Suppose we allow $Q \to 0$ and $\theta_p \to 1$ in such a way that $Q/(1-\theta_p)$ remains finite and $(1-\theta_2)/(1-\theta_1)$ tends to finite ζ. We choose g_1 and g_2 in such a way that

$$\frac{g_1 Q/(1-\theta_1)}{g_2 Q/(1-\theta_2)} = \frac{1/\mu_1(1-\theta_1)}{1/\mu_2(1-\theta_2)}, \text{ i.e. } g_1/g_2 = \mu_2'/\zeta\mu_1' . \tag{7.1}$$

A proportion f_p of service capacity is allocated to type p units in the related model. The propotion f_p of service capacity allocated to type p units is given as

$$f_1 = g_1/(g_1+g_2), = \mu_2'/(\mu_2'+\zeta\mu_1'). \text{ and } f_2 = \zeta\mu_1'/(\mu_2'+\zeta\mu_1').$$

It is easy to see that the partial rates of departure from the system when both types of units are available for service are given by $f_p\mu_p'$ for type p. i.e.

$$f_1\mu_1' = \frac{\mu_1'\mu_2'}{\mu_2'+\zeta\mu_1'} = \mu^* \text{ for type 1, and } f_2\mu_2' = \frac{\zeta\mu_1'\mu_2'}{\mu_2'+\zeta\mu_1'} = \zeta\mu^* \text{ for type 2.}$$

The interpretation for μ^* is that it is the partial rate of departure of type 1 units during a non-solo service in the system. A solo period is one during which all attention is bestowed on a single type of unit. The rate of departure at solo periods will be given as μ_p' for type p units to correspond with the service requirements of the units.

8. **Conclusion.** The general-time steady state system equation for the related queueing model described in sections 6 and 7 can easily be see to coincide with those for the original model in equation (5.5). The processor-sharing approximation to the strictly alternating switch model SAS (PS) is thus equivalent in terms of general time steady state equations to a state-dependent $M/G/2$ queueing situation. This is indicative of a natural extension to the well-known equivalence of a processor-sharing single type model with an $M/M/1$ queueing model. A further examination of the extent of this resemblance with respect to other time-point such as the departure point will be considered in a subsequent paper.

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ON THE SET-VALUED ADDITIVE AND SUBADDITIVE FUNCTIONS

BY

ANCA MARIA PRECUPANU

This paper deals with some continuity properties of the additive multimeasures with respect to suitable topologies defined on their domain. After some general considerations concerning the setting of the problem, in Section 2 are investigated some properties of the o-continuous multimeasures and in Section 3 some basic properties of regular multimeasures on a Hausdorff locally compact space. Finally the special case of regular multimeasures on a normed space is discussed.

1. **Basic notions, terminology and notations.** Let $\mathcal{P}$ be the family of all subsets of a set Ω organized as a Boolean ring for the symmetric difference and intersection of sets, where the sum $\oplus$ in $\mathcal{P}$ is given by the symmetric difference and the product $\odot$ by the intersection. In this ring the empty set, $\emptyset$, is the null element and the set Ω is the unity. We shall also use the following notations: $A' = \Omega \setminus A$, $[0, A] = \{B \in \mathcal{P} ; B \subseteq A\}$, $[A, B] = \{E \in \mathcal{P} ; A \subseteq E \subseteq B\}$, $A \oplus [0, B] = \{A \oplus C ; C \subseteq B\} = [A \setminus B, A \cup B]$ for every $A, B \in \mathcal{P}$. Let $\mathcal{F}$ be the family of all finite sets in Ω and $\mathcal{F}' = \{A ; A' \in \mathcal{F}\}$. It is easy to see that $\mathcal{F}' \neq \emptyset$ and $\mathcal{F}'$ is directed by inclusion. If Ω is infinite, then $\mathcal{F}'$ is a filter on Ω.

Let us consider for every $A \in \mathcal{P}$ the family $\mathcal{B}(A) = \{A + [0, F] ; F \in \mathcal{F}'\}$. From [2] we obtain the following assertions:

(1.1) There exists a unique topology $\mathcal{O}$ on $\mathcal{P}$ for which the family $\mathcal{B}(A)$ is a base of neighbourhoods of A, for every $A \in \mathcal{P}$. We shall call it the *order-topology on* $\mathcal{P}$.

(1.2) The family $\mathcal{W}(F) = \{(A, B) ; A \oplus B \subseteq F, F \in \mathcal{F}'\}$ is a base of vicinities in $\mathcal{P}$ for a uniformity $\mathcal{W}$ on Ω.

(1.3) The operations $(A, B) \to A \cup B$, $(A, B) \to A \setminus B$, $(A, B) \to A \cap B$, $(A, B) \to A \Delta B$ are uniformly continuous. (The topology $\mathcal{O}$ agrees with the topology induced by the uniformity described at (1.2)).

