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REFERENCES

1. Agarwal R.P. and Thandapani E. — *On some new discrete inequalities*, Appl. Math. Comp. 7 (1980), 205—224
2. Agarwal R.P. — *On finite systems of difference inequalities*, J. Math. Phyl. Sci. 10 (1976), 277—288
3. Agarwal R.P. — *Oscillation and asymptotic behaviour of solutions of differential equations with nested arguments*, Boll. U.M.I. (to appear)
4. Agarwal R.P. — *Properties of solutions of higher order nonlinear difference equations*, An. şt. Univ. Iaşi
5. Graff J.R. — *Oscillation, nonoscillation, and growth of solutions of nonlinear functional equations of arbitrary order*, J. Math. Anal. Appl. 60 (1977), 398—409
6. Graff J.R. and Spikas P.W. — *Asymptotic properties of solutions of functional differential equations of arbitrary order*, J. Math. Anal. Appl. 60 (1977), 339—348
7. Kartsatos A.G. — *On the maintenance of oscillations of nth order equations under the effect of a small forcing term*, J. Diff. Equs. 10 (1971), 355—363
8. Kartsatos A.G. and Manougian M.N. — *Further results on oscillation of functional differential equations*, J. Math. Anal. Appl. 53 (1976), 28—37
9. Staikos V.A. and Sficas Y.G. — *Oscillatory and asymptotic behaviour of functional differential equations*, J. Diff. Equs. 12 (1872), 426—437
10. Staikos V.A. and Sficas Y.G. — *Oscillatory and asymptotic properties of differential equations with retarded arguments*, Appl. Anal. 5 (1975), 141—148.
11. Staikos V.A. and Sficas Y.G. — *Forced oscillations for differential equations of arbitrary order*, J. Diff. Equs. 17 (1975), 1—11
12. Thandapani E. and Agarwal R.P. — *Asymptotic behavior and oscillation of solutions of differential equations with deviating arguments*, Boll. U.M.I. 17—B (1980), 82—93

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ROW ORTHONORMAL INFINITE SQUARE MATRICES

BY

ALEXANDER ABIAN

In what follows by *infinite* we mean *denumerably infinite*. By an infinite row (column) vector we mean an infinite sequence written horizontally (vertically). The coordinates of all sequences as well as all matrices are real numbers.

All the matrices and vectors involved in our discussion are infinite; nevertheless, in our statements we constantly prefix them with the word „infinite" so that a statement read independently will not be interpreted to pertain to finite matrices and vectors (although many results mentioned in this paper are obviously valid for the finite case).

Infinite matrices enter in a natural way in the study of Hilbert's space of square summable sequences where emphasis is given to those infinite matrices whose rows form a complete orthonormal set of vectors [4, p. 86—98]. We show here that infinite square matrices whose rows form an orthonormal set of vectors *without necessarily being complete* possess many significant properties. In particular, it is shown that if M is such a matrix (in spite of the fact that multiplication of infinite matrices is not associative) it is the case that $M(M'C)=C$ where M' is the transpose of M and C is any infinite square summable column vector.

All the results are based on the most basic and well known properties of square summable sequences. Almost no Hilbert space terminology is used. In fact, we refer to the *„norm"* of a vector, simply, as to its *„length"*. Thus, if $A=(r_1, r_2, r_3, \ldots)$ is an infinite vector then A is called *of finite length* if $\sum_i r_i^2<\infty$, and, as expected, we denote the length of A by $\|A\|$ where

(1) $$\|A\|=(\sum_i r_i^2)^{1/2}.$$

Obviously, the length of a vector A is equal to zero if and only if A is the zero vector which we denote by $\overline{0}$.

Let $A=(r_i)_{i=1,2,3,\ldots}$ and $B=(q_i)_{i=1,2,3,\ldots}$ be infinite vectors. As usual, we denote the *inner product* of A and B by $A.B$ where

(2) $$A.B=\sum_i r_i q_i.$$

Clearly,

(3) $$\|A\|=(A.A)^{1/2}$$

Moreover, the inner product of two vectors each of finite length exists by virtue of the Cauchy-Schwarz inequality [3, p. 38]:

(4) $$\|A \cdot B\| \leqslant \|A\| \, \|B\|.$$

As expected, a set $(N_i)_{i=1,2,3,\ldots}$ of vectors is called orthonormal if

(5) $$N_i . N_i = 1 \text{ and } N_i . N_j = 0 \text{ for } i \neq j \text{ where } i, j = 1, 2, 3, \ldots$$

We shall frequently use the following basic properties of the orthonormal sets of vectors and the inner products:

Let $(c_1, c_2, c_3, \ldots)$ be an infinite vector of finite length, i.e., $\Sigma_i c_i^2 < \infty$, and let $\{N_1, N_2, N_3, \ldots\}$ be an infinite set of orthonormal vectors. Then

(6) $\sum_i N_i c_i$ *converges pointwise (in fact, normwise) to an infinite vector of finite length* $(\sum_i c_i^2)^{1/2}$.

