11. Hiriart Urruty, J.-B. — ε-*subdifferential calculus*, Proceedings of the Colloquim „Convex Analysis and Optimization", Imperial College, London (28–29 February, 1980) (to appear)
12. Jameson, G.J. — *Ordered Linear Spaces*, Lecture Notes in Mathematics 141, Springer-Verlag, New York, 1970
13. Kutateladze, S.S. — *Formulas for computing subdifferentials*, (Russian) Dokl. Akad. Nauk SSSR, 232(4) (1977), 770–772
14. Kutateladze, S.S. — *Convex operators*, (Russian) Uspehi Mat. Nauk. 34 (1) (1979), 167–196
15. Kutateladze, S.S. — *Convex ε-programming*, (Russian) Dokl. Akad. Nauk SSSR, 245(5) (1979), 1048–1050
16. Nehse, R. — *Some general separation theorems*, Math. Nachr., 84 (1978), 319–327.
17. Nehse, R. — *Separation of two sets in a product space*, Math. Nachr., 97 (1980), 179–187
18. Ponstein, J. — *Approaches to the Theory of Optimization*, Cambridge Univ. Press, 1980
19. Rockafellar, R.T. — *Conjugate duality and optimization*, SIAM Publications, Philadelphia, 1974
20. Rosinger, E.E. — *Multiobjective duality without convexity*, J. Math. Anal. Appl., 66 (1978), 442–450
21. Silverman, R.J., Yen, T. — *The Hahn-Banach theorem and the least upper bound property*, Trans. AM.S., 90(1959), 523–526.
22. Stoer, J., Witzgall, Ch. — *Convexity and Optimization in Finite Dimensions*, Springer-Verlag, 1970
23. Théra, M. — *Subdifferential calculus for convex operators*, J. Math. Anal. Appl. 80(1981), 78–91
24. Théra, M. — *Calcul ε-sous-différentiel des applications convexes vectorielles*, Séminaire d'Analyse Convexe, Montpellier, 1980, Expose'no. 12
25. Tiba, D. — *Subdifferential of composed functions and applications in optimal control*, An. şt. Univ. Iaşi, S. I Mat., 23(1977), 381–386
26. Zălinescu, C. — *A generalization of the Farkas lemma and applications to convex programming*, J. Math. Anal. Appl., 66(1978), 651–678
27. Zălinescu, C. — *On an abstract control problem*, Numer. Funct. Anal. and Optimiz. 2(1980), 531–542
28. Zălinescu, C. — *The Fenchel-Rockafellar duality theory for mathematical programming in order-complete vector lattices and applications*, Preprint 45/1980, INCREST Bucureşti
29. Zălinescu, C. — *Sur les ensembles en position générale*, Linear Algebra and Appl. (to appear)
30. Zowe, J. — *A duality theorem for a convex programming problem in order-complete vector lattices*, J. Math. Anal. Appl. 50 (1975), 273–287
31. Zowe, J. — *Sandwich theorems for convex operators with values in an ordered vector space* J. Math. Anal. Appl., 66(1978), 282–296

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ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII „AL. I. CUZA" DIN IAŞI
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THE PROCESSOR-SHARING APPROXIMATION TO THE STRICTLY ALTERNATING SWITCH MODEL

BY

TITI OBILADE

1. **Introduction.** A prominent example of two queues served by a single server is the strictly alternating switch queueiung (SAS) model. The SAS is basically an endogeneous [1] priority queueing model in which the next customer for service is selected not only on what priority class it belongs but also on what priority class was last serviced. It represents a particular extreme case of switching from one queue to the other after a specified $k \geq 1$ number of units have been served in a two-queue system. The SAS is the $k=1$ case and is of special interest to us by virtue of its added simplicity, the potential practical importance and the relative solvability of the model. This non-zero switch model (NZS) could be contrasted with the zero-switch (ZS) model in which the server only switches from a queue to the other upon exhausting the particular queue. The ZS constitutes the other extreme case of switching after $k=\alpha$ number of units have been served on the particular queue. Previous work on the ZS model includes that of Takacs [2] in which the stability condition and the queue length distributions were obtained. A similar work on the NZS is by Nair [3] in which the difficulties confronting explicit analytic expression for the discipline are highlighted.