Thus $(\mathcal{P}, \oplus, \odot, \mathcal{O})$ is a topological ring of sets.

(1.4) $(\mathcal{P}, \oplus, \odot, \mathcal{O})$ is a totally disconnected compact Hausdorff topological ring.

(1.5) The family $\mathcal{F}$ is dense in $\mathcal{P}$.

(1.6) If $\mathcal{R}$ is a subring of $\mathcal{P}$ and $\mathcal{S}$ is the closure of $\mathcal{R}$ in $\mathcal{O}$-topology, then $(\mathcal{S}, \oplus, \odot)$ is the smallest order complete Boolean subring of $\mathcal{P}$ containing $(\mathcal{P}, \oplus, \odot)$. The symbol O is also used to denote the induced topology on any subfamily $\mathcal{E}$ of $\mathcal{P}$. If $\mathcal{E}$ is a family of subsets of Ω, the symbols $\sigma(\mathcal{E})$, $\delta(\mathcal{E})$ will denote, respectively, the σ-ring, δ-ring generated by $\mathcal{E}$.

We also denote by X a Hausdorff locally convex topological vector space, σ its topology, $\mathcal{V}$ a base of absolutely convex closed neighborhoods of the origin in X. Using notations as in [5], $\mathcal{A}(X)$ denotes the family of all nonempty subsets of X, $\mathcal{B}(X)$, $\mathcal{C}(X)$, $\mathcal{K}(X)$ the families of bounded, closed, closed bounded convex sets of X, respectively. $\bar{A}$ denotes the closure of A in X. If we define the equivalence relation „ρ" on $\mathcal{A}(X)$ by

$$A \rho B \leftrightarrow \bar{A} = \bar{B}, \tag{1}$$

then the quotient space $\mathcal{A}(X)/\rho$ can be identified with the semigroup $(\mathcal{C}(X), \dot{+})$, where the addition $\dot{+}$ is defined by

$$A \dot{+} B = \overline{A + B}. \tag{2}$$

In this way Σ represents a finite sum or a series with respect to $\dot{+}$.

If $\mathcal{U}$ is the invariant uniformity on X compatible with τ then the uniformity $\tilde{\mathcal{U}}$ on $\mathcal{A}(X)$ defined by the following base of vicinities $\mathcal{W}(U) = \{(A, B) \in \mathcal{A}(X) \times \mathcal{A}(X) : A \subset B + U,\ B \subset A + U\}$, $U \in \mathcal{V}$, is said to be the *exponential extension* of $\mathcal{U}$. We shall denote by $\tilde{\tau}$ the topology on $\mathcal{A}(X)$ induced by $\tilde{\mathcal{U}}$. Using the equivalence relation (1), we can identify $(\mathcal{C}(X), \tilde{\tau})$ with $(\mathcal{A}(X), \tilde{\tau})/\rho$ and $(\mathcal{C}(X), \tilde{\mathcal{U}})$ with the separated uniform space associated with $(\mathcal{A}(X), \tilde{\mathcal{U}})$.

If X is metrizable, then $(\mathcal{A}(X), \tilde{\mathcal{U}})$ becomes a semi-metrizable space for the Hausdorff distance between sets (see [7]). If in addition X is complete then so is $\mathcal{C}(X)$. If $(X, \mathcal{U})$ is a complete uniform space then so is $(\mathcal{C}(X), \tilde{\mathcal{U}})$. If (E_n) is a sequence of members of $\mathcal{A}(X)$ we say that the series $\sum_{n=1}^{\infty} E_n$ $\tilde{\tau}$-converges if the sequence $S_n = \sum_{k=1}^{n} E_k$ $\tilde{\tau}$-converges in $\mathcal{A}(X)$ and we write $(\tilde{\tau}) \sum_{n=1}^{\infty} E_n$. In what follows we shall consider correspondences μ from a ring $\mathcal{R}$ of subsets of Ω to X, that is set functions from $\mathcal{R}$ to $\mathcal{A}(X)$ or $\mathcal{C}(X)$ such that $\mu(\varnothing) = 0$, where O denotes the set $\{O\}$ in $\mathcal{A}(X)$.

In fact a correspondence μ may be regarded as a function from $\mathcal{R}$ into either the semigroup $(\mathcal{A}(X), +)$ or, with the above convention about „ρ" which permits to identify two sets with the same closure, the semigroup $(\mathcal{C}(X), \dot{+})$.

If μ is a correspondence from $\mathcal{R}$ to X, then $\bar{\mu}$ denotes the correspondence $\bar{\mu}(A) = \overline{\mu(A)}$ for every $A \in \mathcal{R}$.