Furthermore, if A is any infinite vector of finite length we have [1, p. 49]:

(7) $$A \cdot \sum_i c_i = \sum_i (A \cdot N_i) c_i,$$

and the well-known Bessel's inequality:

(8) $$\|N_1 . A,\ N_2 . A,\ N_3 . A, \ldots\| \leqslant \|A\|.$$

Also, if B is another infinite vector of finite length, we have the familiar triangle inequality:

(9) $$\|A + B\| \leqslant \|A\| + \|B\|.$$

Finally, if $(b_1, b_2, b_3, \ldots)$ and $(c_1, c_2, c_3, \ldots)$ are infinite vectors of finite length, we have:

(10) $$\sum_i N_i B_i \cdot \Sigma_i N_i c_i = \sum_i b_i c_i.$$

Throughout this paper $\overline{0}$ and I stand respectively for the zero (infinite) vector and the unit (infinite) matrix and M' for the transpose of matrix M.

Lemma 1. *Let $(a_{ij})_{i,j=1,2,3,\ldots}$ be a row orthonormal infinite square matrix and $(c_1, c_2, c_3, \ldots)'$ be an infinite column vector of finite length. Then the following identity holds:*

(11) $$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \left(\begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \\ \cdots \end{pmatrix} c_1 + \begin{pmatrix} a_{21} \\ a_{22} \\ a_{23} \\ \cdots \end{pmatrix} c_2 + \begin{pmatrix} a_{31} \\ a_{32} \\ a_{33} \\ \cdots \end{pmatrix} c_3 + \cdots \right) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \cdots \end{pmatrix}.$$

Proof. Let us observe that, in view of the hypothesis of the Lemma, the column vectors appearing in the second factor of the product appearing on the left side of the equality sign in (11) form an orthonormal set $\{N_1, N_2, N_3, \ldots\}$ of infinite vectors. But then from (6) it follows that $\Sigma_i N_i c_i$ is an infinite vector of finite length. Clearly, by (7) the product appearing in (11) yields an infinite column vector whose k-th coordinate, in view of (7) and (5) is given by $N_k . \Sigma_i N_i c_i = (N_k . N_k) c_k = c_k$. Thus, the product appearing in (11) is equal to the column vector $(c_1, c_2, c_3, \ldots)'$, as desired.

Obviously, identity (11) can be rewritten as:

(12) $$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \left(\begin{pmatrix} a_{11} & a_{21} & a_{31} & \cdots \\ a_{12} & a_{22} & a_{32} & \cdots \\ a_{13} & a_{23} & a_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \cdots \end{pmatrix} \right) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \cdots \end{pmatrix}.$$

Remark 1. As (11) shows, the second factor of the products appearing in (11) and (12) is the vector $N_1 c_1 + N_2 c_2 + N_3 c_3 + \ldots$ whose length, in view of (3) and (10) is given by:

(13) $$((\sum_i N_i c_i) \cdot (\sum_i N_i c_i))^{1/2} = (\sum_i c_i^2)^{1/2}.$$

Remark 2. Let M be an infinite square matrix whose rows form an orthonormal set of row vectors. Then obviously, $MM' = I$, whereas $M'M \neq I$, in general. Now, let C be an infinite column vector of finite length. Then trivially, $(MM')C = C$. On the other hand, since the multiplication of infinite matrices is not associative, we cannot readily assert that $M(M'C) = C$. However, the validity of the last equality is ensured by Lemma 1. Indeed, let $M = (a_{ij})_{i,j=1,2,3,\ldots}$ be a row orthonormal infinite square matrix and $C = (c_1, c_2, c_3, \ldots)'$ be an infinite column vector of finite length. But then clearly, the second factor of the product appearing on the left side of the equality sign in (12) is the matrix M' and $M(M'C) = C$ follows directly from (12).

In view of (12), (13) and Remark 2 we have:

Theorem 1. *Let M be a row orthonormal infinite square matrix and C an infinite column vector of finite length. Then*

(14) $$(MM')C = M(M'C) = C \text{ and } \|M'C\| = \|C\|.$$

As shown below, Theorem 1 is most significantly used in solving infinite systems of linear equations each with infinitely many unknowns.