2. **Applications.** The general case of $k \geq 1$ alternating model can be applied to situations where we have non-identical units seeking services simultaneously from a single server. Such situations exist in computer networks, traffic and transportation systems. For computer networks, we may have a situation where memory is commonly shared with equal priority by two or more processors. During the time an I/0 request, for example, is being carried out, all other processes take their turn on a queue. A plausible model is one in which a strictly alternating strategy is used in deciding which processor takes control of the memory. A similar situation exists in telephone traffic as well as in the general traffic situation. It is conceivable that a single lane traffic junction, for example, operates on a strictly alternating basis for, say, north and south bound trains. The same situation applies for a busy airport where one or more sets of aeroplanes have to share a runway. In the case of a shipping port the situation may appear in the form of a single tug being available for moving to and from berths two different classes of ships viz tankers and 'others'. Of course, the necessary assumption is that there is always a berth available for either class

in the case of berthing. The non-zero SAS model is particularly appropriate for situations where the set-up or switching time from one type to the other is negligible.

3. Procesor-Sharing Approximation. Because of the need to keep account of the number on both queues, the analysis of the non-zero SAS is more difficult than that of the zero-switch ZS. We have found it necessary in this paper to introduce an alternation switch processor-sharing approximation SAS (PS) of the system. This represents a fairly general ideal situation in the sense that both unit types now face a virtual appearance, on a fractional mode at least, of the server on a full-time basis. Later on in section 6 we identify a simple multiple-class system with which our limiting case bears a resemblance.

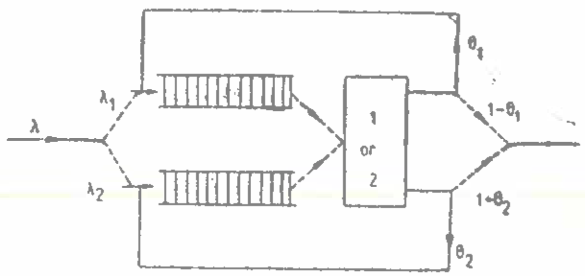

Figure 3.1. Schematic diagram of a one-server two-queue system

4. Description of the model. We consider a system of two parallel queues served by a single server. Units are supposed to arrive into the system in Poisson streams and at rates λ_1, λ_2 for unit types 1 and 2 respectively. We also suppose that the service time distribution is negative exponential and of rates μ_1, μ_2 for unit types 1 and 2 respectively.

In this strictly alternating case, we consider that priority is rotated from one class to the other. A particular unit is followed in service by a unit from the other class. Such is the case insofar as there is at least one unit from the other class present in the system at the moment of departure of the particular unit. In the event of no unit from the other class being available in the system at such moments, the server is assumed to break the strictly alternating rule and allow a unit to be followed in service by its kind.

We consider a feedback model in which either unit may require further service. A type 1 unit may, upon completion of service, leave the system entirely with probability $1-\theta_1$. The alternative is for it to re-join the type 1 queue with probability θ_1. The corresponding probabilities for the type 2 units are $1-\theta_2$ for leaving entirely and θ_2 for rejoining the type 2 queue.

Before proceeding in the next section to derive the steady state equations for the model, we mention in passing that the stability condition for the model is the same for all k, where k is the number of units served on

each queue before switching to the other queue. The stability condition is in fact given as $\rho_1'+\rho_2'<1$ where $\rho_i'=\lambda_i/\mu_i(1-\theta_i)$, $i=1, 2$. The simple argument here is that as far as the server is concerned, it is faced with the same load in the various versions of the non-zero switch $1\leqslant k<\infty$ case as well as the zero-switch $k=\infty$ case. This remains the case as long we take the swapping time from one class to the other as negligible.

5.1. State definition. The state of the system can be adequately defined by the number of units on each of the queues at a particular time. Let $X_j(t)$, $j=1, 2$ denote the number of units on the j^{-th} queue at time $t\in[0, \infty]$. The assumed Poisson arrival and negative exponential service patterns ensure that the random proces $\{X_1(t), X_2(t) : t\in[0, \infty)\}$ is a two-dimensional birth-and-death process with discrete state space $\{(i, j) : i\geqslant 0, j\geqslant 0\}$ where $X_1(t)=i$ and $X_2(t)=j$.

