for every $S\in\mathcal{R}$ with $K_i\subset S\subset A_i$ and $i=1,2,\ldots,n$. Let us denote by $K=\bigcup_{i=1}^{n}K_i$. It is clear that K is compact and $K\subset A$. Consider an arbitrary $E\in\mathcal{R}$ such that $K\subset E\subset A$ and let be $E_i=E\cap A_i$, $i=1,2,\ldots,n$. Obviously $E_i\in\mathcal{R}$, $K_i\subset E_i\subset A_i$ and $E_i\cap E_j=\varnothing$ for $i\neq j$.

Now, using (18) and (19), we obtain

$$\nu(A)-\nu(E)<\sum_{i=1}^{n}\|\mu(A_i)\|+\varepsilon/2-\sum_{i=1}^{n}\|\mu(E_i)\|=\sum_{i=1}^{n}[\|\mu(A_i)\|-\|\mu(E_i)\|]+$$

$$+\varepsilon/2\leqslant\sum_{i=1}^{n}D(\mu(A_i),\ \mu(E_i))+\varepsilon/2<\varepsilon,$$

which, according to proposition 2 of [4], p. 200, assures inner regularity of ν.

This completes the proof.

REFERENCES

1. A r t s t e i n Z. — *Set-valued measures*, Trans. Amer. Math. Soc., 165 (1972), 103—125
2. B o u r b a k i N. — *Topologie générale*, Chap. 1, 2, Paris, Hermann, 1965
3. B r o o k s J.K. — *An integration theory for set-valued measures*, I, II, Bull. Soc. Roy. Sci. de Liège, 37 (1968), 312—319 375—380
4. D i n c u l e a n u N. — *Teoria măsurii și funcții reale*. Ed. did. și ped. — București, 1964
5. D r e w n o w s k i L. — *Additive and countably additive correspondences*, Comment. Math. 19 (1976—1977), 25—54
6. G o d e t-T h o b i e Ch. — *Sur les sélection de mesures généralisées*, C.R. Acad. Sci. Paris 271 (1970), 153—156
7. G o d e t-T h o b i e Ch. — and P h a m T h e L a i — *Sur le plongement de l'ensemble des convex, fermés, bornés d'un espace vectoriel topologique localement convexe*, ibid. 271 (1970), 84—87.
8. G o d e t-T h o b i e Ch. — *Some results about multimeasures and their selectors*. Proceedings of Conf. Oberwolfach 1979, Lecture Notes in Math. 794, Springer-Verlag, Berlin-Heidelberg-New York, 1980, 112—116
9. H a l m o s P. — *Theory of measure* (in Russian), Moskow, 1953
10. K e l l e y J.L. — *General Topology*, Van Nostrand Company-Canada, 1959
11. K e l l e y J.L. and S r i n i v a s a n T.P. — *Pre-measures on Lattices of Sets*. Math. Ann., 190 (1971), 233—241.
12. K i s y n s k i J. — *On the generation of tight measures*, Stud. Math. 30 (1968), 141—151
13. M o r a l e s P. — *Regularity and extension of semigroup-valued Baire measures* — Proc. Conf. Oberwolfach 1979, Lecture Notes in Math. *794*, Springer Verlag, Berlin-Heidelberg-New York, 1980, p. 318—323
14. N e d o g i b c e n k o G.V., S a v e l i e v L.I. — *Complete continuous measures* (in Russian), Sibir. Mat. J. *22* (1981), No 6, p. 126—141
15. S a v e l i e v L.I. — *Extension of continuous measures*. Sibir. Mat. J., 20 (1979), No 5, p. 1 082—1 091
16. S i o n M. — *A Theory of Semigroup Valued Measures*. Lecture Notes in Math. 355, Springer, Berlin, Heidelberg, New York, 1973

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ANALELE ȘTIINȚIFICE ALE UNIVERSITĂȚII „AL. I. CUZA" DIN IAȘI
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ON THE INDEFINITE INTEGRALS OF ABSTRACT FUNCTIONS