Theorem 2. *Let the infinite system of linear equations each with infinitely many unknowns $x_1, x_2, x_3, \ldots$ be given by:*

(15) $$\begin{array}{l} a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \ldots = c_1 \\ a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \ldots = c_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{i1} x_1 + a_{i2} x_2 + a_{i3} x_3 + \ldots = c_i \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

where the coefficient matrix $M = (a_{ij})_{i,j=1,2,3,\ldots}$ is a row orthonormal (infinite square) matrix. Then the system has a solution

(16) $$x_1 = s_1,\ x_2 = s_2,\ x_3 = s_3, \ \ldots$$

with

(17) $$\sum_i s_i^2 < \infty \quad \textit{if and only if} \quad \sum_i c_i^2 < \infty,$$

in which case

(18) $$\sum_i c_i^2 \leqslant \sum_i s_i^2.$$

Proof. Let us rewrite system (15) in the matrix form as:

(19) $$MX = C,$$

where $C=(c_1, c_2, c_3, \ldots)'$ and let $X_1, X_2, X_3, \ldots$ denote the rows of matrix M. From the hypothesis of the Theorem it follows that $\{X_1, X_2, X_3, \ldots\}$ is an orthonormal set of (infinite) row vectors.

Let (16) be a solution of (15) which satisfies the first inequality in (17). Let $S=(s_1, s_2, s_3, \ldots)$. Thus, $\|S\|^2=\Sigma_i s_i^2<\infty$. Also, from (19) we have $MS=(X_1.S, X_2.S, X_3.S, \ldots)=C$, which by (8) implies $\Sigma_i c_i^2 \leqslant \Sigma_i s_i^2 < \infty$ establishing (18) and the second inequality in (17).

Next, we show that the second inequality in (17) implies that (15) has a solution, say, given as in (16) which satisfies (18) and the first inequality in (17). Thus, we assume that $C=(c_1, c_2, c_3, \ldots)'$ is an infinite column vector of finite length. Let $S=M'C$. But then, from (14) and (19) it follows that taking $s_1, s_2, s_3, \ldots$ in (16) as the coordinates of vector S, expressions in (16) yield a solution of (15). Moreover, from the second equality in (14) it follows that $\|S\|=\|C\|$, i.e., $\Sigma_i s_i^2=\Sigma_i c_i^2$ which implies (18) and the first inequality in (17), as desired.

Using matrix notation, the following useful Corollary follows directly from Theorems 1 and 2.

Corollary 1. *Let M be a row orthonormal infinite square matrix and C an infinite column vector of finite length. The equation*

(20) $$MX = C$$

has a solution given by

(21) $$X=M'C \text{ with } \|M'C\|=\|C\|.$$

Moreover, if $X=S$ is any solution of (20) *then*

(22) $$\|C\| \leqslant \|S\|.$$

There are easy examples which show that if additional conditions are not imposed, equation (20) has infinitely many solutions. The Theorem below gives an instance where equation (20) has a unique solution.

Theorem 3. *Let M be a row orthonormal infinite square matrix and C an infinite column vector of finite length. Then $X=M'C$ is the unique solution of length $\|C\|$ of the equation $MX=C$.*

Proof. From (14) it follows directly that $X=M'C$ has length $\|C\|$ and is a solution of equation $MX=C$.

Now, let $X=S_1$ and $X=S_2$ each be a solution of $MX=C$ such that

(23) $$\|S_1\|=\|S_2\|=\|C\|.$$

To complete the proof of the Theorem it suffices to show that $S_1=S_2$. To this end, let us observe that $X=\frac{1}{2}(S_1+S_2)$ is also a solution of $MX=C$

and as such, by (18) its length is greater than or equal to $\|C\|$. But then, from this, (23) and the triangle inequality (9), we have

$$\|C\| \leq \tfrac{1}{2}\|S_1+S_2\| \leq \tfrac{1}{2}\|S_1\| + \tfrac{1}{2}\|S_2\| = \|C\|.$$

From the above it follows that $\|S_1+S_2\| = \|S_1\| + \|S_2\|$ which implies that $S_1=kS_2$ for a real number k. However, if $\|C\|\neq 0$, from (23) it follows that $k=1$ in which case $S_1=S_2$, as desired. On the other hand, if $\|C\|=0$ then again $S_1=S_2$ since in this case by (23) and (1) we have $S_1=S_2=\overline{0}$.