We define $Q_{i,j}(t)=P\{X_1(t)=i, X_2(t)=j\}$. The two dimensional Markov chain is homogeneous, irreducible and aperiodic. Consequently the $\lim_{t\to\infty} Q_{i,j}(t)=Q_{i,j}$ exists for $i\geqslant 0$, $j\geqslant 0$ and it is independent of the initial state. In the same way, we introduce the limiting probability $Q_{i,j,k}$ $i\geqslant 0$, $j\geqslant 0$, $k=1, 2$, for the equilibrium probability of i type 1 and j type 2 units in the system during the service of a type k unit. The conditions $\sum_{i,j} Q_{i,j}=\sum_{i,j,k} Q_{i,j,k}=1$ and $Q_{i,j}\, Q_{i,j,k}>0$ for all i, j, k imply the existence of the unique distribution for the Markov chain.

5.2. General-Time Steady-State Equations. The steady-state difference equations are given as

$$\lambda_1 Q_{i-1,j,1}+\lambda_2 Q_{i,j-1,1}+\mu_2(1-\theta_2)Q_{i,j+1,2}+\mu_2\theta_2 Q_{i,j,2}=(\lambda_1+\lambda_2+\mu_1)Q_{i,j,1},\ i\geqslant 1,\ j\geqslant 1$$

$$\lambda_1 Q_{i-1,j,2}+\lambda_2 Q_{i,j-1,2}+\mu_1(1-\theta_1)Q_{i+1,j,1}+\mu_1\theta_1 Q_{i,j,1}=(\lambda_1+\lambda_2+\mu_2)Q_{i,j,2},\ i\geqslant 1,\ j\geqslant 1$$

$$\lambda_1 Q_{i,1-1,0,1}+\mu_1(1-\theta_1)Q_{i+1,0,1}+\mu_2(1-\theta_2)Q_{i,1,2}=(\lambda_1+\lambda_2+$$

$$+\mu_1(1-\theta_1))Q_{i,0,1},\ i\geqslant 1,\ j=0. \tag{5.1}$$

$$\lambda_2 Q_{0,j-1,2}+\mu_1(1-\theta_1)Q_{i,j,1}+\mu_2(1-\theta_2)Q_{0,j+1,2})=(\lambda_1+\lambda_2+$$

$$+\mu_2(1-\theta_2))Q_{0,j,2}\mu_1(1-\theta_1)Q_{1,0,1}+\mu_2(1-\theta_2)Q_{0,1,2}=(\lambda_1+\lambda_2)Q_{0,0}$$

Adding together the first two of equations (5.1), the steady-state equations for $i\geqslant 1$, $j\geqslant 1$ become,

$$\lambda_1 Q_{i-1,j}+\lambda_2 Q_{i,j-1}+\mu_1(1-\theta_1)Q_{i+1,j,1}+\mu_2(1-\theta_2)Q_{i,j+1,2}=$$

$$=(\lambda_1+\lambda_2+\mu_1(1-\theta_1))Q_{i,j,1}+(\lambda_1+\lambda_2+\mu_2(1-\theta_2))Q_{i,j,2}\ i\geqslant 1,\ j\geqslant 1 \tag{5.2}$$

Now

$$Q_{i,0,1}=Q_{i,0},\ i\geqslant 0,\ Q_{i,j,1}=\frac{\mu_2}{\mu_1+\mu_2}Q_{i,j},\ i\geqslant 1,\ j\geqslant i,$$

$$Q_{0,j,2}=Q_{0,j},\ j\geqslant 0,\ Q_{i,j,2}=\frac{\mu_1}{\mu_2+\mu_1}Q_{i,j},\ i\geqslant 1,\ j\geqslant 1,$$

In the limit as $\theta_1, \theta_2 \to 1$ and $\mu_1, \mu_2 \to \infty$ but in such a way that $\mu_1(1-\theta_1) = \mu_1'$ and $\mu_2(1-\theta_2) = \mu_1'$ and $(1-\theta_2)/(1-\theta_1) = \zeta$ all remain finite, we have

$$(5.3) \qquad Q_{i,0,1} = Q_{i,0}, \ i \geq 0, \ Q_{0,j,2} = Q_{0,j}, \ j \geq 0, \ Q_{i,j,1} = \frac{\mu^*}{\mu_1'} Q_{i,j}, \ i \geq 1, \ j \geq 1,$$

$$(5.4) \qquad Q_{i,j,2} = \frac{\zeta\mu^*}{\mu_2'} Q_{i,j}, \ i \geq 1, \ j \geq 1, \text{ where } \mu^* = \frac{\mu_1'\mu_2'}{\zeta\mu_1' + \mu_2}$$