Definition 1.7. *A correspondence μ from $\mathcal{R}$ to X, that is a function $\mu : \mathcal{R} \to \mathcal{C}(X)$, is said to be a multimeasure iff* a) $\mu(\varnothing) = 0$ *and*

$$\text{b) } \mu(A \cup B) = \mu(A) \dot{+} \mu(B) = (\overline{\mu(A) + \mu(B)}). \tag{3}$$

for every A, $B \in \mathcal{R}$ such that $A \cap B = \varnothing$.

Definition 1.8. *A correspondence μ from $\mathcal{R}$ to $\mathcal{C}(X)$ is said to be exhaustive (locally exhaustive) iff $\mu(A_n) \to 0$, for every disjoint sequence $(A_n) \subset \mathcal{R}$ (for every $T \in \mathcal{R}$, and every disjoint sequence $(A_n) \subset \mathcal{R}$ such that $A_n \subseteq T$).*

2. **O-continuous multimeasures.** Let $(A_i)_{i \in I}$ be a disjoint family in the ring $\mathcal{R}$ on Ω ($A_i \cap A_j = \varnothing$ for $i \neq j$) and let $\mathcal{J}$ be the family of all finite subsets of I. Suppose $\mathcal{J}$ directed by inclusion and $\varphi : \mathcal{J} \to \mathcal{R}$ a net. For every $J \in \mathcal{J}$ we agree to take $\varphi(J) = \sum_{j \in J} A_j = \bigcup_{j \in J} A_j$. The limit of φ for the O-topology of $\mathcal{R}$ is said to be the *sum* of the family $(A_i)_{i \in I}$ and it is denoted by $\sum_{i \in I} A_i$.

The $\tilde{\tau}$-limit of the net $\mu \circ \varphi : \mathcal{J} \to \mathcal{C}(X)$ defines the sum $(\tilde{\tau}) \sum_{i \in I} \mu(A_i)$.

Definition 2.1. *A multimeasure $\mu : \mathcal{R} \to \mathcal{C}(X)$ is said to be:*

I) *completely additive iff for every disjoint family $(A_i)_{i \in I}$ in $\mathcal{R}$ holds*

$$\mu\Big(\sum_{i \in I} A_i\Big) = (\tilde{\tau}) \sum_{i \in I} \mu(A_i); \tag{4}$$

II) *σ-additive or countably additive iff*

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = (\tilde{\tau}) \sum_{n=1}^{\infty} \mu(A_n), \tag{5}$$

for every disjoint sequence $(A_n)_{n \in N}$ in $\mathcal{R}$;

III) *summable iff for every disjoint family $(A_i)_{i \in I}$ in $\mathcal{R}$ with $\bigcup_{i \in I} A_i \in \mathcal{R}$ the family $\{\mu(A_i)\}_{i \in I}$ for which the sum exists and*

$$\mu\Big(\bigcup_{i \in I} A_i\Big) = (\tilde{\tau}) \sum_{i \in I} \mu(A_i). \tag{6}$$

Definition 2.2. *A multimeasure $\mu : (\mathcal{R}, O) \to (\mathcal{C}(X), \tilde{\tau})$ is said to be o-exhaustive iff for every infinite disjoint family $(A_i)_{i \in I}$ of $\mathcal{R}$ (I without maximal element) converging to $\varnothing$ with respect to the filter $\mathcal{J}'$ of all the complements of finite sets in I implies $\mu(A_i) \to O$ with respect to the same filter.*

Definition 2.3. *The multimeasure $\mu : \mathcal{R} \to \mathcal{C}(X)$ is said to be o-continuous at $A \in \mathcal{R}$ (on $\mathcal{R}$), briefly continuous, iff the function $\mu : (\mathcal{R}, O) \to (\mathcal{C}(X), \tilde{\tau})$ is continuous at A (on $\mathcal{R}$).*

Using the results of [15], § 3, we obtain.

Proposition 2.4. *The multimeasure $\mu : \mathcal{R} \to \mathcal{C}(X)$ is o-continuous on $\mathcal{R}$ if and only if μ is continuous at $\varnothing$.* Assuming that $(X, \mathcal{U})$ is complete and using [15] (theorem 1) we also obtain.

Theorem 2.5. *If $(X, \mathcal{U})$ is complete and μ is a o-continuous multimeasure from $\mathcal{R}$ to $\mathcal{C}(X)$, then there exists a unique o-continuous multimeasure $\tilde{\mu}$ from $\mathcal{S}$, the closure of $\mathcal{R}$ for the O-topology, to $\mathcal{C}(X)$, which extends μ.*

Using the properties (1.1)—(1.6) of the O-topology we obtain.