Remark 3. We would like to observe that Theorem 3 is equivalent to the well known Riesz-Fischer Theorem [3, p. 153]. Indeed, let $\{X_1, X_2, X_3, \ldots\}$ be an orthonormal set of infinite vectors and $C=(c_1, c_2, c_3, \ldots)$ be an infinite vector of finite length, i.e., $\Sigma_i c_i^2<\infty$. Then the unique infinite vector $S=(s_1, s_2, s_3, \ldots)$ with $\Sigma_i s_i^2<\infty$ whose existence is ensured by the Riesz-Fischer Theorem is precisely the unique solution $X=S$ of the equation $MX=C$ mentioned in Theorem 3 where M is the infinite square matrix whose rows are $X_1, X_2, X_3, \ldots$.

Remark 4. Let M be a row orthonormal infinite square matrix. As mentioned in Remark 2, we have trivially $MM'=I$, however, $M'M\neq I$, in general. For instance if the rows of M are $(0, 1, 0, 0, 0, \ldots)$, $(0, 0, 1, 0, 0, \ldots)$, $(0, 0, 0, 1, 0, \ldots)$, ... then obviously, $M'M\neq I$.

Below we introduce a special type of row orthonormal matrices called *row complete orthonormal* matrices such that if K is a row complete orthonormal infinite matrix then $KK'=K'K=I$ which, in view of (14) implies that K' is the unique inverse of K in the usual sense of matrix algebra.

First, however, let us observe that if M is a row orthonormal infinite square matrix then the columns of M need not form an orthonormal set of column vectors. This is obvious because if M is any row orthonormal infinite square matrix which is also column orthonormal then the matrix M_1 obtained by eliminating a finite number of rows of M is such that M_1 is still a row orthonormal infinite square matrix, but M_1 is obviously not column orthonormal. In this connection, we have the following:

Lemma 2. *Let M be a row orthonormal infinite square matrix. Then the columns of M are vectors of length ≤ 1.*

Proof. Let $X_1, X_2, X_3, \ldots$ be the rows of M. Without loss of generality, we show that the first column of M is of length ≤ 1. Let

(24) $$E_1=(1, 0, 0, 0, \ldots)'$$

Clearly, the first column of M is given by

(25) $$ME_1=(X_1E_1, X_2E_1, X_3E_1, \ldots)'.$$

But then, from (8), (24) and (1) it follows that $\|ME_1\|\leqslant 1$, as desired.

Let us recall that an orthonormal set $\{K_1, K_2, K_3, \ldots\}$ of (infinite) vectors is called *complete* if and only if $\overline{0}$ is the only (infinite) vector which is orthogonal to every K_i, i.e., for every $i=1, 2, 3, \ldots$

(26) $$K_i . V = 0 \quad implies \quad V = \overline{0}$$

It is well known [2, p. 237] that any complete orthonormal set of vectors is denumerable and that every orthonormal set of vectors can be enlarged to a complete orthonormal set.

Also, it is well known [5, p. 165] that if $(K_i)_{i=1,2,3,\ldots}$ is a complete orthonormal set of vectors and V any vector of finite length then

(27) $$V = (K_1 . V)K_1 + (K_2 . V)K_2 + (K_3 . V)K_3 + \ldots$$

and the corresponding Bessel's equality (given below) holds:

(28) $$\|(K_1 . V, K_2 . V, K_3 . V, \ldots)\| = \|V\|.$$

As expected, a matrix is called a *row complete orthonormal* matrix if its rows form a complete orthonormal set of vectors. We prove below a curious fact, i.e., a (finite or infinite) matrix is row complete orthonormal if and only if it is column complete orthonormal.

Theorem 4. *Let K be an infinite row complete orthonormal matrix. Then its transpose K' is also row complete orthonormal.*

Proof. Let $K_1, K_2, K_3, \ldots$ represent the rows of K and $T_1, T_2, T_3, \ldots$ those of K'. First we show that $\{T_1, T_2, T_3, \ldots\}$ is an orthonormal set of vectors. To this end, without loss of generality, it is enough to show that

(29) $$T_1 . T_1 = 1 \quad and \quad T_1 . T_2 = 0.$$

Let

(30) $$E_1 = (1, 0, 0, 0, \ldots)' \text{ and } E_2 = (0, 1, 0, 0, \ldots)'$$

As in the case of (25), the first row T_1 of K' is given by:

(31) $$T_1 = (K_1 . E_1, K_2 . E_1, K_3 . E_1, \ldots).$$

However, since $\{K_1, K_2, K_3, \ldots\}$ is a complete orthonormal set of vectors, by (31), (28) and (30) we have:

$$\|T_1\| = \|(K_1 . E_1, K_2 . E_1, K_3 . E_1 \ldots)\| = \|E_1\| = 1$$

which establishes the first equality in (29).