5.3. Processor-Sharing Limiting Equations. We consider a situation where types 1 and 2 units are processor-shared alternatingly by the server. This is to say that the service rates μ_1, μ_2 are approaching infinity such that the units are processed infinitely quickly by the server. This situation further implies that both θ_1 and θ_2 tend to unity. As the mean number of times a type k unit circulates round the servicing facility is $1-\theta_k$, $k = 1, 2$, the mean service times for types 1 and 2 units (i.e. $1/\mu_k(1-\theta_k)$ for type k unit) are constrained to remain finite as $\mu_1, \mu_2 \to \infty$ and $\theta_1, \theta_2 \to 1$. We further assume that the ratios $1-\theta_2/1-\theta_1$ and μ_2/μ_1 remain finite.

The SAS (PS) limiting case can be visualized as a conveyorbelt system where units are automatically arranged alternatingly by types. Any unit type in excess is allowed a space(s) behind the last 'couple'. Such units are displaced as soon as a unit(s) of the rare type appears. In this way the underlying alternating discipline of the model is preserved.

Taking limits in equations (5.1) and (5.2) as $\theta_1, \theta_2 \to 1$ and $\mu_1, \mu_2 \to \infty$ but in such a way that $\mu_1(1-\theta_1)$, $\mu_2(1-\theta_2)$ and $(1-\theta_2)/(1-\theta_1)$ all remain finite, and applying equations (5.3), we obtain

$$\lambda_1 Q_{i-1,j} + \lambda_2 Q_{i,j-1} + \mu^* Q_{i+1,j} + \zeta\mu^* Q_{i,j+1} = (\lambda_1 + \lambda_2 + \mu^*(\lambda + \zeta))Q_{i,j}, \ i \geq 1, \ j \geq 1,$$

$$(5.5) \qquad \lambda_1 Q_{i-1,0} + \mu_1' Q_{i+1,0} + \zeta\mu^* Q_{i,1} = (\lambda_1 + \lambda_2 + \mu_1')Q_{i,0}, \ i \geq 1,$$

$$\lambda_2 Q_{0,j-1} + \mu^* Q_{1,j} + \mu_2' Q_{0,j+1} = (\lambda_1 + \lambda_2 + \mu_2')Q_{0,j}, \ j \geq 1, \ \mu_1' Q_{10} + \mu_2 Q_{01} = (\lambda_1 + \lambda_2)Q_{00}.$$

We hasten to mention that all terms in the general-time steady-state equations (5.5), and indeed al subsequent equations now represent limiting versions of their previous definitions. The set of equations (5.5) is familiar for this sort of queueing systems. Our aim in this paper is to identify a related state dependent $M/G/2$ queueing system with identical set of equations. This theme is taken up in section 6. For now, we obtain for the sake of completeness the queue-length generating functions for the SAS (PS) system. We define the following generating functions

$$(5.6) \qquad Q_1(x, y) = \sum_{i=1} \sum_{j=0} Q_{i,j,1} x^i y^j + Q_{001}, \quad Q_2(x, y) = \sum_{i=0} \sum_{j=1} Q_{i,j,2} x^i y^j + Q_{002},$$

$$Q(x, y) = \sum_{i=0} \sum_{j=0} Q_{i,j} = Q_1(x, y) + Q_2(x, y) - Q_{00}.$$

We note that $Q_k(x, j) - Q_k(0, 0)$, $k = 1, 2$, is the generating function for the distribution of the number of types 1 and 2 units in the system at an arbitrary time point during the service of a type k unit. Similarly $Q(x, y)$ is the generating function for the distribution of types 1 and 2 units in the system at an arbirtrary time point in the original model. It is clear, as well, that $Q_{00} = Q_{00k} = Q_k(0, 0) = Q(0, 0)$. Of course, $Q_k(1, 1) = \lambda_k/\mu_k + Q_{00}$ since the probability of a type k unit being in service is given by λ_k/μ_k.