Theorem 2.6. *If μ is a o-continuous multimeasure from $\mathcal{R}$ to $\mathcal{C}(X)$ ((X, U) complete) then:*

a) *μ is completely additive;* b) *μ is summable;* c) *μ is o-exhaustive.*

Proof. a) Since the O-topology is compatible with the order given by inclusion and the topology $\tilde{\tau}$ on $\mathcal{C}(X)$ is separated, we can apply the results of [14], § 2.2, and hence μ is completely additive.

b) It is obvious that every summable multimeasure is completely additive. On the other hand, if $\mathcal{R}$ is closed for the O-topology then for every family $(A_i)_{i\in I}$ in $\mathcal{R}$ we have $\bigcup_{i\in I} A_i \in \mathcal{R}$, hence in this case the complete additivity is equivalent to the summability of μ. If $\mathcal{S}$ is the closure of $\mathcal{R}$ for the O-topology then $\mathcal{S}$ is, by virtue of (1.6), a closed subring of $\mathcal{P}$ on which the complete additivity of μ coincides with the summability.

But, according to the theorem 2.5, the multimeasure μ can be extended to $\mathcal{S}$ and the extension $\tilde{\mu}$ of μ is o-continuous on $\mathcal{S}$. From a) it follows that $\tilde{\mu}$ is completely additive, hence summable on $\mathcal{S}$. Consequently μ is summable on $\mathcal{R}$.

c) Since the convergence with respect to the O-topology and the order convergence of the nets in $\mathcal{P}$ are equivalent, we obtain immediately from the o-continuity of μ that for every infinite disjoint family $(A_i)_{i\in I}$ of $\mathcal{R}$ (I without maximal element) with $A_i \to \varnothing$ with respect to the filter $\mathcal{F}'$ of the complements of finite subsets of I holds $\mu(A_i)\to\mu(\varnothing)=0$ with respect to the same filter. This completes the proof.

Proposition 2.7. *The range of every o-continuous multimeasure $\mu: \mathcal{R}\to\mathcal{C}(X)$ is a relatively compact set.*

Proof. We have seen (the proposition (1.4)) that $(\mathcal{P}, O)$ is a Hausdorff compact space. If $\mathcal{S}$ is the closure of $\mathcal{R}$, then $\mathcal{S}$ is also a compact set in $(\mathcal{P}, O)$. On the other hand, by virtue of theorem 2.5, μ can be extended to an o-continuous multimeasure $\tilde{\mu}$ on $\mathcal{S}$. From here we obtain that $\tilde{\mu}(\mathcal{S})$ is a compact set of the Hausdorff space $(\mathcal{C}(X), \tilde{\tau})$ and hence $\mu(\mathcal{R})$ is relatively compact.

3. **Regular multimeasures.** Let $(\Omega, \mathcal{T})$ be a Hausdorff locally compact space, $\mathcal{K}$, $\mathcal{K}_0$, respectively the family of compact, compact G_δ subsets of Ω, and $\mu: \mathcal{R}\to\mathcal{C}(X)$ a multimeasure.

Definition 3.1. I) *A set $A\in\mathcal{R}$ is said to be outer regular with respect to the multimeasure μ iff for every vicinity $W\in\tilde{\mathcal{U}}$, there exists an open set $D\supset A$ such that $(\mu(E)\dot{+}T, T)\in W$ for every $E\in\mathcal{R}$ with $E\subset D\setminus A$ and every $T\in\mathcal{C}(X)$.*

II) *A set $A\in\mathcal{R}$ is said to be inner regular with respect to the multimeasure μ iff for every vicinity $W\in\tilde{\mathcal{U}}$, there exists a compact set $K\subset A$ such that $(\mu(E)\dot{+}T, T)\in W$ for every $E\in\mathcal{R}$ with $E\subset A\setminus K$ and every $T\in\mathcal{C}(X)$.*

III) *A set $A\in\mathcal{R}$ is said to be regular with respect to the multimeasure μ iff for every vicinity $W\in\tilde{\mathcal{U}}$ there exist an open set $D\supset A$ and a compact set $K\subset A$ such that $(\mu(E)\dot{+}T, T)\in W$ for every $E\in\mathcal{R}$ with $E\subset D\setminus K$ and every $T\in\mathcal{C}(X)$.*

Definition 3.2. *A multimeasure $\mu:\mathcal{R}\to\mathcal{C}(X)$ is said to be regular (outer regular, inner regular) iff every $A\in\mathcal{R}$ is a regular set (respectively outer regular, inner regular set).*

Example. If μ is a σ-additive and locally exhaustive multimeasure from $\delta(\mathcal{K}_0)$ to $\mathcal{C}(X)$ then by virtue of [13] (lemma 3.2, p. 319) it follows that μ is a regular measure.

