 \circ L$, hence we are in position to satisfy (3.9) with p' and $u = (x_0, q'_0)$ specified above.

Remark 6. Let us point out the weak form (3.6) of the Slater type hypothesis and especially the construction of $g+p'(u, v, w)$ appropriate to use Theorem 1 with $E = S \times Z'$, $C_1 = C \times Q^+$, $C_2 = D \times Q^+$, $C_3 = S \times Q^+$, C_1 and C_3 closed in E.

Theorem 3. *Let $\mathcal{F}$ and $\mathcal{L}$ be convex families of (non-identically null) real functions on a closed convex subset C of a reflexive Banach space S. We consider $X = \{x \in \operatorname{int} C ;\ h(x) \leqslant 0,\ \forall h \in \mathcal{L}\}$ defined in $\operatorname{int} C \neq \varnothing$ by the constraints h and suppose that:*

(3.11) *the cost functions $j \in \mathcal{F}$ and all $h \in \mathcal{L}$ are convex on C;*

(3.12) *$j \in \mathcal{F}$, $h \in \mathcal{L}$ are bounded on (bounded subsets of) C;*

(3.13) *$j \in \mathcal{F}$, $h \in \mathcal{L}$ are C^1–Fréchet differentiable on $\operatorname{int} C$, $h \in \mathcal{L}$ are continuous on C and $\forall h \in \mathcal{L}$, $\exists x \in \operatorname{int} C$ with $h(x) < 0$;*

(3.14) *$\mathcal{F}$ and $\mathcal{L}$ are compact with respect to the topology of uniform convergence on the bounded subsets of C, in the space of real functions bounded on (bounded subsets of) C. If there exist $j \in \operatorname{co} \mathcal{F} \setminus \{0\}$, C_0 compact $\subset \operatorname{int} C$, $\bar{\lambda}\bar{h} \in \operatorname{conv}(\mathcal{L} \cup \{0\})$, $\bar{x} \in C_0$ so that*

(3.15) $dj(\bar{x} ; x-\bar{x}) + \alpha dh(\bar{x} ; x-\bar{x}) - \alpha h(x) + \bar{\lambda}\bar{h}(x) > 0$, $\forall (x, \alpha h) \in (C \times \operatorname{co} \mathcal{L}) \setminus [C_0 \times \operatorname{conv}(\mathcal{L} \cup \{0\})]$,

then there exists a w.P.m. associated with $(X, \mathcal{F})$.

Proof. At first, let us frame the problem into the data of Th. 2, taking $Y' = Z'$ the space of real functions bounded on $\operatorname{int} C$ endowed with the topology mentioned in (3.14), $Y = Z$ the adjoint of the locally convex space Y and $J = L : C \to Y$ with $J(x) = w_x$, $\forall x \in C$, where $< w_x, u > = u(x)$, $\forall u \in Y'$. Clearly, by (3.12) the conical hulls $\mathcal{J} = \operatorname{co} \mathcal{F}$ and $\operatorname{co} \mathcal{L}$ are convex cones in Y'. We take also $P = \mathcal{J}^+$, $Q = (\operatorname{co} \mathcal{L})^+$ the polar cones in the duality $< Y, Y' >$. One can show that (3.14) and $0 \notin \mathcal{F}$, $0 \notin \mathcal{L}$ imply that $\mathcal{J}$, $\operatorname{co} \mathcal{L}$ are closed in the pointwise convergence topology and so much the more in Y'; therefore $P^+ = \mathcal{J}^{++} = \mathcal{J}$, $Q^+ = (\operatorname{co} \mathcal{L})^{++} = \operatorname{co} \mathcal{L}$ and by definition of J one has $\{u : C \to R ;\ u = p' \circ J,\ p' \in P^+\} = \mathcal{J}$, similarly $\{u : C \to R ; u = q' \circ L,\ q' \in Q^+\} = \operatorname{co} \mathcal{L}$. Thus $X = \{x \in \operatorname{int} C ;\ h(x) = < Lx, h > \leqslant 0,\ \forall h \in \mathcal{L}\} = \{x \in \operatorname{int} C ;\ Lx \in -(\operatorname{co} \mathcal{L})^+ = -Q\}$.

