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PROPERTIES OF SOLUTIONS OF HIGHER ORDER NONLINEAR DIFFERENCE EQUATIONS II

BY

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1. Introduction. This is in continutation to our work [4] on asymptotic and oscillatory properties of solutions of the nonlinear difference equation

$$\Delta^n x(t) + f(t, \hat{x}(t)) = 0,\ t \in I, \tag{1.1}$$

where I is the discrete set $\{0, 1, \ldots\}$ and $\hat{x}(t)$ stands for $(x(t), \Delta x(t), \ldots, \Delta^{n-1}x(t))$, Δ is the difference operator $\Delta x(t) = x(t+1) - x(t)$. Besides (1.1), we shall also consider its variant

$$\Delta^n x(t) + p(t)g(t, \hat{x}(t)) = h(t, \hat{x}(t)),\ t \in I, \tag{1.2}$$

the form of which will be clear from the type of conditions we shall impose on p, g and h.

In (1.1) and (1.2), we shall assume that f, p, g and h are defined in their domain of definition and for all initial values of the type

$$\Delta^i x(t_1) = A_i;\ 0 \leqslant i \leqslant n-1,\ t_1 \in I$$

solutions of (1.1), (1.3) and (1.2), (1.3) exist for all $t \geqslant t_1$, $t \in I$.

Throughout the paper, by a solution we shall refer to a nontrivial solution. A solution $x(t)$ will be called nonoscillatory if it is eventually of fixed sign; the solution is called oscillatory if there is no end of t_1 and $t_2 (t_1 < t_2)$ in I such that if $x(t_1) > 0 (x(t_1) < 0)$ then $x(t_2) < 0$ $(x(t_2) > 0)$; and it is called Z-type if it changes sign arbitrarily but is ultimately nonnegative or nonpositive. We shall denote by F_k where k is a nonnegative integer, the class of all functions $u(t)$ defined on I and $|u(t)| = O((t)^{(k)})$ as $t \to \infty$. A solution $x(t)$, which belongs to F_k will be called F_k solution. For example, the equation

$$\Delta^2 x(t) + 8/3\Delta x(t) + 4/3x(t) = 0,\ t \in I \tag{1.4}$$

has an oscillatory solution $x(t) = (-1)^t$, and a nonoscillatory solution $x(t) = (1/3)^t$, whereas $x(t) = (-1)^t$ for all finite t and $x(t) = 0$ otherwise, is a F_0 solution.

Section 2, contains five lemmas, first three of which are adopted from our earlier work [1, 2, 4], whereas fourth lemma is the discrete analogue of L'Hospital rule, the last lemma is about a particular first order difference equation. In section 3, we prove several results about asymptotic and oscillatory properties of solutions of equations (1.1) and (1.2). Most of the results are known for the continuous case e.g. see [3,5–12] and cover some results known for the discrete case ($n=2$) e.g. see references in [4].

In what follows, we shall denote the set $\{t_1, t_1+1, \ldots\}$ where $t_1 \in I$ by I_{t_1}; $(t)^{(m)} = t(t-1)\ldots(t-m+1)$ the usual factorial notation and $\nabla x(t) = x(t) - x(t-1)$.

2. Some Lemmas. Lemma 2.1. [4]. *Let $u(t)$ be some function defined on I_{t_1}. Then, for $0 \leqslant k \leqslant n-1$*

$$\Delta^k u(t) = \sum_{i=k}^{n-1} \frac{(t-t_1)^{(i-k)}}{(i-k)!} \Delta^i u(t_1) + \frac{1}{(n-k-1)!} \sum_{s=t_1}^{t-n+k} (t-s-1)^{(n-k-1)} \Delta^n u(s).$$

Lemma 2.2 [4]. *Let $1 \leqslant q \leqslant n-1$ and $u(t)$ be some function defined on I. Then,*

(a) $\liminf_{t\to\infty} \Delta^q u(t) > 0$ *implies* $\lim_{t\to\infty} \Delta^i u(t) = \infty$, $0 \leqslant i \leqslant q-1$

(b) $\limsup_{t\to\infty} \Delta^q u(t) < 0$ *implies* $\lim_{t\to\infty} \Delta^i u(t) = -\infty$, $0 \leqslant i \leqslant q-1$.

Lemma 2.3. *Let $A \geqslant 0$, $f(t) \geqslant 0$ and $u(t) \geqslant 0$ on I_{t_1} and $u(t)$ satisfies an inequality*

$$u(t) \leqslant A + \sum_{s=t_1}^{t-1} f(s) u^{\alpha}(s). \tag{2.1}$$

Then, for $t \in I_{t_1}$

$$u(t) \leqslant A \prod_{s=t_1}^{t-1} (1+f(s)), \text{ if } \alpha = 1, \quad u(t) \leqslant \left[A^{1-\alpha} + (1-\alpha) \sum_{s=t_1}^{t-1} f(s)\right]^{1/(1-\alpha)}, \text{ if } \alpha \neq 1.$$

If $\alpha > 1$, it is assumed that $A^{1-\alpha} + (1-\alpha) \sum_{s=t_1}^{t-1} f(s) > 0$.

For the proof and several linear and nonlinear generalizations of (2.1) see [1, 2].