Again, since $\{K_1, K_2, K_3, \ldots\}$ is a complete orthonormal set, by (27) we have:

(32) $$E_1 = (K_1 . E_1)K_1 + (K_2 . E_1)K_2 + (K_3 . E_1)K_3 + \ldots$$

As in the case of (31), the second row T_2 of K' is given by:

$$T_2 = (K_1 . E_2, K_2 . E_2, K_3 . E_2, \ldots).$$

Thus, by (31) we have:

(33) $$T_1 . T_2 = (K_1 . E_1)(K_1 . E_2) + (K_2 . E_1)(K_2 . E_2) + (K_3 . E_1)(K_3 . E_2) + \ldots$$

On the other hand, from (32) and (30) it follows that

$$E_1 . E_2 = 0 = (K_1 . E_1)(K_1 . E_2) + (K_2 . E_1)(K_2 . E_2) + (K_3 . E_1)(K_3 . E_2) + \ldots$$

which, in view of (33), establishes the second equality in (29).

It remains to show that K' is row complete. To this end, we show that for every vector V of finite length we have:

(34) $$K'V = \overline{0} \quad implies \quad V = \overline{0}.$$

Now, let $K'V = \overline{0}$. But then by (14) and (26) we have $K(K'V) = V = \overline{0}$ which establishes (34). But then, clearly, (34), in view of (26) implies that K' is a row complete orthonormal matrix.

Observing that any (finite or infinite) row complete matrix must be square, Theorem 4 can be rephrased as follows:

Corollary 2. *The rows of a (finite or infinite square) matrix form a complete orthonormal set of vectors if and only if its columns form a complete orthonormal set of vectors.*

From Remarks 2 and 4 it follows that, in general, the transpose M' of a row orthonormal matrix M (in spite of (14)) cannot be called the *inverse* of M in the usual algebraic sense. However, in the case of row complete orthonormal matrices we have:

Lemma 3. *Let K be a row complete orthonormal infinite square matrix. Then the transpose K' of K is the inverse of K in the usual sense, i.e.,*

(35) $$K'K = KK' = I \quad and \quad K(K'H) = K'(KH) = H$$

for any matrix H whose columns are infinite vectors of finite length.

Proof. Clearly, $K'' = K$. Moreover, from Theorem 4 it follows that K' is also a row (complete) orthonormal matrix. But then (35) is a direct consequence of (14).

A significant application of Lemma 3 is given by:

Theorem 5. *Let $K = (a_{ij})_{i,j=1,2,3,\ldots}$ be a row orthonormal infinite square matrix and $C = (c_1, c_2, c_3, \ldots)'$ an infinite column vector of finite length and $X = (x_1, x_2, x_3, \ldots)'$ an infinite column vector of unknowns $x_1, x_2, x_3, \ldots$. Then the infinite system of linear equations in infinitely many unknowns:*

(36) $$KX = C$$

has a unique solution if and only if K is row complete orthonormal. Otherwise, the system has infinitely many (systems of) solutions.

Proof. Let K be row complete orthonormal. Then from (35) it follows that $X = K'C$ is a solution of (36) and in fact the unique solution of (36).

To prove the converse, we first observe that from (21) it follows that $X = K'C$ is a solution of (36). Now, let us assume to the contrary that $X = K'C$, is the unique solution of (36) and that K is not row complete. But then, by (26), there exists an infinite column vector $V \neq \overline{0}$ such that $KV = \overline{0}$. Obviously, $X = K'C + V \neq K'C$ is another solution of (36), contradicting the uniqueness of the solution of (36). Hence our assumption is false and K is row complete.

To complete the proof of the Theorem, it remains to show that if K is row orthonormal but not complete then (36) has infinitely many solutions. Again, by (21) we see that $X = K'C$ is a solution of (36). However, since K is not complete, by (26) there exists an infinite column vector $V \neq \bar{0}$ such that $KV = \bar{0}$. Let s by any real number. Then clearly, $X = (K'C + Vs)$ gives infinitely many solutions of (36).