Now, we note that by Lemma 1 x_0 is a w.P.m. associated with $(X, \mathcal{F})$ iff it is a w.P.m. associated with $(X, \mathcal{J} \setminus \{0\})$ and further, in consideration of Rem. 5, iff it is a w.P.m. of J in the above construction. Applying Th. 2 (its hypotheses are verified thanks to (3.11)$\to$(3.14)), x_0 is a w.P.m. of J on X iff there exist $j \in \mathcal{J}$, $j \neq 0$ such that

(3.16) $g_j(x_0, \alpha_0 h_0, x_0, \alpha_0 h_0, x, \alpha h) \geqslant 0$, $\forall (x, \alpha h) \in \operatorname{int} C \times \operatorname{co} \mathcal{L}$ has a solution of the form $(x_0, \alpha_0 h_0) \in \operatorname{int} C \times \operatorname{co} \mathcal{L}$. Here

$$g_j(x_1, \alpha_1 h_1, x_2, \alpha_2 h_2, x_3, \alpha_3 h_3) = d_j(x_2 ; x_3 - x_2) + \alpha_1 dh_1(x_2 ; x_3 - x_2) - \\ - \alpha_3 h_3(x_1) + \alpha_2 h_2(x_1),$$

with $(x_1, \alpha_1 h_1) \in C \times \operatorname{co} \mathcal{L}$, $(x_2, \alpha_2 h_2) \in \operatorname{int} C \times \operatorname{co} \mathcal{L}$, $(x_3, \alpha_3 h_3) \in S \in \operatorname{co} \mathcal{L}$.

At last, we must show that for the j form (3.15), the conditions of Th. 1 are satisfied for the QVI (3.16) in the space $E = S \times Z'$. It is easy to check (2.1) (2.2((2.3) (2.5), using the monotonicity of dj, dh_1 for (2.2), the continuity of $dj(\cdot)$ $dh(\cdot)$ from $\operatorname{int} C$ into the conjugate space S' for (2.3)

and taking $K = C_0 \times \operatorname{conv}(\mathcal{L} \cup \{0\})$, $v_0 = (\bar{x}, \bar{\lambda}\bar{h})$ for (2.5). With regard to (2.4), we insist on the continuity of $g_j(x, \alpha h, x_2, \alpha_2 h_2, x, \alpha h)$ in the argument $(x, \alpha h) \in C \times \operatorname{co} \mathcal{L}$. If $\{\alpha_\delta h_\delta\}_{\delta \in \Delta}$ is a sequence in $\operatorname{co} \mathcal{L}$ which converges to αh uniformly on the bounded subsets of C then

$$|d\alpha_\delta h_\delta(x_2 ; k) - d\alpha h(x_2 ; k)| \leqslant \\ \leqslant \lim_{t \to 0} \left(\left| \frac{(|\alpha_\delta h_\delta(x_2 + tk) - \alpha h(x_2 + tk)}{t} \right| + \left| \frac{\alpha_\delta h_\delta(x_2) - \alpha h(x_2)}{t} \right| \right) \leqslant \lim_{t \to 0} (\varepsilon/|t| + \varepsilon/|t|)$$