Lemma 2.4. *Let $u(t)$ and $v(t)$ be defined on I and $v(t) > 0$, $\nabla v(t) < 0$ for all sufficiently large t in I. Then, if $u(t) \to 0$ and $v(t) \to 0$*

$$\lim_{t\to\infty} \frac{\nabla u(t)}{\nabla v(t)} = r \text{ implies } \lim_{t\to\infty} \frac{u(t)}{v(t)} = r. \tag{2.2}$$

Proof. Let t_1 be sufficiently large so that for all $t \geqslant t_1$, in I, $v(t) > 0$ and $\nabla v(t) < 0$. If r is finite, $\lim_{t\to\infty} \nabla u(t)/\nabla v(t) = r$ implies that, there exists t_2 large enough so that $t_2 \geqslant t_1$ and for all $t \geqslant t_2$

$$(r-\varepsilon) \leqslant \frac{\nabla u(t)}{\nabla v(t)} \leqslant (r+\varepsilon).$$

Since, $\nabla v(t) < 0$ for all $t \geqslant t_2$, the above inequality implies that $-(r-\varepsilon)\nabla v(t) \leqslant -\nabla u(t) \leqslant -(r+\varepsilon)\nabla v(t)$. Let $t_3 \geqslant t_2$ be arbitrary in I, then summing the above inequality from t_3+1 to t, we find

$$-(r-\varepsilon)[v(t) - v(t_3)] \leqslant -[u(t) - u(t_3)] \leqslant -(r+\varepsilon)[v(t) - v(t_3)],$$

which from $v(t_3) > 0$, implies that

$$(r-\varepsilon)\left[1 - \frac{v(t)}{v(t_3)}\right] \leqslant \frac{u(t_3)}{v(t_3)} - \frac{u(t)}{v(t_3)} \leqslant (r+\varepsilon)\left[1 - \frac{v(t)}{v(t_3)}\right]$$

and in the above inequality as $t \to \infty$, we get $(r-\varepsilon) \leqslant u(t_3)/v(t_3) \leqslant r+\varepsilon$. Since t_3 is arbitrary, the above inequality holds for all $t_3 \geqslant t_2$ and thus (2.2) is true for finite r.

If r is infinite say $+\infty$ (the case $-\infty$ can be treated similarly) then, for arbitrary $K > 0$, there exists $t_2 \geqslant t_1$ such that for all $t \geqslant t_2$ $\nabla u(t)/\nabla v(t) \geqslant K$, and from $\nabla v(t) < 0$, we find $\nabla u(t) \leqslant K \nabla v(t)$. Let $t_3 \geqslant t_2$ be arbitrary in I, then summing the above inequality from t_3+1 to t, we find $u(t) - u(t_3) \leqslant K[v(t) - v(t_3)]$ and in the above inequality as $t \to \infty$, we get $u(t_3)/v(t_3) \geqslant K$. Since K and t_3 are arbitrary, the above inequality holds for all $K > 0$ and $t_3 \geqslant t_2$ which implies (2.2) for r being infinite.

Lemma 2.5. *Consider the difference equation*

$$\nabla z(t) - \frac{\mu}{t} z(t) + \frac{f(t)}{t} = 0, \tag{2.3}$$

where μ is a positive integer and f is defined on I_{t_1}, $t_1 > \mu$ and nonoscillatory. If $\lim_{t\to\infty} |f(t)| = \infty$, and $z(t)$ is the solution of (2.3) with $z(t_1) = 0$, then $\lim_{t\to\infty} z(t) = \pm\infty$.

Proof. By direct substitution, it is easy to verify that

$$z(t) = -(t)^{(\mu)} \sum_{s=t_1+1}^{t} \frac{f(s)}{s^{(\mu+1)}}, \quad t \geqslant t_1$$

is the solution of (2.3) satisfying $z(t_1) = 0$. Since, f is of constant sign for all large t, the summation

$$\sum_{s=t_1+1}^{\infty} \frac{f(s)}{s^{(\mu+1)}}$$

exists in the extended real line. If the value of this summation is different from zero, the result follows. If it is zero, then let

$$u(t) = \sum_{s=t_1+1}^{t} \frac{f(s)}{(s)^{(\mu+1)}}$$

and $v(t) = 1/(t)^{\mu}$, so that lemma 2.4 is applicable and, we find

$$\lim_{t\to\infty} \frac{u(t)}{v(t)} = \lim_{t\to\infty} \left[\frac{f(t)}{(t)^{(\mu+1)}} \times -\frac{(t)^{(\mu+1)}}{\mu}\right] = -\lim_{t\to\infty} \frac{f(t)}{\mu} = \pm\infty.$$

Thus, $\lim_{t\to\infty} z(t) = \pm\infty$.

3. Main Results. Theorem 3.1. *Assume that there exist integers k, l, m, such that $0 \leqslant m \leqslant n-1$, $0 \leqslant 1 \leqslant k \leqslant n-m-1$; and for every nonoscillatory $u(t) \in F_k$ with*

$$\lim_{t\to\infty} \inf \left(\frac{|u(t)|}{(t)^{(l)}}\right) \neq 0, \quad \sum^{\infty} (s)^{(m)} f(s, \hat{u}, (s)) = \pm\infty. \tag{3.1}$$

Then, for all nonoscillatory F_k - solutions $x(t)$ of (1.1), $\lim_{t\to\infty}\inf |x(t)|/(t)^{(l)} = 0$.