Proposition 3.3. *If $A\in\mathcal{R}$ is an outer regular set with respect to multimeasure $\mu:\mathcal{R}\to\mathcal{C}(X)$, then for every vecinity $W\in\tilde{\mathcal{U}}$ there exists an open set $D\supset A$ such that $(\mu(S), \mu(A))\in W$ for every $S\in\mathcal{R}$ with $A\subset S\subset D$.*

Proof. Since $A\in\mathcal{R}$ is outer regular, for every $W\in\tilde{\mathcal{U}}$, there exists an open set $D\supset A$ such that

$$(\mu(E)\dot{+}T, T)\in W \tag{7}$$

for every $E\in R$ with $E\subset D\setminus A$ and every $T\in\mathcal{C}(X)$. Take $S\in\mathcal{R}$ such that $A\subset S\subset D$. Because $S\setminus A\subset D\setminus A$ and $S\setminus A\in\mathcal{R}$ from (7) we have

$$(\mu(S\setminus A)\dot{+}T, T)\in W, \tag{8}$$

for every $T\in\mathcal{C}(X)$.

On the other hand, $S=A\cup(S\setminus A)$, where $A\cap(S\setminus A)=\varnothing$, which yields $\mu(S)=\mu(A)\dot{+}\mu(S\setminus A)$. From here and from (8) written for $T=\mu(A)$ we have $(\mu(S), \mu(A))=(\mu(A)\dot{+}\mu(S\setminus A), \mu(A))\in W$ as claimed. In the same way we can prove.

Proposition 3.4. *If $A\in\mathcal{R}$ is an inner regular set with respect to multimeasure $\mu:\mathcal{R}\to\mathcal{C}(X)$, then for every vicinity $W\in\tilde{\mathcal{U}}$ there exists a compact set $K\subset A$ such that $(\mu(A), \mu(S))\in W$ for every $S\in\mathcal{R}$ with $K\subset S\subset A$.*

Example. If $\mathcal{P}$ is the family of all subsets of the Hausdorff locally compact space Ω, $\mathcal{F}$ the family of all finite sets in Ω and μ and o-continuous multimeasure from $\mathcal{P}$ to $\mathcal{C}(X)$, then μ satisfies the property mentioned in the proposition 3.4. Indeed, let $A\in\mathcal{P}$. Since μ is o-continuous at A it follows that for every $W\in\tilde{\mathcal{U}}$ there exists an O-neighborhood of A, that is an interval $[A\cap K, A\cup cK]$ where $K\in\mathcal{F}$ such that $(\mu(S), \mu(A))\in W$ for every $S\in\mathcal{R}$ with $S\in[A\cap K, A\cup cK]$. But $K_0=A\cap K$ is again a compact set and $K_0\subset A\subset A\cup cK$. Now it is easy to observe that $(\mu(S), \mu(A))\in W$ whenever $S\in\mathcal{R}$, $S\in[K_0, A]$.

Proposition 3.5. *If $A\in\mathcal{R}$ is a regular set with respect to the multimeasure $\mu:\mathcal{R}\to\mathcal{C}(X)$, then for every vecinity $W\in\tilde{\mathcal{U}}$ there exist an open set $D\supset A$ and a compact set $K\subset A$ such that $(\mu(S), \mu(A))\in W$ for every $S\in\mathcal{R}$ with $K\subset S\subset D$.*

Proof. If $W\in\tilde{\mathcal{U}}$ there exists a symmetric vecinity $W_1\in\tilde{\mathcal{U}}$ such that $W_1\circ W_1\subset W$. From the regularity of $A\in\mathcal{R}$, for W_1 there exist $D\in\mathcal{T}$ and $K\in\mathcal{K}$ such that $K\subset A\subset D$ and

$$(\mu(E)\dot{+}T, T)\in W_1, \tag{9}$$

for every $E\in\mathcal{R}$ with $E\subset D\setminus K$ and every $T\in\mathcal{C}(X)$. Take $S\in\mathcal{R}$ with $K\subset\subset S\subset D$. We observe that $A\setminus S\subset D\setminus K$, $S\setminus A\subset D\setminus K$ and $S=(A\cap S)\cup\cup(S\setminus A)$, where $A\cap S$, $S\setminus A$, $A\setminus S\in\mathcal{R}$. From (9) we have $(\mu(A\setminus S)\dot{+}T, T)\in W_1$ and $(\mu(S\setminus A)\dot{+}T, T)\in W_1$ for every $T\in\mathcal{C}(X)$, so that

$$(\mu(A\setminus S)\dot{+}T, \mu(S\setminus A)\dot{+}T)\in W_1\circ W_1\subset W, \tag{10}$$

for every $T\in\mathcal{C}(X)$. If $T=\mu(A\cap S)$ we obtain from (10) $(\mu(A\setminus S)\dot{+}\mu(A\cap S), \mu(S\setminus A)\dot{+}\mu(S\cap A))=(\mu(A), \mu(S))\in W$, as claimed.