BY

BOLIS BASIT*

1. **Introduction.** Let X be a locally convex complete linear space and X^* its dual space. The function $\varphi: R\to X$ is said to be locally Lebesgue integrable if $x^*(\varphi(t))$ is locally Lebesgue integrable ((l.L.i.) for each $x^*\in X^*$. If $\psi: R\to\mathbf{C}$ is a l.L.i. function, then the spectrum of ψ will be denoted by $\sigma(\psi)$ and it is equal to the support of the Fourier transform in the sense of distributions ([10] p. 408) of ψ, i.e. $\sigma(\psi)=\operatorname{supp}\hat{\psi}$. If $\varphi: R\to X$ is a l.L.i. function, then $\sigma(\varphi)$ is equal to the closure of the set $\bigcup_{x^*\in X^*}\sigma(x^*(\varphi))$. If $\psi: R\to\mathbf{C}$ is a bounded l.L.i. function, then one can define the B e u r l i n g spectrum $\sigma_0(\psi)$ to be hull $I_0(\psi)$, where $I_0(\psi)=\}f: f*\psi=0,\ f\in L^1(R)$ (see [12] ch. 7). One can prove that $\sigma(\psi)=\sigma_0(\psi)$. If $\varphi: R\to X$ is a bounded l.L.i. function, then $\sigma_0(\varphi)$ can be defined also as in the scalar case and it is proved in [5] (theorem 4.1) that $\sigma_0(\varphi)=\overline{\bigcup_{x^*\in X^*}\sigma(x^*(\varphi))}$ and hence $\sigma_0(\varphi)=\sigma(\varphi)$. We denote by Λ any class of functions from $C_{ub}(R,C)$ or $C_{ub}(R,X)$ (the set of all bounded uniformly continuous functions from R to $\mathbf{C}$ or X) having the following properties.

P.1. Λ is a linear space containing constants and invariant under translation.

P.2. Λ is closed with respect to uniform convergence, i.e. a uniform limit of a sequence, or a net of elements of Λ is also an element of Λ.

Using *P*.1. and *P*. 2. it is proved in [5], that $f*\varphi\in\Lambda$, for each $\varphi\in\Lambda$ and $f\in L^1(R)$. Also it is shown that if $\varphi\in\Lambda$, then $\psi(x)=\int_{x}^{x+s}\varphi(t)dt\in\Lambda$ for each $s\in R$. One can define the Λ-spectrum $\sigma_\Lambda(\varphi)$ of a function $\varphi\in C_{ub}(R,X)$ to be equal to the hul lof the ideal $I_\Lambda(\varphi)=\{f: f*\varphi\in\Lambda\ f\in L^1(R)\}$. One can prove that if $\lambda\in\sigma_\Lambda(\varphi)$, then there exists $f\in L^1(R)$ such that $f*\varphi\in\Lambda$ and $\hat{f}(\lambda)\neq 0$. Since $0\in\Lambda$, then $I_0(\varphi)\subset I_\Lambda(\varphi)$. Hence $\sigma_\Lambda(\varphi)\subset\sigma(\varphi)$. Using only *P*.1. and *P*.2. it is proved in [5] (theorem 2.1 and 4.2) that $\varphi\in C_{ub}(R,X)$ is an element of Λ iff $\sigma_\Lambda(\varphi)=\varnothing$ (empty set). The spectrum $\sigma(\varphi)$ is isolated from zero iff there exists $\delta>0$ such that $\sigma(\varphi)\cap\,]-\delta,\ \delta[\,=\varnothing$. We shall use the space S of all functions $f: R\to\mathbf{C}$ having all derivatives and rapidly decreasing to zero together with all their derivatives ([10] p. 408). We notice that the Fourier transform of a function $f\in S$ is also an element of S.