for $\delta > \delta_0$, where δ_0 is chosen corresponding to ε and a bounded set that contains x_2 and $x_2 + tk$; it follows that $\forall k \in S$, $\lim\limits_{\delta \in \Delta} d\alpha_\delta h_\delta (x_2 ; k) = d\alpha h(x_2 ; k)$ (worthy to note: uniformly with respect to x_2 in a bounded set), hence $d\alpha h(x_2)$ is continuous in the variable αh from $\operatorname{co} \mathcal{L}$ to S'-weak*. This shows the continuity of the term $\alpha dh(x_2 ; - x_2)$ in αh. The continuity of the term $-\alpha h(x)$ in x and αh is a consequence of Lagrange's theorem: if $\lim\limits_{\delta \in \Delta} x_\delta = x$ in C and again $\lim\limits_{\delta \in \Delta} \alpha_\delta h_\delta = \alpha h$ in $\operatorname{co} \mathcal{L}$, then $|\alpha_\delta h_\delta(x_\delta) - \alpha h(x)| \leqslant \sup\limits_{0 \leqslant \theta \leqslant 1} \|d\alpha_\delta h_\delta(x + \theta(x_\delta - x))\|_{S'} \|x_\delta - x\|_S + |\alpha_\delta h_\delta(x) - \alpha h(x)|$; because αh is of class C^1, for each fixed k, for sufficiently large δ, $d\alpha_\delta h_\delta(x + \theta(x_\delta - x))$ (its convergence was emphasized before) is bounded in R by a constant which depends only on k, αh and the bounded set B of the combinations $x + \theta(x_\delta - x)$ thus, $d\alpha_\delta h_\delta(x + \theta(x_\delta - x))$ is bounded in S' by a constant which depends on αh and B, hence the second side of the last inequality is however small for sufficiently large δ.

4. A complementarity type problem as a Q.V.I. For a mapping F of a subset $D(F)$ of R^N, following an idea from [16], let us consider the problem

(4.1) *find $y, z \in R^N$ such that $z \geqslant Fy$, $y \geqslant 0$, $z \geqslant 0$, $< y, z > = 0$.*

$< \cdot, \cdot >$ denotes the usual inner product in the Euclidean space.

Theorem 4. i) *The problem* (4.1) *has a solution $(y, z) \in R^N \times R^N$ if and only if y is a solution of the QVI*

$$y \in R^N, \ y \geqslant 0,$$
$$< w - y, \ Fy > \geqslant 0, \ \forall w \in R^N \text{ with } 0 \leqslant w \leqslant y. \tag{4.2}$$

ii) *Let f, g_i, $i = 1, \dots, N$ be real strictly convex functions on a normed space X and consider the univalent F defined by*

$$Fy = (g_i(x))_{i=1,\dots,N} \Leftrightarrow 0 \in \partial \left(f + \sum_{i=1}^{N} y_i g_i \right)(x). \tag{4.3}$$

The problem (4.1) *with F* (4.3) *can be solved if and only if there exist $y \geqslant 0$ and an element $x_0 \in X$ such that*

(4.4) $f + \sum\limits_{i=1}^{N} y_i g_i$ *attains its minimum an X at the point x_0,*

(4.5) $y_i g_i(x_0) \leqslant 0$, $i = 1, \dots, N$.

Proof. i) If (y, z) is a solution of (4.1), then the vector y may have components $y_i \neq 0$ and components $y_k = 0$. For w with $0 \leqslant w \leqslant y$, one has

$<w-y, Fy> = \sum_i (w_i - y_i)(Fy)_i$, since $0 \leqslant w_k \leqslant y_k = 0$; from (4.1) it follows that for $y_i \neq 0$ one must have necessarily $z_i = 0$ and further $z_i \geqslant (Fy)_i$ imply $(Fy)_i \leqslant 0$; as we have also $w_i \leqslant y_i$, we may assert that $<w-y, Fy> \geqslant 0$.

Conversely, if y is a solution of (4.2), one can find z such that (y, z) verifies (4.1) : for the components $y_k = 0$, take arbitrary $z_k \geqslant (Fy)_k$, $z_k \geqslant 0$; for the components $y_i > 0$, as suitable choices of w in (4.2) show that $(Fy)_i \leqslant \leqslant 0$, it is clear that we are in position to take $z_i = 0$.

ii) Because we must have $z \geqslant Fy$, it is understood that Fy is non-void i.e. there exists x_0 such that 0 belongs to the subdifferential $\partial\left(f + \sum_{i=1}^{N} y_i g_i\right)(x_0)$ which means just (4.4). Next, $y \geqslant 0$, $z \geqslant 0$, $<y, z> = 0$ and $z \geqslant (g_i(x_0))_{i=1,\dots,N}$ can hold simultaneously if and only if, when a component $y_i \neq 0$ then one has correspondingly $g_i(x_0) \leqslant 0$; thus, interfere in our problem the relations (4.5).