Remark. It is easy to see that if $A\in\mathcal{R}$ is regular with respect to the multimeasure μ, then the function μ defined on $\mathcal{R}$ with values in $(\mathcal{C}(X), \tilde{\mathcal{U}})$ is continuous at $A\in\mathcal{R}$ for a topology Γ on $\mathcal{R}$ which has as a base the family $\mathcal{B}=\{[K, D]\,;\ \mathcal{K}\in K,\ D\in\mathcal{T}\}$.

Theorem 3.6. *A set $A\in R$ is regular with respect to the multimeasure μ if and only if A is simultaneously outer regular and inner regular with respect to μ.*

Proof. „*If part*“ is trivial.

To prove the „*only if part*“, assume $A\in\mathcal{R}$ simultaneously outer regular and inner regular. Let $W\in\tilde{\mathcal{U}}$ and let $W_1\in\tilde{\mathcal{U}}$, such that $W_1\circ W_1\subset W$.

From the outer regularity of $A\in\mathcal{R}$ it follows that there exists an open set $D\supset A$ such that

(11) $$(\mu(E)\dot{+}T,\ T)\in W,$$

for every $E\subset D\setminus A$ and every $T\in\mathcal{C}(X)$.

By inner regularity of $A\in\mathcal{R}$ we obtain a compact set $K\subset A$ such that

(12) $$(\mu(E)\dot{+}T,\ T)\in W_1,$$

for every $E\subset A\setminus K$ and every $T\in\mathcal{C}(X)$. Now let us take $S\in\mathcal{R}$ with $S\subset\subset D\setminus K$.

We observe that $S\setminus A\subset D\setminus A$, $S\cap A\subset A\setminus K$ and $S=(S\cap A)\cup(S\setminus A)$ where $(S\cap A)\cap(S\setminus A)=\varnothing$ and $S\setminus A,\ S\cap A\in\mathcal{R}$.

From (11) and (12) we have

(11′) $$(\mu(S\setminus A)\dot{+}T,\ T)\in W_1,\ \text{for every } T\in\mathcal{C}(X) \text{ and}$$

(12′) $$(\mu(S\cap A)\dot{+}T,\ T)\in W_1,\ \text{for every } T\in\mathcal{C}(X).$$

Taking $T=\mu(S\cap A)$ in (11′) we obtain

(13) $$(\mu(S),\ \mu(S\cap A))=(\mu(S\setminus A)\dot{+}\mu(S\cap A),\ \mu(S\cap A))\in W_1,$$

which implies, by virtue of the invariance of $\widetilde{\mathcal{U}}$,

(14) $$(\mu(S)\dot{+}T,\ \mu(S\cap A)\dot{+}T)\in W_1,\ \text{for every } T\in\mathcal{C}(X).$$

Now from (12′) and (14) it follows $(\mu(S)\dot{+}T,\ T)\in W_1\circ W_1\subset W$, for every $T\in\mathcal{C}(X)$, which completes the proof. Let $A\in\mathcal{R}$. We remark that the families $\{\mu(K)\,;\,K\subset A,\ K\in\mathcal{R}\cap\mathcal{K}\}$ and $\{\mu(D)\,;\,D\supset A,\ D\in\mathcal{T}\cap\mathcal{R}\}$ are nets with respect to inclusion.

If, moreover, the set $A\in\mathcal{R}$ satisfies the condition

(*) *„for every compact set $K\subset A$ there exists a compact set $K_0\in\mathcal{R}$ such that $K\subset K_0\subset A$"*, then we have

Proposition 3.7. *If $A\in\mathcal{R}$ is inner regular with respect to the multimeasure μ and satisfies the property (*) then*

(15) $$\mu(A)=\lim\{\mu(K)\,;\,K\subset A,\ K\in\mathcal{R}\cap\mathcal{K}\}.$$

Proof. If $A\in\mathcal{R}$ is inner regular, according to the proposition 3.4 for every $W\in\widetilde{\mathcal{U}}$, there exists a compact set $K_0\subset A$ such that

(16) $$(\mu(A),\ \mu(S))\in W,\ \text{for every } S\in\mathcal{R} \text{ with } K_0\subset S\subset A.$$

In particular (16) holds for every $K\in\mathcal{R}\cap\mathcal{K}$ such that $K_0\subset K\subset A$. Hence for every $W\in\widetilde{\mathcal{U}}$ there is a $K_0\in\mathcal{K}$, such that $(\mu(A),\ \mu(K))\in W$ for every $K\in\mathcal{R}\cap\mathcal{K}$ with $K_0\subset K\subset A$, that is the equality (15) holds.