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In the first result of this paper we prove that if $\varphi: R \to X$ is a weakly uniformly continuous function with isolated from zero spectrum, then φ is bounded. The second result of the paper says that if $\varphi: R \to X$ is a I.L.i. function with isolated from zero spectrum, then there exists $f \in L^1(R)$ such that $\psi(x) = \int_0^x \varphi(t)dt = f*\varphi(x) - f*\varphi(0)$. This result extends a classical Favard's theorem (which states that if $\varphi: R \to \mathbf{C}$ is an almost periodic (A.P.) function with isolated from zero spectrum, then its indefinite integral $\psi(x) = \int_0^x \varphi(t)dt$ is also A.P.), to any class of I.L.i. functions invariant under the operation of involution. Finally, the third result of this paper gives a Λ-spectrum characterization of the indefinite integrals of the elements of Λ. This extends a theorem of G ü n z l e r to the case of Λ classes (see [9]. theorem 6).

2. **Uniformly continuous functions.** In this section we prove three lemmas concerning uniformly continuous functions.

Lemma 2.1. *Let $\varphi: R \to \mathbf{C}$ be a uniformly continuous function. Then there exists a constant K such that* $\|\varphi_s - \varphi\|_\infty = \sup_{t \in R} \|\varphi(t+s) - \varphi(t)\| \leqslant K(|s|+1)$, $s \in R$.

Proof. Let $\varepsilon_0 > 0$, there exists $0 < \delta_0 < 1$ such that $\|\varphi(t+h) - \varphi(t)\|_\infty \leqslant \leqslant \varepsilon_0$, $|h| \leqslant \delta_0$. We can assume that $s > 0$. Choose the integer n_0 satisfying the inequality $s - \delta_0 \leqslant n_0$, $\delta_0 < s$. We have the equality $\varphi(t) - \varphi(t+s) = \sum_{k=1}^{n_0} [\varphi(t + (k-1)\delta_0) - \varphi(t + k\delta_0)] + \varphi(t + n_0\delta_0) - \varphi(t+s)$, from which we get $\|\varphi(t) - \varphi(t+s\|_\infty \leqslant (n_0+1)\varepsilon_0 \leqslant (s+1)\varepsilon_0\delta_0^{-1} = K(s+1)$.

Corollary 2.1. *Let $\varphi: R \to X$ be weakly uniformly continuous. Then $\varphi_s - \varphi$ is bounded for each $s \in R$.*

Proof. Using lemma 2.1, we get that $x^*(\varphi_s - \varphi)$ is bounded for each $x^* \in X^*$, $s \in R$. Since X is locally convex, we find that $\varphi_s - \varphi$ is bounded, $s \in R$.

We remark that lemma 2.1 is not true for functions from R^2 to $\mathbf{C}$. Indeed, let $\varphi: R^2 \to \mathbf{C}$ be a bounded uniformly continuous function such that $\psi(x, y) = \int_0^x \varphi(t, y)dt$ is not bounded. Then the differences $\psi(x, y+s) - \psi(x, y)$ are not necessarily bounded. In fact the following can be proved for any locally compact group G taking $f \in L^1(G)$ with compact support.

Lemma 2.2. *Let $\varphi: R \to \mathbf{C}$ be a uniformly continuous function and let $f \in S$, $\int_{-\infty}^{\infty} f(t)dt = 1$. Then $\varphi - f*\varphi$ is bounded.*

Proof. We have $|\varphi(x) - \varphi*f(x)| = \int_{-\infty}^{\infty} [\varphi(x) - \varphi(x-t)]f(t)dt \geqslant \int_{-1}^{1} |\varphi(x) - \varphi(x-t)|\,|f(t)|dt + \int_{|t|\geqslant 1} |\varphi(x) - \varphi(x-t)|f(t)dt = J_1 + J_2$. Using lemma 2.1 and the assumption that $f \in S$, we get $J_1 \leqslant K_1$, and $J_2 \leqslant K \int_{|t|\geqslant 1} (|t|+1)|f(t)|\,dt = K_2$. Hence $\|\varphi - \varphi*f\|_\infty \leqslant K_1 + K_2$.

Corollary 2.2. *Lemma 2.2 is true for functions from $R \to X$.* The proof is the same as in corollary 2.1.