Multiplying equations (5.5) by appropriate powers of x and y and then summing, we have

$$Q(x, y)(\lambda_1(x-1) + \lambda_2(y-1) - \mu^*(1-1/x) - \zeta\mu^*(1-1/y)) = Q(x, 0)((\mu_1' -$$

$$(5.7) \qquad -\mu^*)(1-1/x) - \zeta\mu^*(1-1/y)) + Q(0, y)((\mu_2' - \zeta\mu^*)(1-1/y) -$$

$$-\mu^*(1-1/x)) + Q(0, 0)((\mu^* - \mu_1')(1-1/x) + (\zeta\mu^* - \mu_2')(1-1/y))$$

and

$$Q_1(x, y)(\lambda_1(x-1) + \lambda_2(y-1) - \mu_1'(1-1/x)) + Q_2(x, y)(\lambda_1(x-1) + \lambda_2(y-1).$$

$$-\mu_0'(1-1/y)) = Q(0, 0)(\lambda_1(x-1) + \lambda_2(y-1) - \mu_1'(1-1/x) - \mu_2'(1-1/y)).$$

where

$$(5.8) \qquad \mu_1' = \mu_1(1-\theta_1) \text{ and } \mu_2' = \mu_2(1-\theta_2).$$

Equation (5.7) gives the generating functions $Q(x, y)$ for the number of types 1 and 2 units in the system at an arbitrary time point in terms of unknown generating functions $Q(x, 0)$ and $Q(0, y)$. The coefficient of $Q(x, y)$ could be exploited to obtain and solve functional equations for the system [2]. We do not intend to pursue this further in this paper.

6. A Related Queueing Network. We consider a simple queueing network in which two unit types 1 and 2 arrive at independent Poisson streams of intensities λ_1, λ_2 respectively and with $\lambda_1 + \lambda_2 = \lambda$. We also suppose that service in the system is provided by a single server at a certain fixed capacity. The allocation of serving resource is such that a proportion f_k is reserved for a type k unit with $f_2 = 1 - f_1$. The simultaneous services follow exponential distributions of rates μ^* and $\zeta\mu^*$ for types 1 and 2 units respectively, where μ^*, ζ are functions of the service requirements of the two types. We make the additional provision that service is given at certain specified rates in the event that only one type of unit is available for service. The system can alternatively be described as consisting of two servers confronted with two different queues but in which a server comes in to lend a helping hand whenever he is without a customer. It is basically a state-dependent service $M/G/2$ queueing situation.

7. Queueing Parameters for Related Model. Our starting point is a round-robin system. We follow the traditional path of taking limits as the quantum of service becomes infinitely small [4]. This leads us into the processor-sharing approximation.

Suppose, in the round-robin system, time is quantized into segments such that g_pQ seconds service is given to a type p unit, $p = 1, 2$, $Q >> 0$, each time it cycles round. Suppose also that the amounts of service received by the units are geometric with respect to the number of circulations and such that the mean number of circulations is $1/(1-\theta_p)$ for a type p unit. The mean length of service given to a type p unit is therefore $g_pQ/(1-\theta_p)$ seconds. This compares with the average time requirement of $1/\mu_p(1-\theta_p)$ seconds for a type p unit.

Suppose we allow $Q \to 0$ and $\theta_p \to 1$ in such a way that $Q/(1-\theta_p)$ remains finite and $(1-\theta_2)/(1-\theta_1)$ tends to finite ζ. We choose g_1 and g_2 in such a way that

$$\frac{g_1 Q/(1-\theta_1)}{g_2 Q/(1-\theta_2)} = \frac{1/\mu_1(1-\theta_1)}{1/\mu_2(1-\theta_2)}, \text{ i.e. } g_1/g_2 = \mu_2'/\zeta\mu_1' . \tag{7.1}$$

A proportion f_p of service capacity is allocated to type p units in the related model. The propotion f_p of service capacity allocated to type p units is given as

$$f_1 = g_1/(g_1+g_2), = \mu_2'/(\mu_2'+\zeta\mu_1'), \text{ and } f_2 = \zeta\mu_1'/(\mu_2'+\zeta\mu_1').$$

It is easy to see that the partial rates of departure from the system when both types of units are available for service are given by $f_p\mu_p'$ for type p. i.e.

$$f_1\mu_1' = \frac{\mu_1'\mu_2'}{\mu_2'+\zeta\mu_1'} = \mu^{\cdot} \text{ for type 1, and } f_2\mu_2' = \frac{\zeta\mu_1'\mu_2'}{\mu_2'+\zeta\mu_1'} = \zeta\mu^{\cdot} \text{ for type 2.}$$

The interpretation for $\mu^{\cdot}$ is that it is the partial rate of departure of type 1 units during a non-solo service in the system. A solo period is one during which all attention is bestowed on a single type of unit. The rate of departure at solo periods will be given as μ_p' for type p units to correspond with the service requirements of the units.