Proof. Let $x(t)$ be a nonoscillatory F_k-solution with $\lim_{t\to\infty}\inf |x(t)|/(t)^{l} = 0$. Without loss of generality, we assume that $x(t) > 0$ on I_{t_1}, where $t_1 > \max\{l, m\}$. The case $x(t) < 0$ can be treated similarly.

We define

$$q^p_{ij}(t) = \sum_{s=t_1}^{t-1} (s)^{(i)} \nabla^j x(t+p)$$

and find

$$q^p_{ij}(t) = (s)^{(i)} \nabla^{j-1} x(s+p) \Big|_{s=t_1}^{t} - \sum_{s=t_1}^{t-1} \Delta (s)^{(i)} \Delta^{j-1} x(s+p+1) = (t)^{(i)} \Delta^{j-1} x(t+p) - (t_1)^{(i)} \Delta^{j-1} x(t_1+p) - i \sum_{s=t_1}^{t-1} (s)^{(i-1)} \Delta^{j-1} x(s+p+1) \tag{3.2}$$

also, since $\nabla q^{p+1}_{i-1,j-1}(t) = (t-1)^{(i-1)} \nabla^{j-1} x(t+p)$, equation (3.2) takes the form

$$q^p_{ij}(t) = t\nabla z(t) - (t_1)^{(i)} \nabla^{j-1} x(t_1+p) - iz(t).$$

where $z(t) = q^{p+1}_{i-1,j-1}(t)$. Equation (3.3) can be written as

$$\nabla z(t) - \frac{i}{t} z(t) + \frac{f^p_{ij}(t)}{t} = 0, \quad z(t_1) = 0 \, (t_1 > m) \tag{3.4}$$

where

$$f^p_{ij}(t) = -(t_1)^{(i)} \Delta^{j-1} x(t_1+p) - q^p_{ij}(t). \tag{3.5}$$

Let $i = m$, $j = n$ and $p = 0$ then, (3.5) from (1.1) is in fact

$$f^0_{mn}(t) = -(t_1)^{(m)} \Delta^{n-1} x(t_1) + \sum_{s=t_1}^{t-1} (s)^{(m)} f(s, \hat{x}(s)),$$

and from (3.1), $f^0_{mn}(t)$ is nonoscillatory and $\lim_{t\to\infty} |f^0_{mn}(t)| = \infty$.

Thus, from lemma 2.5, we find

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} q^1_{m-1,n-1}(t) = \pm\infty. \tag{3.6}$$

Next, since

$$f^1_{m-1,n-1}(t) = -(t_1)^{(m-1)} \nabla^{n-2} x(t_1+1) - q^1_{m-1,n-1}(t)$$

and, from (3.6) we find, $f^1_{m-1,n-1}(t)$ is nonoscillatory and $\lim_{t\to\infty} |f^1_{m-1,n-1}(t)| = \infty$. Thus, lemma 2.5 is again applicable and, we get $\lim_{t\to\infty} q^2_{m-2,n-2}(t) = \pm\infty$. Continuing this way, we find

$$\lim_{t\to\infty} q^m_{0,n-m}(t) = \pm\infty. \tag{3.7}$$

From the definition of $q^p_{ij}(t)$, we have

$$q^m_{0,n-m}(t) = \Delta^{n-m-1} x(t+m) - \nabla^{n-m-1} x(t_1+m)$$

and from (3.7), we conclude $\lim_{t\to\infty} \Delta^{n-m-1} x(t+m) = \pm\infty$. The case $\lim_{t\to\infty} \Delta^{n-m-1} x(t+m) = -\infty$ is impossible from lemma 2.2, since it contradicts the fact that $x(t)$ is positive, and thus

$$\lim_{t\to\infty} \Delta^{n-m-1} x(t+m) = \infty, \text{ i.e., } \lim_{t\to\infty} \Delta^{n-m-1} x(t) = \infty.$$

Since, $x(t) > 0$ and belongs to F_k, there exists $C > 0$ such that $x(t) < C(t)^{(k)}$ for large $t \in I$. Thus, the function $v(t) = x(t) - C(t)^{(k)}$ is negative for large $t \in I$, but since $k \leqslant n-m-1$, we find $\lim_{t\to\infty} \Delta^{n-m-1} v(t) = \lim_{t\to\infty} \Delta^{n-m-1} x(t) = \infty$, which from lemma 2.2, leads to a contradiction that $v(t)$ is negative. This completes the proof.

Remark. The conclusion in theorem 3.1 is only for nonoscillatory F_k-solutions. In fact, for the equation (1.4) the hypothesis for $m = k = 0$ are satisfied and for nonoscillatory solution $u(t) = (1/3)^t$, $\lim_{t\to\infty}\inf |u(t)| = 0$, whereas for the oscillatory solution $u(t) = (-1)^t$, $\lim_{t\to\infty}\inf |u(t)| = 1$. It will be desirable to find corresponding result for all the solutions.

Remark. If in theorem 3.1, $k = 0$ then, as conclusion we have that for all bounded nonoscillatory solutions of (1.1), $\lim_{t\to\infty}\inf |x(t)| = 0$. The next result is a corresponding result for all nonoscillatory solutions.