REFERENCES

1. Berberian, S.K. — *Introduction to Hilbert Space*, Oxford University Press, 1961
2. Cooke, R.G. — *Infinite Matrices and Sequence Spaces*, Dover Pub., New York, 1965
3. Kolmogorov, A.N. and Fomin, S.V. — *Introductory Real Analysis*, Dover Pub., 1970
4. Stone, M.H. — *Linear Transformations in Hilbert Space*, Amer. Math. Soc., 1932
5. Vulikh, B.Z. — *Functional Analysis*, Pergamon Press, New York, 1963

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RECENZII

H. KURKE, J. MECKE, H. TRIEBEL, R. THIELE (Editors): *Recent Trends in Mathematics*, Reinhardsbrunn, 1982. Teubner Texte zur Mathematic. Band 50. B.S.B. B.G. Teubner Verlagsgesellschaft, Leipzig, 1983, 332 p., DDR 30, M

Cartea conţine lucrări asupra tendinţelor recente din matematică, susţinute de autori la conferinţa de la Reinhardsbrunn, R.D.G., din 11—13 oct. 1982. Din tematica acestor conferinţe, cităm: tendinţe în geometria riemanniană, (M. Berger), grafice computerizate în cercetarea geometrică (T. F. Banchoff), geometrie analitică (G. Pickert), fizică matematică (P. Möbius, V. Mustonen, J. Vasely, C. G. Simader), teoria grupurilor — problema deciziei (V. Huber-Dyson), procese aleatoare şi aplicaţii în biologie (R. Ambartzumian, J. Grandell, K. P. Hadeler, W. Nöbauer, W. Timischl, W. v. Wahl), spaţii analitice (S. V. Bochkarev, Z. Ciesielski, L. D. Kudryavtsev, K. Ueno, H. Wallin), calcule cu matrice (C. Procesi), cercetări operaţionale (K.-L. Elster, A. Göpfert), învăţămînt matematic — cercetare (H. Dinges). Fiecare conferinţă aduce informaţii utile cercetătorilor asupra stadiului problemelor tratate, asupra ultimelor idei de cercetare, asupra unor metode ce au un grad de aplicabilitate mare într-o ramură de matematică.

Mirela Ştefănescu

Leonard Euler 1707—1783. Beiträge zu Leben und Werk. Gedenkband des Kantons Basel-Stadt. Birkhäuser Verlag, Basel, 1983, 555 p., S Fr. 58

Cartea de faţă reprezintă un volum comemorativ cu ocazia a 200 ani de la moartea lui Euler şi conţine articole a 30 de specialişti din 10 ţări şi 4 continente. Nu-i vorba de o reunire întîmplătoare de articole fără legătură între ele, ci volumul are o structură unitară reuşind să dea o vedere de ansamblu nu numai asupra vastei sale activităţi din domeniul matematic (teoria numerelor, algebră şi analiză), ci şi din aproape toate ştiinţele naturii din acea vreme (în special fizică, astronomie şi teoria navigaţiei) atrăgînd atenţia totodată şi asupra consecinţelor durabile ale rezultatelor sale pînă-n ziua de azi. De asemenea, un articol al lui W. Breidert tratează şi despre preocupările filozofice ale lui Euler. O parte importantă (scrisă de E. A. Fillmann) este consacrată vieţii sale, articole speciale referindu-se apoi la desfăşurarea activităţii sale în cadrul celor 2 academii pe care le-a creat: cea din Petersburg şi cea din Berlin, cît şi la legăturile sale cu contemporani de marcă (I. Bernoulli, Lagrange, Cramer). În sfîrşit, J. J. Burckhardt dă o listă completă (circa 700 titluri) ale scrierilor despre Euler. Se dă de asemenea o listă cronologică a operelor fundamentale ale lui Euler. Cartea se-ncheie cu un indice de figuri.

Petru Caraman

P. J. BIKEL, K. DOKSUM, J. L. HODGES, Jr. (Editors): *A Festschrift for Erich L. Lehmann. In Honor of His Sixty-Fifth Birthday*, Wadsworth International Group Belmont California, A Division of Wadsworth Inc., 1983, VIII + 461 p.

Volumul celebrează cea de a 65-a aniversare a profesorului E. L. Lehmann, care s-a născut la Strasbourg la 20 noiembrie 1917, a studiat matematicile la Universităţile din Zürich şi Cambridge şi şi-a dat doctoratul la profesorul J. Neyman de la Universitatea Berkeley din California. Din 1946 este cetăţean al Statelor Unite ale Americii. Este profesor la Universitatea Berkeley din California (între anii 1973 şi 1976 a condus Facultatea