Finally, we prove

Lemma 2.3. *Let φ, $\varphi' \in C_{ub}(R, X)$ and let $\sigma_\Lambda(\varphi) = \{0\}$. Then $\varphi' \in \Lambda$.*

Proof. For each $\lambda \neq 0$, there exists $f \in L^1(R)$ such that $|\hat{f}*(\lambda) \neq 0$ and $f*\varphi \in \Lambda$. Since $\varphi' \in C_{ub}(R, X)$, we get $\varphi'*f(x) = (\varphi*f(x))' = \lim_{n\to\infty} n[\varphi*f(x+1/n) - \varphi*f(x)]$ uniformly with respect to $x \in R$. Using P. 2, we obtain $\varphi*f \in \Lambda$. This implies that $\sigma_\Lambda(\varphi') \subset \{0\}$. Since $M\varphi' = \lim_{T\to\infty} 1/2T \int_{-T+a}^{T+a} \varphi'(t)dt = 0$ and the limit exists uniformly with respect to $a \in R$, then by [6] (p. 23) or [3] (lemma 1) or [13] (§7) φ' can be obtained as a uniform limit of functions of the form $\varphi' - \varphi'*f$, where $f_\varepsilon \in L^1(R)$ and $\hat{f}_\varepsilon(\lambda) = 1$, $|\lambda| \leqslant \varepsilon$. We have $\sigma(\varphi' - \varphi'*f) \cap (-\varepsilon, \varepsilon) = \varnothing$ and $\sigma_\Lambda(\varphi' - \varphi'*f_\varepsilon) \cup \{0\}$ and this implies that $\sigma_\Lambda(\varphi' - \varphi'*f_\varepsilon) = \varnothing$. By [5] (theorem 4.2), we get $(\varphi' - \varphi'*f_\varepsilon) \in \Lambda$. Using P. 2, we obtain $\varphi' \in \Lambda$.

Corollary 2.3. *Let $\varphi \in C_{ub}(R, X)$ be such that $\sigma_\Lambda(\varphi) = \{0\}$. Then $\varphi_s - \varphi \in \Lambda$, for each $s \in R$.*

Proof. Choose $f \in L^1(R)$ such that $\hat{f}(\lambda) = 1$, $|\lambda| \leqslant 1/2$ and supp. $\hat{f} \subset [-1, 1]$. Then $\psi' = (f*\varphi)' \in C_{ub}(R, X)$ and hence by lemma 2.3 $\psi' \in \Lambda$ and therefore $\psi_s(x) - \psi(x) - \int_x^{x+s} \psi'(t)dt \in \Lambda$. Since the Beurling spectrum $\sigma(\varphi - \psi)$ is isolated from zero, the Λ-spectrum $\sigma_\Lambda(\varphi - \psi) = \varnothing$. By [5] (theorem 4.2) we obtain $\varphi - \psi \in \Lambda$. This implies that $\varphi_s - \psi_s \in \Lambda$, for each $s \in R$ and hence $\varphi_s - \varphi = [(\varphi_s - \psi_s) + (\psi_s - \psi) + (\psi - \varphi)] \in \Lambda$.

3. **Favards's theorem.** In this section we extend F a v a r d 's theorem ([1] p. 56, [11]) in two directions. Firstly, we have the extension to functions with values in X and the proof of the boundedness of these functions. Secondly, we have the extension to any class F of I.L.i. functions containing constants and invariant under involution (i.e. $f*\varphi \in F$ for each $\varphi \in F$ and $f \in L^1(R)$). To explain the first extension we notice that almost periodic functions are always bounded and that the Beurling spectrum $\sigma(\varphi)$ of an a.p. function $\varphi: R \to \mathbf{C}$ is equal to the closure of the set $\{c : \lim_{T\to\infty} (1/T)\int_0^T e^{itct}\varphi(t)\,dt \neq 0,\ c \in R\}$.