Theorem 3.2. *If there exist $K \geqslant 0$ such that for any $u(t)$ defined on I $\lim_{t\to\infty}\inf u(t) > K$ ($\lim_{t\to\infty}\inf u(t) < -K$) we have*

$$\sum^{\infty} f(s, \hat{u}(s)) = +\infty \, (-\infty), \tag{3.8}$$

then, every nonoscillatory solution $x(t)$ of (1.1) *satisfies* $\lim_{t\to\infty}\inf |x(t)| \leqslant K$.

Proof. Let $x(t)$ a be nonoscillatory solution, say $x(t) > 0$ for all $t \geqslant t_1$ in I, and assume that $\lim_{t\to\infty}\inf x(t) > K \geqq 0$. The case $x(t) < 0$ can be treated similarly. Summing the equation (1.1) from t_1 to t, we find

$$\Delta^{n-1} x(t) = \Delta^{n-1} x(t_1) - \sum_{t_1}^{t-1} f(s, \hat{x}(s)) \to -\infty$$

as $t \to \infty$. Thus, $\lim_{t\to\infty}\sup \Delta^{n-1} x(t) < 0$ and lemma 2.2 implies that $\lim_{t\to\infty} x(t) = -\infty$, which is a contradiction to our assumption that $x(t) > 0$.

Theorem 3.3. *Assume that there exist integers k, m such that $0 \leqslant m \leqslant n-1$, $0 \leqslant k \leqslant n-m-1$; and for every nonoscillatory $u(t) \in F_k$ with $\lim_{t\to\infty}\inf |u(t)| \neq 0$ there exist constants A, B ($AB > 0$), such that for all large $t \in I$, $A \leqslant$*

$\leqslant g(t, \hat{u}(t)) \leqslant B$; *and for such* $u(t)$ *there exists a nonnegative function* $H(t)$ *such that for all large* $t \in I$, $|h(t, \hat{u}(t))| \leqslant H(t)$.

If for every $\mu_1 > 0$ *and* $\mu_2 > 0$

$$\sum^{t} (t)^{(m)} [\mu_1 p_+(t) - p_-(t) - \mu_2 H(t)] = \infty$$

or

$$\sum^{t} (t)^{(m)} [\mu_1 p_-(t) - p_+(t) - \mu_2 H(t)] = \infty$$

where $p_+(t) = \max[p(t), 0]$, $p_-(t) = \max[-p(t), 0]$; *then, for all nonoscillatory* F_k*-solutions* $x(t)$ *of* (1.2) $\liminf_{t\to\infty} |x(t)| = 0$.

Proof. For any nonoscillatory F_k-solution with $\liminf_{t\to\infty} |x(t)| \neq 0$, we find for all large $t \geqslant t_1$ in I, that

$$\sum_{t_1}^{t} (s)^{(m)} [A p_+(s) - B p_-(s) - H(s)] \leqslant \sum_{t_1}^{t} (s)^{(m)} [p(s) g(s, \hat{u}(s)) - h(s, \hat{u}(s))] \leqslant$$

$$\leqslant \sum_{t_1}^{t} (s)^{(m)} [B p_+(s) - A p_-(s) + H(s)].$$

Thus, for A and B positive

$$B \sum_{t_1}^{t} (s)^{(m)} \left[\frac{A}{B} p_+(s) - p_-(s) - \frac{1}{B} H(s)\right] \leqslant \sum_{t_1}^{t} (s)^{(m)} [p(s) g(s, \hat{u}(s)) - h(s, \hat{u}(s))] \leqslant$$

$$\leqslant -B \sum_{t_1}^{t} (s)^{(m)} \left[\frac{A}{B} p_-(s) - p_+(s) - \frac{1}{B} H(s)\right]$$

and, for A and B negative

$$-A \sum_{t_1}^{t} (s)^{(m)} \left[\frac{-B}{-A} p_-(s) - p_+(s) - \frac{1}{-A} H(s)\right] \leqslant \sum_{t_1}^{t} (s)^{(m)} [p(s) g(s, \hat{u}(s)) -$$

$$- h(s, \hat{u}(s))] \leqslant A \sum_{t_1}^{t} (s)^{(m)} \left[\frac{B}{A} p_+(s) - p_-(s) - \frac{1}{-A} H(s)\right].$$

Now, it is obvious that for $f(t, \hat{u}(t)) = p(t) g(t, \hat{u}(t)) - h(t, \hat{u}(t))$, $i = 0$; the hypothesis of theorem 3.1 is satisfied, thus the conclusion follows.

Theorem 3.4. *Let for any* $u(t)$ *defined on* I *and* $\liminf_{t\to\infty} u(t) > 0$ ($\limsup_{t\to\infty} u(t) < 0$) *there exist constants* A, B ($AB > 0$), *such that for all* $t \in I$, $A \leqslant g(t, \hat{u}(t)) \leqslant B$; *and for such* $u(t)$ *there exists a nonnegative function* $H(t)$ *such that for all large* t, $|h(t, \hat{u}(t))| \leqslant H(t)$.