Remark 7. We conclude that the QVI (4.2) with F (4.3) characterizes the minimum points $x_0 \in X$ of all $f + \sum_k y_k g_k$, $y_k > 0$, with $g_k(x_0) \leqslant 0$; compare to the Kuhn-Tucker conditions ([3]–III § 1.1).

Remark 8. (4.2) is a particular case of the problems of type

$$\text{(4.6)} \qquad \begin{aligned} & u \in P, \ u \leqslant Mu, \\ & g(u, w) \geqslant 0, \ \forall w \in P, \ w \leqslant Mu. \end{aligned}$$

In (4.6) P is a closed convex subset of a reflexive Banach space $(E, \|\cdot\|)$, „$\leqslant$" is an order induced in E by a closed convex cone. $M : E \to E$ is an implicit constraint operator and g is a real function defined on $P \times P$. Note that (4.6) is a QVI obtained from (1.1) by taking $h(u, w) = \begin{cases} 0 \text{ if } w \leqslant Mu, \\ +\infty \text{ otherwise.} \end{cases}$

Let us denote the norm (weak) topology of E by $\tau(\tau_2)$ and state.

Theorem 5. *In the above situation assume also that E has linear continuous injection in a linear topological space (E_1, τ_1). If P has compact injection in E_1 and*

(4.7) *M is continuous from each (closed) ball B_0 of E taken with the τ_1-topology, to (E, τ) ;*

(4.8) *for any sequence $(u_n)_n \subset B_0$ with $u_n \to u \in B_0$ and $w \leqslant Mu$, $w \in P$, there exist $w_n \leqslant Mu_n$, $w_n \in P$ with $\tau - \lim_{n \to \infty} w_n = w$.*

(4.9) *g is τ u.s.c. in the first variable, convex in the second one, $\tau_1 \times \tau$-u.s.c. on each $(P \cap B_0) \times P$ and satisfies $g(v, v) \leqslant 0$, $\forall v \in P$;*

(4.10) *there are $k > 0$ and $w_0 \in P$ such that $w_0 \leqslant Mu$, $\forall u \in P$ with $\|u\| \leqslant k$ and $g(v, w_0) < 0$, $\forall v \in P$ with $\|v\| > k$; then the QVI (4.6) admits at least a solution.*

Proof. We consider the selection map for (4.6), that is to say $S : \{u \in \in P ; \|u\| \leqslant k\} = C_0 \to 2^P$ with $v \in S(u)$ if v is a solution of the variational problem ;

(4.11) $\quad v \in P, v \leqslant Mu ; \ g(v, w) \geqslant 0, \ \forall w \in P, \ w \leqslant Mu.$

Let us note at once that $S(u)$ are non-empty : according to Th. 2.1 from

[19]-Ch. 1, the solutions of (4.11), which is a $V.I.$ on the closed convex set $C = \{v \in P ; v \leqslant Mu\}$ in the space (E, τ_2), form a non-empty convex τ_2-compact subset in $\{u \in E ; \|u\| \leqslant k\} \cap C$. We prove in the sequel that S has $\tau_2 \times \tau_2$-closed graph in $C_0 \times C_0$. By our hypotheses, C_0 is both τ_2-compact in E and τ_1-relatively compact in E_1 ; since τ_2 and τ_1 are both stronger on E than the weak topology of E_1, it follows that C_0 is even τ_1-compact and τ_2, τ_1 are equal on C_0. Let us take a sequence $(u_n, v_n)_n \subset S$ which has the τ_1 or τ_2-limit $(u, v) \in C_0 \times C_0$. On the one hand, by (4.7) the inequalities $w_n \leqslant Mu_n$ imply $w \leqslant Mu$; on the other hand, for w fixed in P with $w \leqslant Mu$ and for w_n given by (4.8), we have $g(v_n, w_n) \geqslant 0$ and by (4.9) these inequalities imply $g(v, w) \geqslant \geqslant 0$; thus, $v \in S(u)$ holds.