Theorem 3.1. *Let $\varphi: R \to \mathbf{C}$ be uniformly continuous function with isolated from zero spectrum. Then $\varphi(x)$ and $\psi(x) = \int_0^x \varphi(t)dt$ are bounded on R.*

Proof. Let $\delta > 0$ be such that $[-\delta, \delta] \cap \sigma(\varphi) = \varnothing$. Choose $f \in S$ such that supp. $\hat{f} \subset [-\delta, \delta]$ and $\int_{-\infty}^{\infty} f(t)dt = 1$. Since $\sigma(f*\varphi) \subseteq$ supp. $\hat{f} \cap \sigma(\varphi)$, we get $f*\varphi = 0$. By lemma 2.2, we find $\varphi = \varphi - f*\varphi$ is bounded. This implies that $\psi(x)$ is an uniformly continuous function. We have $\psi'(x) = \varphi(t)$ and hence $\hat{\varphi}(\lambda) = 2i\pi\lambda\hat{\psi}(\lambda)$ in the sense of distributions ([10], p. 410). This implies that supp. $\hat{\psi}$ = supp. $\hat{\varphi}$ which means that $\sigma(\varphi) = \sigma(\psi)$. Hence $\sigma(\psi) \cap [-\delta, \delta] = \varnothing$ and therefore $f*\psi = 0$. By lemma 2.2, we get $\psi = \psi - f*\psi$ is bounded.

Corollary 3.1. *Theorem 3.1 is true for the functions with values in X.* Indeed, if $\varphi : R \to X$ satisfies all the conditions of theorem 3.1 ; then $x^*(\varphi)$ and $x^*(\psi)$ are bounded, for each $x^* \in X^*$. Hence φ, ψ are bounded functions.

Theorem 3.2. *Let $\varphi : R \to \mathbf{C}$ be a l.L.i. function with isolated from zero spectrum. Then there exists $f \in L^1(R)$ such that $\psi(x) = \int_0^x \varphi(t)dt = f*\varphi(x) - f*\varphi(0)$.*

Proof. Let $\delta > 0$ be such that $[-\delta, \delta] \cap \sigma(\varphi) = \varnothing$. Choose $f \in L^1(R)$ such that $\hat{f}(\lambda) = 1/(i\lambda)$, $\lambda \in [-\delta, \delta]$. Then the equality $f*\varphi(x) = f*\varphi(0) + \int_0^x \varphi(t)dt$ is true in the sense of distributions and hence it is true pointwise, for both sides are continuous functions.

Corollary 3.2. *Theorem 3.2 is true in the case of functions from R to X.* The proof is the same as in corollary 3.1.

We note that the sets of almost automorphic [7], recurrent [8], distal [8] and Poisson functions [2] from R to $\mathbf{C}$ give examples of classes of type F. One can define easily almost periodic, almost automorphic, recurrent, distal and Poisson functions from R to X, and also it is easy to verify that these sets are classes of type F. Other type F classes are the of Stepanov, Weil and Besicovitch (S^1, W^1, B^1) functions.

4. **A theorem of Günzler** [9]. In this section we give Λ-spectral characterization of the indefinite integrals of functions from Λ which do not necessarily belong to Λ. This study completes and extends a theorem of Günzler ([9] theorem 6), we prove

Theorem 4.1. *Let $\varphi \in C_{ub}(R, X)$. Then the following conditions are equivalent :*

(c.1) $\sigma_\Lambda(\varphi) \subset \{0\}$.

(c.2) $\varphi_s - \varphi \in \Lambda$ *for each* $s \in R$

(c.3) *there exists* ψ, $\zeta \in \Lambda$ *such that* $\varphi(x) = \psi(x) + \int_0^x \zeta(t)dt$.

(c.4) *there exists a sequence* $\{\varphi_n\} \subset \Lambda$ *such that*

$\varphi(x) = \varphi(0) + \lim_{n\to\infty} \int_0^x \varphi_n(t)dt$, *and the limit exists uniformly with respect to* $x \in R$.

Proof. (c.1)$\Rightarrow$(c.2). This follows from corollary 2.3 (c.2)$\Rightarrow$(c.1). Let $\lambda \neq 0$. Choose $s \in R$ such that $e^{i\lambda s} \neq 1$. Using (c.2) for each $f \in L^1(R)$ and $\hat{f}(\lambda) \neq 0$, we have $(\varphi_s - \varphi)*f = \varphi*(f_s - f) \in \Lambda$. Since $\hat{f_s}(\lambda) - \hat{f}(\lambda) = \hat{f}(\lambda)(e^{i\lambda s} - 1) \neq \neq 0$, we conclude that $\lambda \notin \sigma_\Lambda(\varphi)$. Hence $\sigma_\Lambda(\varphi) \subset \{0\}$.