Similarly if $A\in\mathcal{R}$ satisfies the property:

(**) *„for every open set $D\supset A$ there exists an open set $D_0\in\mathcal{R}$ such that $A\subset D\subset D_0$"*.

We can prove:

Proposition 3.8. *If $A\in\mathcal{R}$ is outer regular with respect to multimeasure μ and satisfies (**) then*

(16) $$\mu(A)=\lim\{\mu(D)\,:\,D\supset A,\ D\in\mathcal{R}\cap\mathcal{T}\}.$$

Also the following result is true.

Proposition 3.9. *If $A\in\mathcal{R}$ satisfies the properties (*) and (**) and it is regular with respect to the multimeasure μ, then*

(17) $$\mu(A)=\lim\{\mu(D)\,;\,D\supset A,\ D\in\mathcal{R}\cap\mathcal{T}\}=\lim\{\mu(K)\,;\,K\subset A,\ K\in\mathcal{K}\cap\mathcal{R}\}.$$

Remark. If $\mathcal{R}$ is the ring generated by the family $\mathcal{K}$ then every $A\in\mathcal{R}$ satisfies simultaneously (*) and (**) and hence every regular multimeasure μ on $\mathcal{R}$ satisfies (17).

Special case. If X is a real normed space and $\mathcal{C}_b(X)$ is the family of all bounded closed sets of $\mathcal{A}(X)$, endowed with the Hausdorff metric D we define $\|A\|=D(A,\{O\})=D(A,O)$ for every $A\in\mathcal{C}_b(X)$. Taking into account the special form of the vicinities of $\widetilde{\mathcal{U}}$ in this case, we observe that if $A\in\mathcal{R}$ is a regular set (respectively outer regular, inner regular) with respect a ot multimeasure $\mu:\mathcal{R}\to\mathcal{C}_b(X)$, then the definition 3.1 can be restated as

Definition 3.1.′ *$A\in\mathcal{R}$ is said to be regular (outer regular inner regular) with respect to the multimeasure $\mu:\mathcal{R}\to\mathcal{C}_b(X)$ iff for every $\varepsilon>0$ there exist(s) a compact set $K\subset A$ and an open set $D\supset A$ (respectively an open set $D\supset A$, a compact set $K\subset A$) such that $D(\mu(S)\dot{+}T,\ T)<\varepsilon$ for every $S\in\mathcal{R}$ with $S\subset\subset D\setminus K$ (resp. $S\subset D\setminus A$, $S\subset A\setminus K$) and every $T\in\mathcal{C}_b(X)$.*

Definition 3.10. *Let $\mu:\mathcal{R}\to\mathcal{C}_b(X)$ be a multimeasure*

I) *We call the total variation of μ the real valued set function ν defined on $\mathcal{R}$ by $\nu(A)=\sup\{\Sigma\|A_i\|\}$ where supremum is extended over all finite partitions $(A_i)_{i=1}^n$ of A with $A_i\in\mathcal{R}$. (ν can be infinite).*

II) *We say that ν has finite variation iff $\nu(A)<\infty$ for every $A\in\mathcal{R}$.*

We observe that ν is a non-negative additive set function defined on $\mathcal{R}$.

In the following we shall suppose: μ *with finite variation*. It is clear that

(17) $$\|A\|\leqslant\nu(A),\ \text{for every } A\in\mathcal{R}.$$

Taking into account this last inequality we obtain

Proposition 3.11. *If the total variation ν of the multimeasure μ is regular (respectively outer regular, inner regular) then μ is regular (resp. outer regular inner regular).*

Generally the converse does not hold except for the inner regularity. Indeed the following theorem is true.

Theorem 3.12. *The multimeasure $\mu:\mathcal{R}\to\mathcal{C}_b(X)$ is inner regular if and only if the total variation ν is inner regular on $\mathcal{R}$.*

Proof. The *„only if part"* is trivial by proposition 3.11. To prove the *„if part"*, consider $A\in\mathcal{R}$ inner regular. By the definition of the total variation ν we find for every $\varepsilon>0$ a finite partition $(A_i)_{i=1}^n\subset\mathcal{R}$ of A for which

(18) $$\nu(A)<\sum_{i=1}^{n}\|\mu(A_i)\|+\varepsilon/2.$$

But μ is inner regular at every A_i, $i=1,2,\ldots,n$, and then, by virtue of proposition 3.4 there exists a compact $K_i\subset A_i$, $i=1,2,\ldots,n$, such that

(19) $$D(\mu(A_i),\ \mu(S))<\varepsilon/2n,$$

for every $S \in \mathcal{R}$ with $K_i \subset S \subset A_i$ and $i = 1, 2, \ldots, n$. Let us denote by $K = \bigcup_{i=1}^{n} K_i$. It is clear that K is compact and $K \subset A$. Consider an arbitrary $E \in \mathcal{R}$ such that $K \subset E \subset A$ and let be $E_i = E \cap A_i$, $i = 1, 2, \ldots, n$. Obviously $E_i \in \mathcal{R}$, $K_i \subset \subset E_i \subset A_i$ and $E_i \cap E_j = \varnothing$ for $i \neq j$.