If for every $\mu_1 > 0$ *and* $\mu_2 > 0$, $\sum^{\infty} [\mu_1 p_+(t) - p_-(t) - \mu_2 H(t)] = \infty$ then, every nonoscillatory solution $x(t)$ of (1.2) satisfies $\liminf_{t\to\infty} |x(t)| = 0$.

Proof. The proof is similar to theorem 3.3, here we need to show that hypothesis of theorem 3.2 is satisfied for $K = 0$.

Next, we shall consider equation (1.2) subject to some of these conditions:

(i) $p(t) = 1$ for all $t \in I$,
(ii) $p(t) > 0$ for all large $t \in I$,
(iii) $p(t) \geqslant 0$ for all $t \in I$,
(iv) $h(t, \hat{x}(t)) = h(t)$,
(v) there exists a function $H_1(t)$ defined on I, such that $h(t, \hat{x}(t)) \geqslant H_1(t)$ for all $t \in I$ and $\hat{x}(t) \in R^n$,
(vi) there exists a function $H_2(t)$ defined on I, such that $H_2(t) \geqslant h(t, \hat{x}(t))$ for all $t \in I$ and $\hat{x}(t) \in R^n$,
(vii) there exists a nonnegative function $H(t)$ defined on I, such that $|h(t, \hat{x}(t))| \leqslant H(t)$ for all $t \in I$ and $\hat{x}(t) \in R^n$,
(viii) if $x_0 \neq 0$, then $x_0 g(t, x_0, \ldots, x_{n-1}) \geqslant 0$,
(ix) if $x_0 \neq 0$, then $x_0 g(t, x_0, \ldots, x_{n-1}) \leqslant 0$,
(x) condition (viii) and $g(t, x_0, \ldots, x_{n-1})$ is bounded away from zero if x_0 is bounded away from zero,
(xi) $g(t, \hat{x}(t)) \leqslant K$ for all $t \in I$ and $\hat{x}(t) \in R^n$,
(xii) $g(t, \hat{x}(t)) \geqslant N$ for all $t \in I$ and $\hat{x}(t) \in R^n$.

Theorem 3.5. *Let in equation* (1.2); p, h *and* g *satisfy* (ii), (vii), (x) *and*

$$\text{(a) } \sum^{\infty} p(s) = \infty, \quad \text{(b) } \frac{H(t)}{p(t)} \to 0 \text{ as } t \to \infty, \text{ or } \sum^{\infty} H(s) < \infty.$$

Then, any nonoscillatory solution $x(t)$ *of* (1.2) *satisfies* $\liminf_{t\to\infty} |x(t)| = 0$.

Proof. We shall show that the hypotheses of theorem 3.2 with $K = 0$ are satisfied. Let $u(t)$ be some function defined on I such that $\liminf_{t\to\infty} u(t) > 0$. Then, from (x) there exist $A > 0$ and $t_1 \in I$ such that $g(t, \hat{u}(t)) \geqslant A$ for all $t \geqslant t_1$. Let $t_2 \geqslant t_1$ and large enough so that $p(t) > 0$ for all $t \geqslant t_2$ in I. If $\sum^{\infty} H(s) < \infty$, then

$$\sum_{t_1}^{\infty} [p(s) g(s, \hat{u}(s)) - h(s, \hat{u}(s))] \geqslant \sum_{t_1}^{t} [A p(s) - H(s)] \to \infty$$

as $t \to \infty$ and the conclusion follows.

If $H(t)/p(t) \to 0$ as $t \to \infty$, then we choose $t_3 \geqslant t_2$ such that $H(t)/p(t) \leqslant A/2$ for all $t \geqslant t_3$ in I, and we find

$$\sum_{t_3}^{\infty} [p(s) g(s, \hat{u}(s)) - h(s, \hat{u}(s))] \geqslant \sum_{t_3}^{t} p(s) \left[A - \frac{H(s)}{p(s)}\right] \geqslant \frac{A}{2} \sum_{t_3}^{t} p(s) \to \infty,$$

as $t \to \infty$. The case $\limsup_{t\to\infty} u(t) < 0$ is treated similarly.

Theorem 3.6. *Let in equation* (1.2); p *and* h *satisfy* (i), (vii) *and for all* $t \in I$, $(x_0, x_1, \ldots, x_{n-1}) \in R^n$

$$\text{(3.9)} \qquad |g(t, x_0, x_1, \ldots, x_{n-1})| \leqslant \sum_{i=0}^{n-1} g_i(t) |x_i|^{r_i}$$

where $g_i(t)$, $0 \leqslant i \leqslant n-1$ *are nonnegative functions defined on* I *and* α_i, $0 \leqslant i \leqslant n-1$ *are nonnegative constantt. Then, if*

$$\sum^{\infty} H(s) < \infty \tag{3.10}$$

and

$$\prod^{\infty}\left[1+\sum_{i=0}^{n-1} g_i(s)[(s)^{n-1-i}]^{\alpha_i}\right]<\infty \tag{3.11}$$

the equation (1.2) *has solutions which is asymptotic to* $\sum_{i=0}^{n-1} a_i(t)^{(i)}$ *as* $t \to \infty$, *where* a_i, $1 \leqslant i \leqslant n$ *are real constants and* $a_{n-1} \neq 0$.