We come to the conclusion that one may apply Kakutani-Ky Fan's theorem to the multi-valued map S and find $u \in S(u)$, that is just a solution of (4.6).

Remark 9. We have no monotony assumptions on g : compare (4.9) to (7.3) (7.8) from [19]-Ch. 1. The hypothesis (4.10) of lower boundedness for M (on a ball with sufficiently large radius, see also [23]) takes the place of different monotonicity properties of M (from e.g. [5] [8] [15] [19]) or abstract assumptions on the existence of a compact set stable under the selection (e.g. in [10]-Th. 1).

Corollary 2. *The QVI (4.2) has at least a solution, provided $<w, Fv>$ is u.s.c. in the argument (v, w), $<v, Fv>$ is convex l.s.c. in the argument v and $<v, Fv> > 0$, $\forall v \geqslant 0$, $v \neq 0$.*

Proof. Use Th. 5. with $E = E_1 = R^N$, $P = \{v \in R^N ; v \geqslant 0\}$, $g(v, w) = = <w - v, Fv>$, $Mu = u$, $\forall u \in R^N$, $w_0 = 0$, k arbitrary > 0.

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Polytechnic Institute
Department of Mathematics,
6600-Iaşi, R. S. România

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DUALITY FOR VECTORIAL NONCONVEX OPTIMIZATION BY CONVEXIFICATION AND APPLICATIONS

BY

CONSTANTIN ZĂLINESCU

Introduction. Many papers concerning (strong) vectorial optimization have appeared in the last few years. In the convex case, the main attempt was to generalize the results from the scalar one as, f.i., the Fenchel-Rockafellar duality theorem ([1], [10], [23], [30]), the formulae for conjugate operators and ε-subdifferentials ([10], [13], [14], [15], [23], [24]) and theorems of Kuhn-Tucker type ([10], [30]). The non-convex case was also considered and for this case Fenchel type theorems ([8] for the scalar case, [20]) were established and the Lagrangeans asociated with such problems were studied ([7], [16]). We elaborated ourselves a paper ([28]) in which we extended the Fenchel-Rockafellar duality theory for (strong) vectorial programming. The objective functions were non-convex, the imposed conditions assured the existence of their convex hulls and the validity of duality results for the relaxed problems. Thus it was reobtained all the formulae for the calculus of conjugate operators and subdifferentials established in [13], [14], some of them in more general conditions, in an unified and simpler manner. Meantime some papers have appeared with similar results (see [3], [4], [24]), so that our mentioned paper could not be published in the initial form. So, the main ideas are taken from [28], but we insist mainly on the non-convex case. The paper is divided in four sections. The first section presents the preliminary results and notations needed in the sequel. The second one is devoted to duality theory. The main result is Theorem 2.3 which states the existence of the convex hull of the considered function and gives a duality result. Theorem 2.3 will help to obtain many duality results and formulae for calculating conjugate operators and ε-subdifferentials. In fact, as mentioned in [11], as soon as a duality result is valid, it is easy to establish such formulae. We also apply Theorem 2.3 to obtain the same kind of results for functions of the form $g \circ f$, where f, g are convex and g is increasing, getting the corresponding formulae from [14], [15] in more general conditions. In the same manner there are obtained a „Farkas lemma" and a „sandwich" theorem. It is also given a simple proof of [17, Th. 1] concerning the separation of sets in product spaces. In the third section there are given conditions which assure that the primal problem and the relaxed primal problem have the same value, or an optimal solution of the primal problem is also optimal for the relaxed one. As applications of these conditions and the results of Section 2, we reobtain the results concerning the non-convex programming in [8], [16], [20], some of them in more general conditions. We also give a Kuhn-Tucker theorem for convex programming. Finally, in Section 4 it is underlined the continuous case.