Now, using (18) and (19), we obtain

$$\nu(A) - \nu(E) < \sum_{i=1}^{n} \|\mu(A_i)\| + \varepsilon/2 - \sum_{i=1}^{n} \|\mu(E_i)\| = \sum_{i=1}^{n} [\|\mu(A_i)\| - \|\mu(E_i)\|] + $$
$$+ \varepsilon/2 \leq \sum_{i=1}^{n} D(\mu(A_i), \mu(E_i)) + \varepsilon/2 < \varepsilon,$$

which, according to proposition 2 of [4], p. 200, assures inner regularity of ν.

This completes the proof.

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ON THE INDEFINITE INTEGRALS OF ABSTRACT FUNCTIONS

BY

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1. Introduction. Let X be a locally convex complete linear space and X^* its dual space. The function $\varphi : R \to X$ is said to be locally Lebesgue integrable if $x^*(\varphi(t))$ is locally Lebesgue integrable ((l.L.i.) for each $x^* \in \in X^*$. If $\psi : R \to \mathbf{C}$ is a l.L.i. function, then the spectrum of ψ will be denoted by $\sigma(\psi)$ and it is equal to the support of the Fourier transform in the sense of distributions ([10] p. 408) of ψ, i.e. $\sigma(\psi) = \operatorname{supp} \hat{\psi}$. If $\varphi : R \to X$ is a l.L.i. function, then $\sigma(\varphi)$ is equal to the closure of the set $\bigcup_{x^* \in X^*} \sigma(x^*(\varphi))$. If $\psi : R \to \mathbf{C}$ is a bounded l.L.i. function, then one can define the Beurling spectrum $\sigma_0(\psi)$ to be hull $I_0(\psi)$, where $I_0(\psi) = \}f : f*\psi = 0, f \in L^1(R)$ (see [12] ch. 7). One can prove that $\sigma(\psi) = \sigma_0(\psi)$. If $\varphi : R \to X$ is a bounded l.L.i. function, then $\sigma_0(\varphi)$ can be defined also as in the scalar case and it is proved in [5] (theorem 4.1) that $\sigma_0(\varphi) = \overline{\bigcup_{x^* \in X^*} \sigma(x^*(\varphi))}$ and hence $\sigma_0(\varphi) = \sigma(\varphi)$. We denote by Λ any class of functions from $C_{ub}(R, C)$ or $C_{ub}(R, X)$ (the set of all bounded uniformly continuous functions from R to $\mathbf{C}$ or X) having the following properties.

P.1. Λ is a linear space containing constants and invariant under translation.

P.2. Λ is closed with respect to uniform convergence, i.e. a uniform limit of a sequence, or a net of elements of Λ is also an element of Λ.

Using *P*.1. and *P*. 2. it is proved in [5], that $f*\varphi \in \Lambda$, for each $\varphi \in \Lambda$ and $f \in L^1(R)$. Also it is shown that if $\varphi \in \Lambda$, then $\psi(x) = \int_{K}^{X+S} \varphi(t)dt \in \Lambda$ for each $s \in R$. One can define the Λ-spectrum $\sigma_\Lambda(\varphi)$ of a function $\varphi \in C_{ub}(R, X)$ to be equal to the hul lof the ideal $I_\Lambda(\varphi) = \{f : f*\varphi \in \Lambda\ f \in L^1(R)\}$. One can prove that if $\lambda \in \sigma_\Lambda(\varphi)$, then there exists $f \in L^1(R)$ such that $f*\varphi \in \Lambda$ and $\hat{f}(\lambda) \neq 0$. Since $0 \in \Lambda$, then $I_0(\varphi) \subset I_\Lambda(\varphi)$. Hence $\sigma_\Lambda(\varphi) \subset \sigma(\varphi)$. Using only *P*.1. and *P*.2. it is proved in [5] (theorem 2.1 and 4.2) that $\varphi \in C_{ub}(R, X)$ is an element of Λ iff $\sigma_\Lambda(\varphi) = \varnothing$ (empty set). The spectrum $\sigma(\varphi)$ is isolated from zero iff there exists $\delta > 0$ such that $\sigma(\varphi) \cap \exists -\delta, \delta = \varnothing$. We shall use the space S of all functions $f : R \to \mathbf{C}$ having all derivatives and rapidly decreasing to zero together with all their derivatives ([10] p. 408). We notice that the Fourier transform of a function $f \in S$ is also an element of S.

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