(c.1)$\Rightarrow$(c.3). Choose $f \in L^1(R)$ as in corollary 2.3. Let $\psi(x) = \varphi(x) - f*\varphi(x) + f*\varphi(0)$ and $\zeta(x) = (f*\varphi)' = f'*\varphi$. Then using lemma 2.3, we get $\zeta \in \Lambda$ and since $\sigma(\psi) \cap (-1/2, 1/2) = \varnothing$, we get $\sigma_\Lambda(\psi) = \varnothing$ and by theorem 4.2 of [5] we conclude that $\psi \in \Lambda$. We have $\varphi(x) = \psi(x) + \int_0^x \zeta(t)dt$ and ψ, $\zeta \in \Lambda$.

(c.3)$\Rightarrow$(c.2). We have $\varphi_s(x) - \varphi(x) = \psi_s(x) - \psi(x) + \int_x^{x+s} \zeta(t)dt \in \Lambda$. for each $s \in R$.

(c.1)$\Rightarrow$(c.4). Take $f_n(x) = (1 - \cos nx/nx^2)$. Then by [6] or [3] or [13] we find that $\lim_{n\to\infty} f_n*\varphi'(x) = \varphi(x)$ uniformly with respect to $x \in R$, moreover, $\sigma(f_n) \subset [-n, n]$. Put $\varphi_n(x) = [f_n*\varphi(x)]'$. Using lemma 2.3, we get $\varphi_n \in \Lambda$, $n \in N$. We have $\varphi(x) = \varphi(0) + \lim_{n\to\infty} \int_0^x \varphi_n(t)dt$ (uniformly with respect to $x \in R$).

(c.4)$\Rightarrow$(c.2) We have $\varphi_s(x) - \varphi(x) = \lim_{n\to\infty} \int_x^{x+s} \varphi_n(t)dt$ (uniformly with respect to $x \in R$) and hence $\varphi_s - \varphi \in \Lambda$, for each $s \in R$.

Corollary 4.1. *Let $\varphi \in \Lambda$ and let $\psi(x) = \int_0^x \varphi(t)dt$ be a bounded function. Then either $\psi \in \Lambda$ or $\sigma_\Lambda(\psi) = \{0\}$.*

Proof. The difference $\varphi_s(x) - \varphi(x) = \int_x^{x+s} \varphi(t)dt \in \Lambda$, for each $s \in R$. By theorem 4.1 we get $\sigma_\Lambda(\psi) \subset \{0\}$. Hence either $\sigma_\Lambda(\psi) = \varnothing$ (and hence $\psi \in \Lambda$) or $\sigma_\Lambda(\psi) = \{0\}$.

Corollary 4.2. *In addition to P.1 and P.2, let Λ satisfy the following :*

P.3. $\gamma_\Lambda \varphi \in \Lambda$ for each $\gamma_\Lambda(t) = e^{it\lambda}$ and $\varphi \in \Lambda$.

P.4. If $\varphi \in C_{ub}(R, X)$ and $\varphi_s - \varphi \in \Lambda$, for each $s \in R$, then $\varphi \in \Lambda$.

In this case we have

Corollary 4.3. *Let $\varphi \in \Lambda$. Then either $\psi(x) = \int_0^x \varphi(t)dt$ is an element of Λ or ψ is unbounded.*

Proof. We have $\psi_s(x) - \psi(x) = \int_x^{x+s} \varphi(t)dt \in \Lambda$, $s \in R$. If ψ is bounded, then by property P.4, we get $\psi \in \Lambda$.

We notice that if X satisfies certain conditions ([2], [4]), then the known classes type Λ satisfying P.1, P.2 and P.3 also satisfy P. 4.