Proof. Let $x(t)$ be any solution of (1.2), then from lemma 2.1 for any $t_1 \in I$, it follows that

$$\Delta^k x(t)=\sum_{i=k}^{n-1}\frac{(t-t_1)^{(i-k)}}{(i-k)!}\Delta^i x(t_1)-\frac{1}{(n-k-1)!}\sum_{s=t_1}^{t-n+k}(t-s-1)^{(n-k-1)}\times \tag{3.12}$$

$$\times[g(t,\hat{x}(s))-h(s,\hat{x}(s))];\ t\in I_{t_1},\ 0 \leqq k \leqq n-1.$$

In (3.12), using (3.9) and (3.10) to obtain

$$|\Delta^k x(t)| \leqslant A_k(t)^{(n-1-k)}+B_k(t)^{(n-1-k)}\sum_{s=t_1}^{t-n+k}\sum_{i=0}^{n-1} g_i(s)|\Delta^i x(s)|^{\alpha_i} \tag{3.13}$$

where

$$A_k=(t)^{-(n-1-k)}\sum_{i=k}^{n-1}\frac{(t)^{(i-k)}}{(i-k)!}|\Delta^i x(t_1)|+\frac{1}{(n-k-1)!}\sum_{s=t_1}^{t-n+k} H(s)$$

and

$$B_k=\frac{1}{(n-k-1)!}\ .$$

Let $A=\max\limits_{\substack{0\leqq k\leqq n-1\\ t\in I_{t_1}}} A_k$ then since $B_k \leqslant 1$; inequality (3.13) implies that

$$|\Delta^k x(t)| \leqslant (t)^{(n-1-k)}F(t),\ 0 \leqslant k \leqslant n-1 \tag{3.14}$$

where

$$F(t)=A+\sum_{s=t_1}^{t-n+k}\sum_{i=0}^{n-1} g_i(s)|\Delta^i x(s)|^{\alpha_i} \tag{3.15}$$

Using (3.14) in (3.15), to find

$$F(t) \leqslant A+\sum_{s=t_1}^{t-n+k}\sum_{i=0}^{n-1} g_i(s)[(s)^{(n-1-i)}]^{\alpha_i}F^{\alpha_i}(s). \tag{3.16}$$

Let $\alpha=\max\limits_{0\leqslant i\leqslant n-1}\alpha_i$, then (3.16) implies that

$$F(t) \leqslant A+\sum_{s=t_1}^{t-n+k}\sum_{i=0}^{n-1} g_i(s)[(s)^{(n-1-i)}]^{\alpha_i}F^{\alpha}(s). \tag{3.17}$$

Using lemma 2.3, to obtain

$$F(t) \leqslant A\prod_{s=t_1}^{t-n+k}\left[1+\sum_{i=0}^{n-1} g_i(s)[(s)^{(n-1-i)}]^{\alpha_i}\right],\ \text{if } \alpha=1 \tag{3.18}$$

$$F(t) \leqslant [A^{1-\alpha}+(1-\alpha)\sum_{s=t_1}^{t-n+k}\sum_{i=0}^{n-1} g_i(s)[(s)^{(n-1-i)}]^{\alpha_i}]^{1/1-\alpha},\ \text{if } \alpha\neq 1. \tag{3.19}$$

Thus, from (3.11), (3.18) and (3.19), $F(t)$ is bounded if $\alpha \leqslant 1$ and for $\alpha > 1$, if

$$A^{1-\alpha} > (\alpha-1)\sum_{s=t_1}^{t-n+k}\sum_{i=0}^{n-1} g_i(s)[(s)^{(n-i-1)}]^{\alpha_i}. \tag{3.20}$$

Since $F(t_1)=A$, condition (3.20) corresponds to an appropriate choice of initial conditions for the solution $x(t)$ of (1.2).

Thus, if (3.20) is satisfied for any α, $F(t) \leqslant c$ (some constant) and inequality (3.14) takes the form

$$|\Delta^k x(t)| \leqslant (t)^{(n-1-k)}c,\ 0 \leqslant k \leqslant n-1. \tag{3.21}$$

Next, from (1.2), we have

$$\Delta^{n-1}(x(t)=\Delta^{n-1}x(t_1)-\sum_{s=t_1}^{t-1} g(s,\hat{x}(s))+\sum_{s=t_1}^{t-1} h(s,\hat{x}(s)) \tag{3.22}$$

and from (vii), (3.9), (3.10) and (3.21) we find that sums in the right side of (3.22) converge as $t\to\infty$ and therefore $\lim \Delta^{n-1}x(t)$ exists and is a finite number. To ensure that $t\to\infty$ this limit is not zero, we choose t_1 so large that $\left|\sum_{s=t_1}^{t-1}[g(s,\hat{x}(s))-h(s,\hat{x}(s))]\right|<\varepsilon$. Then, the solution $x(t)$ with the condition $\Delta^{n-1}x(t_1)=A$ will have the desired type of asymptotic behaviour.