8. **Conclusion.** The general-time steady state system equation for the related queueing model described in sections 6 and 7 can easily be see to coincide with those for the original model in equation (5.5). The processor-sharing approximation to the strictly alternating switch model SAS (PS) is thus equivalent in terms of general time steady state equations to a state-dependent $M/G/2$ queueing situation. This is indicative of a natural extension to the well-known equivalence of a processor-sharing single type model with an $M/M/1$ queueing model. A further examination of the extent of this resemblance with respect to other time-point such as the departure point will be considered in a subsequent paper.

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REFERENCES

1. J a i s w a l, N.K. — *Priority Queues*, Academic Press, London and New York, 1968
2. T a k a c s, L. — *Two queues attended by a single server*, Ops. Res. Vol. 16 pp. 639—650, 1968
3. N a i r, S.S. — *Alternating priority queues with non-zero switch rule*, Comput. and Ops. Res., Vol. 3, pp. 337—346, 1976
4. K l e i n r o c k, L. — *Time-shared systems: a theoretical treatment*, J. ACM. vol. 14, pp. 242—261, 1967

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ON THE SET-VALUED ADDITIVE AND SUBADDITIVE FUNCTIONS

BY

ANCA MARIA PRECUPANU

This paper deals with some continuity properties of the additive multimeasures with respect to suitable topologies defined on their domain. After some general considerations concerning the setting of the problem, in Section 2 are investigated some properties of the o-continuous multimeasures and in Section 3 some basic properties of regular multimeasures on a Hausdorff locally compact space. Finally the special case of regular multimeasures on a normed space is discussed.

1. **Basic notions, terminology and notations.** Let $\mathcal{P}$ be the family of all subsets of a set Ω organized as a Boolean ring for the symmetric difference and intersection of sets, where the sum $\oplus$ in $\mathcal{P}$ is given by the symmetric difference and the product $\odot$ by the intersection. In this ring the empty set, $\varnothing$, is the null element and the set Ω is the unity. We shall also use the following notations: $A' = \Omega \setminus A$, $[0, A] = \{B \in \mathcal{P}\,;\ B \subseteq A\}$, $[A, B] = \{E \in \mathcal{P}\,;\ A \subseteq E \subseteq B\}$, $A \oplus [0, B] = \{A \oplus C\,:\ C \subseteq B\} = [A \setminus B,\ A \cup B]$ for every $A, B \in \mathcal{P}$. Let $\mathcal{F}$ be the family of all finite sets in Ω and $\mathcal{F}' = \{A\,;\ A' \in \mathcal{F}\}$. It is easy to see that $\mathcal{F}' \neq \varnothing$ and $\mathcal{F}'$ is directed by inclusion. If Ω is infinite, then $\mathcal{F}'$ is a filter on Ω.

Let us consider for every $A \in \mathcal{P}$ the family $\mathcal{B}(A) = \{A + [0, F]\,;\ F \in \mathcal{F}'\}$. From [2] we obtain the following assertions:

(1.1) There exists a unique topology O on $\mathcal{P}$ for which the family $\mathcal{B}(A)$ is a base of neighbourhoods of A, for every $A \in \mathcal{P}$. We shall call it the *order-topology on* $\mathcal{P}$.

(1.2) The family $\mathcal{W}(F) = \{(A, B)\,;\ A \oplus B \subseteq F,\ F \in \mathcal{F}'\}$ is a base of vicinities in $\mathcal{P}$ for a uniformity $\mathcal{W}$ on Ω.

(1.3) The operations $(A, B) \to A \cup B$, $(A, B) \to A \setminus B$, $(A, B) \to A \cap B$, $(A, B) \to A \Delta B$ are uniformly continuous. (The topology O agrees with the topology induced by the uniformity described at (1.2)).

Thus $(\mathcal{P}, \oplus, \odot, O)$ is a topological ring of sets.

(1.4) $(\mathcal{P}, \oplus, \odot, O)$ is a totally disconnected compact Hausdorff topological ring.

(1.5) The family $\mathcal{F}$ is dense in $\mathcal{P}$.

(1.6) If $\mathcal{R}$ is a subring of $\mathcal{P}$ and $\mathcal{S}$ is the closure of $\mathcal{R}$ in O-topology, then $(\mathcal{S}, \oplus, \odot)$ is the smallest order complete Boolean subring of $\mathcal{P}$ contai-