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REFERENCES

1. Amerio, L., and Prouse, G. — *Almost periodic functions and functional equations.* Van Nostrand R.C. (1971)
2. Basit, B — *Generalization of two theorems of Kadets on indefinite integral of abstract almost periodic functions.* Math. Notes 9, No. 3 (1971) 311—321
3. Basit, B. — *Note on a theorem of Levitan for the integral of almost periodic functions.* Rend. Ist. di Matem. Univ. Trieste Vol. V (1973) fasc. 1, 9—14
4. Basit, B. and Emam, M. — *Differences of functions in locally convex spaces and applications to almost periodic and almost automorphic functions.* To appear in „Annales Polonici Math.", XLI No. 3 (1983)
5. Basit, B. — *Spectral characterization of abstract functions.* An. şt. Univ. Iaşi, XXVIII (1982), S. I a Matematică, f. 1, 25—35
6. Bochner, S. — *Verlesungen uber Fourier'sche Integrale,* Chelsea pub. C. New York (1948)
7. Bochner, S. — *A new approach to almost periodicity,* Proc. Nat. Acad. Sci. U.S.A. 48 (1962) 2 039—2 043
8. Flor, P. — *Rhythmische Abbildungen abelscher Gruppen II,* Z. Wahrescheinlichkeits theorie Verw. Geb. 7 (1967) 17—28

9. G ü n z l e r, H. — *Integration of almost periodic functions*, Math. Zeitsch. 102 (1967) 253—287

10. H o r v a t h, J. — *Topological vector spaces and distributions*, Vol. I, Addison-Wesley Pub. Comp., 1966

11. L e v i t a n, B.M. — *Integration of almost periodic functions with values in Banach spaces*, Izv. Acad. Nayk U.S.S.R., 30 (1966) 1 101—1 110

12. R u d i n, W. — *Fourier Analysis on groups*, Interscience Pub. New York, 1967

13. Z i k o v, B.B. and L e v i t a n, B.M. — *Favard's theory*, Uspechi Math. Nauk. V. 32 No. 2, (194) (1977), 123—171

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THE MEAN VALUE OF AN ENTIRE FUNCTION REPRESENTED BY DIRICHLET SERIES

BY

SATENDRA KUMAR VAISH

1. Consider the Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$, where $\lambda_{n+1} > \lambda_n$, $\lambda_1 \geqslant 0$, $\lim_{n\to\infty} \lambda_n = \infty$, $s = \sigma + it$ and

$$\limsup_{n\to\infty} \frac{(\log) n}{\lambda_n} = 0. \tag{1.1}$$

Let σ_c and σ_a be the abscissa of convergence and the abscissa of absolute convergence, respectively, of $f(s)$. Let $\sigma_c = \infty$, then σ_a will also be infinite, since according to a known result [1, p. 4] a Dirichlet series which satisfies (1.1) has its abscissa of convergence equal to its abscissa of absolute convergence and therefore $f(s)$ represents an entire function.

We define the mean value of $f(s)$ as:

$$I_\delta(\sigma) = \lim_{T\to\infty} \left\{ \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^\delta \, dt \right\}, \quad \delta > 0. \tag{1.2}$$

Since $\log I_\delta(\sigma)$ is an increasing convex function of σ [3], we are able to write $\log I_\delta(\sigma)$ in the following form:

$$\log I_\delta(\sigma) = \log I_\delta(\sigma_0) + \int_{\sigma_0}^{\sigma} V(x)dx, \quad \sigma > \sigma_0, \tag{1.3}$$

where $V(x)$ is an increasing function of x, continuous in adjacent intervals.

The R i t t-order ρ and lower order λ (in the sence of Ritt) for $f(s)$ are defined by [4, p. 78]:

$$\lim_{\sigma\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log \log M(\sigma)}{\sigma} = \begin{matrix}\rho \\ \lambda\end{matrix}, \quad 0 \leqslant \lambda \leqslant \rho \leqslant \infty, \tag{1.4}$$

where $M(\sigma) = \sup\{|f(\sigma + it)| : -\infty < t < \infty\}$.

In this paper we obtain a few relations with the help of (1.2).