Corollary 3.7. *Under the hypothesis of theorem* 3.6, (1.2) *has nonoscillatory solutions.*

Theorem 3.8. *Let in equation* (1.2) ; p, h *and* g *satisy* (i), (vii) *and* (viii). *Then, any nonoscillatory or Z-type solution* $x(t)$ *of* (1.2) *satisfies*

$$|x(t)|=O((t)^{(n-1)}+\sum^{t-1}(t-s-1)^{(n-1)}H(s)),\ t\to\infty. \tag{3.23}$$

Proof. Let $x(t)$ be any solution of (1.2), then from lemma 2.1 for any $t_1 \in I$, it follows that

$$x(t)=\sum_{i=0}^{n-1}\frac{(t-t_1)^{(i)}}{(i)!}\Delta^i x(t_1)+\frac{1}{(n-1)!}\sum_{s=t_1}^{t-n}(t-s-1)^{(n-1)}\times[- \tag{3.24}$$

$$-g(s,\hat{x}(s))+h(s,\hat{x}(s))].$$

If $x(t)$ is nonoscillatory or Z-type, then $x(t) \geqslant 0$ or $x(t) \leqslant 0$ for all $t \geqslant t_2$ in I. If $x(t) \geqslant 0$, then from (3.24), we find

$$0 \leqq x(t) \leqq \sum_{i=0}^{n-1}\frac{(t-t_2)^{(i)}}{(i)!}\Delta^i x(t_2)+\frac{1}{(n-1)!}\sum_{s=t_2}^{t-1}(t-s-1)^{(n-1)}H(s)$$

and, if $x(t) \leqq 0$

$$0 \geq x(t) \geq \sum_{i=0}^{n-1} \frac{(t-t_2)^{(i)}}{(i)\,!} \Delta^i x(t_2) - \frac{1}{(n-1)\,!} \sum_{s=t_2}^{t-n} (t-s-1)^{(n-1)} H(s)$$

also, in either case

$$|x(t)| \leq \sum_{i=0}^{n-1} \frac{(t-t_2)^{(i)}}{(i)\,!} |\Delta^i x(t_2)| + \frac{1}{(n-1)!} \sum_{s=t_2}^{t-n} (t-s-1)^{(n-1)} H(s).$$

This completes the proof.

Remark. If we impose additional condition

$$\sum^{t-n} (t-s-1)^{(n-1)} H(s) = O((t)^{(m)}), \qquad t \to \infty,$$

for some nonnegative integer m, then every nonoscillatory or Z-type solution $x(t)$ of (1.2) satisfies $|x(t)| \leq O((t)^{(k)})$, $t \to \infty$, where $k = \max\{n-1, m\}$.

Theorem 3.9. *Let in equation* (1.2) ; *p, h and g satisfy* (i), (iv), (ix) *and for every* $k > 0$

$$\limsup_{t \to \infty} [\sum^{t-n} (t-s-1)^{(n-1)} h(s) - k(t)^{(n-1)}] = +\infty$$

and

$$\liminf_{t \to \infty} [\sum^{t-n} (t-s-1)^{(n-1)} h(s) + k(t)^{(n-1)}] = -\infty.$$

Then, every bounded solution of (1.2) *is oscillatory.*

Proof. If $x(t)$ is bounded and nonoscillatory or Z-type, then $x(t) \geq 0$ or $x(t) \leq 0$ for all $t \geq t_2$, in I. If $x(t) \geq 0$, then from (1.2) and (iv), we find

$$(3.25) \qquad 0 \leq x(t) \geq \sum_{i=0}^{n-1} \frac{(t-t_2)^{(i)}}{(i)\,!} \Delta^i x(t_2) + \frac{1}{(n-1)\,!} \sum_{s=t_2}^{t-n} (t-s-1)^{(n-1)} h(s)$$

and, if $x(t) \leq 0$

$$(3.26) \qquad 0 \geq x(t) \leq \sum_{i=0}^{n-1} \frac{(t-t_2)^{(i)}}{(i)\,!} \Delta^i x(t_2) + \frac{1}{(n-1)\,!} \sum_{s=t_2}^{t-n} (t-s-1)^{(n-1)} h(s).$$

Next, let $t_3 \geq t_2$ be sufficiently large so that for some $k > 0$.

$$-k(t)^{(n-1)} \leq \sum_{i=0}^{n-1} \frac{(t-t_2)^{(i)}}{(i)\,!} \Delta^i x(t_2) \leq k(t)^{(n-1)} \text{ for all } t \geq t_3$$

in I, then from (3.25) and (3.26)

$$0 < x(t) \leq \frac{1}{(n-1)\,!} \sum_{s=t_3}^{t-n} (t-s-1)^{(n-1)} h(s) - k(t)^{(n-1)}$$

$$0 \geq x(t) \leq \frac{1}{(n-1)\,!} \sum_{s=t_3}^{t-n} (t-s-1)^{(n-1)} h(s) + k(t)^{(n-1)}$$

and from the hypothesis

$$0 \leq \limsup_{t \to \infty} x(t) \geq \limsup_{t \to \infty} \frac{1}{(n-1)\,!} \left[\sum_{s=t_3}^{t-n} (t-s-1)^{(n-1)} h(s) - k(n-1)\,(t)^{(n-1)} \right] = +\infty$$

$$0 \geq \liminf_{t \to \infty} x(t) \leq \liminf_{t \to \infty} \frac{1}{(n-1)!} \left[\sum_{s=t_3}^{t-n} (t-s-1)^{(n-1)} h(s) + k(n-1)\,(t)^{(n-1)} \right] = -\infty$$

hence in either case we get a contradiction to our assumption that $x(t)$ is bounded.

Corollary 3.10. *Let in equation* (1.2) ; *p, h and g satisfy* (i), (iv), (vii) *and for every* $k > 0$

$$\limsup_{t \to \infty} [\sum^{t-n} (t-s-1)^{(n-1)} h(s) - k(t)^{(n-1)}] \geq 0$$

$$\liminf_{t \to \infty} [\sum^{t-n} (t-s-1)^{(n-1)} h(s) + k(t)^{(n-1)}] \leq 0.$$

Then, every solution of (1.2) *is oscillatory or Z-type.*

Proof. The proof is similar to that of theorem 3.9.

Theorem 3.11. *Let in equation* (1.2) ; *p, h and g satisfy* (iii), (v), (xi) *and for every* $k > 0$

$$(3.27) \qquad \liminf_{t \to \infty} [\sum^{t-n} (t-s-1)^{(n-1)} [H_1(s) - Kp(s)] - k(t)^{(n-1)}] \geq 0.$$

Then, all solutions of (1.2) *are nonoscillatory or nonnegative Z-type. Also, if strict inequality holds in* (3.27), *then all solutions of* (1.2) *are nonoscillatory.*

Proof. From the equation (1.2), we obtain $\Delta^n x(t) \geq H_1(t) - Kp(s)$. Thus, from lemma 2.1, we find for all $t_1 \in I$

$$x(t) \geq \sum_{i=0}^{n-1} \frac{(t-t_1)^{(i)}}{(i)\,!} \Delta^i x(t_1) + \frac{1}{(n-1)!} \sum_{s=t_1}^{t-n} (t-s-1)^{(n-1)} \times [H_1(s) - Kp(s)], \quad t \geq t_1.$$

Let $t_2 \geq t_1$ be sufficiently large so that for some $k > 0$.

$$-k(t)^{(n-1)} < \sum_{i=0}^{n-1} \frac{(t-t_1)^{(i)}}{(i)\,!} \Delta^i x(t_1), \text{ for all } t \geq t_2. \text{ Thus, we have}$$

$$\liminf_{t \to \infty} x(t) \geq \liminf_{t \to \infty} \frac{1}{(n-1)!} \left[\sum_{s=t_2}^{t-n} (t-s-1)^{(n-1)} [H_1(s) - Kp(s)] - k(n-1)\,!\,(t)^{(n-1)} \right] \geq 0.$$

Hence, the conclusion follows.

Theorem 3.12. *Let in equation* (1.2) ; *p, h and g satisfy* (iii), (vi), (xii) *and for every* $k > 0$

$$(3.28) \qquad \limsup_{t \to \infty} [\sum^{t-n} (t-s-1)^{(n-1)} [H_2(s) - Np(s)] + k(t)^{(n-1)}] \leq 0.$$

Then, all solutions of (1,2) *are nonoscillatory or nonnegative Z-type. Also, if strict inequality holds in* (3.28), *then all solutions of* (1,2) *are nonoscillatory.*

Proof. The proof is similar to that of theorem 3.11.

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ROW ORTHONORMAL INFINITE SQUARE MATRICES

BY

ALEXANDER ABIAN

In what follows by *infinite* we mean *denumerably infinite*. By an infinite row (column) vector we mean an infinite sequence written horizontally (vertically). The coordinates of all sequences as well as all matrices are real numbers.

All the matrices and vectors involved in our discussion are infinite; nevertheless, in our statements we constantly prefix them with the word „infinite" so that a statement read independently will not be interpreted to pertain to finite matrices and vectors (although many results mentioned in this paper are obviously valid for the finite case).

Infinite matrices enter in a natural way in the study of Hilbert's space of square summable sequences where emphasis is given to those infinite matrices whose rows form a complete orthonormal set of vectors [4, p. 86—98]. We show here that infinite square matrices whose rows form an orthonormal set of vectors *without necessarily being complete* possess many significant properties. In particular, it is shown that if M is such a matrix (in spite of the fact that multiplication of infinite matrices is not associative) it is the case that $M(M'C)=C$ where M' is the transpose of M and C is any infinite square summable column vector.

All the results are based on the most basic and well known properties of square summable sequences. Almost no Hilbert space terminology is used. In fact, we refer to the *„norm"* of a vector, simply, as to its *„length"*. Thus, if $A=(r_1, r_2, r_3, \ldots)$ is an infinite vector then A is called *of finite length* if $\sum_i r_i^2 < \infty$, and, as expected, we denote the length of A by $\|A\|$ where

$$\|A\| = (\textstyle\sum_i r_i^2)^{1/2}. \tag{1}$$

Obviously, the length of a vector A is equal to zero if and only if A is the zero vector which we denote by $\overline{0}$.

Let $A=(r_i)_{i=1,2,3,\ldots}$ and $B=(q_i)_{i=1,2,3,\ldots}$ be infinite vectors. As usual, we denote the *inner product* of A and B by $A.B$ where

$$A.B = \textstyle\sum_i r_i q_i. \tag{2}$$

Clearly,

$$\|A\| = (A.A)^{1/2} \tag{3}$$

Moreover, the inner product of two vectors each of finite length exists by virtue of the Cauchy-Schwarz inequality [3, p. 38]: