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THE MEAN VALUE OF AN ENTIRE FUNCTION REPRESENTED BY DIRICHLET SERIES

BY

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1. Consider the Dirichlet series $f(s)=\sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$, where $\lambda_{n+1}>\lambda_n$, $\lambda_1 \geqslant 0$, $\lim_{n\to\infty} \lambda_n=\infty$, $s=\sigma+it$ and

$$\lim_{n\to\infty}\sup \frac{(\log)n}{\lambda_n}=0. \tag{1.1}$$

Let σ_c and σ_a be the abscissa of convergence and the abscissa of absolute convergence, respectively, of $f(s)$. Let $\sigma_c=\infty$, then σ_a will also be infinite, since according to a known result [1, p. 4] a Dirichlet series which satisfies (1.1) has its abscissa of convergence equal to its abscissa of absolute convergence and therefore $f(s)$ represents an entire function.

We define the mean value of $f(s)$ as:

$$I_\delta(\sigma)=\lim_{T\to\infty}\left\{\frac{1}{2T}\int_{-T}^{T}|f(\sigma+it)|^{\delta}\,dt\right\},\quad \delta>0. \tag{1.2}$$

Since $\log I_\delta(\sigma)$ is an increasing convex function of σ [3], we are able to write $\log I_\delta(\sigma)$ in the following form:

$$\log I_\delta(\sigma)=\log I_\delta(\sigma_0)+\int_{\sigma_0}^{\sigma} V(x)\,dx,\quad \sigma>\sigma_0. \tag{1.3}$$

where $V(x)$ is an increasing function of x, continuous in adjacent intervals.

The R i t t-order ρ and lower order λ (in the sence of Ritt) for $f(s)$ are defined by [4, p. 78]:

$$\lim_{\sigma\to\infty}\sup_{\inf}\frac{\log\log M(\sigma)}{\sigma}=\begin{matrix}\rho\\ \lambda\end{matrix},\quad 0\leqslant\lambda\leqslant\rho\leqslant\infty, \tag{1.4}$$

where $M(\sigma)=\sup\{|f(\sigma+it)|:-\infty<t<\infty\}$.

In this paper we obtain a few relations with the help of (1.2).

2. If $f(s)$ is an entire function of non-zero finite Ritt-order ρ, type T and lower type t, $0\leqslant t\leqslant T\leqslant\infty$, then for $\delta\geqslant 1$, we have [2, p. 15]:

$$\lim_{\sigma\to\infty}\ \sup_{\inf}\ \frac{\log I_\delta(\sigma)}{e^{\rho\sigma}}=\frac{\delta T}{\delta t}. \tag{2.1}$$

Let us set

$$\lim_{\sigma\to\infty}\ \sup_{\inf}\frac{V(\sigma)}{e^{\rho\sigma}}=\frac{\alpha}{\beta},\quad 0\beta\leqslant\alpha\leqslant\infty\leqslant. \tag{2.2}$$

We prove the following results:

Theorem 1. *If $f(s)$ is an entire function of non-zero finite Ritt-order ρ, with and T, t; α, β the constants defined as in* (2.1) *and* (2.2), *then*

$$\beta\leqslant\delta\rho\leqslant\delta\rho T\leqslant\alpha, \tag{2.3}$$

$$\delta\rho T\geqslant(\alpha/e)^{\beta/\alpha}\geqslant\beta, \tag{2.4}$$

$$\delta\rho t\leqslant\beta\{1+\log(\alpha/\beta)\}\leqslant\alpha, \tag{2.5}$$

and

$$\alpha+\delta\rho T\leqslant e\delta\rho T. \tag{2.6}$$

Proof. From (1.3), we have, for any $k\geqslant 0$,

$$\log I_\delta(\sigma+k/\rho)=0(1)+\left\{\int_{\sigma_0}^{\sigma}+\int_{\sigma}^{\sigma+\frac{k}{\rho}}\right\}V(x)dx,\quad \sigma>\sigma_0. \tag{2.7}$$

We first suppose that $0<\beta<\infty$. Then, from (2.2), for any $\varepsilon>0$, and σ sufficiently large, we get

$$V(\sigma)/e^{\rho\sigma}>\beta-\varepsilon.$$

Hence,

$$\frac{e^k\log I_\delta(\sigma+k/\rho)}{e^\rho(\sigma+k/\rho)}>O(e^{-\rho\sigma})+\frac{\beta-\varepsilon}{\rho}(1-o(1))+\frac{k}{\rho}\frac{V(\sigma)}{e^{\rho\sigma}}. \tag{2.8}$$

Taking inferior limits of both sides in (2.8), as $\sigma\to\infty$, we get

$$\delta te^k\geqslant\beta/\rho+(k/\rho)\beta.$$

For $k=0$, this gives

$$\delta\rho t\geqslant\beta, \tag{2.9}$$

which also holds when $\beta=0$. If β were infinite, the above argument, with an arbitrary large number instead of $(\beta-\varepsilon)$, gives $t=\infty$. Thus (2.9) holds when $0\leqslant\beta\leqslant\infty$. Further, from (2.2), we get, for any $\varepsilon>0$ and $\sigma>\sigma_0$,

$$V(\sigma)<(\alpha+\varepsilon)e^{\rho\sigma}. \tag{2.10}$$

This estimate for $V(\sigma)$ on account of (1.3) yields

$$\frac{\log I_\delta(\sigma)}{e^{\rho\sigma}}<O(e^{-\rho\sigma})+\frac{\alpha+\varepsilon}{\rho}(1-o(1)),$$

whence,

$$\delta\rho T\leqslant\alpha. \tag{2.11}$$

Combining (2.9) and (2.11), since $\delta\rho T\geqslant\delta\rho t$, we get (2.3).

In order to establish (2.4), we take superior limits, as $\sigma\to\infty$, of both sides in (2.8), and get

$$\delta Te^k\geqslant\beta/\rho+(k/\rho)\alpha,$$

which gives for $k=1-\beta/\alpha$,

$$\delta\rho T\geqslant\frac{\alpha}{e}e^{\beta/\alpha}\geqslant\beta; \tag{2.12}$$

the last inequality in (2.12) follows from the fact that $x\geqslant 0\Rightarrow e^x\geqslant ex$.

Further, from (2.7) and (2.10), we get

$$\log I_\delta(\sigma+k/\rho)<o(1)+\frac{\alpha+\varepsilon}{\rho}(e^{\rho\sigma}-e^{\rho\sigma}{}_\sigma)+k/\rho V(\sigma+k/\rho),$$

$$\text{or,}\quad \frac{e^k\log I_\delta(\sigma+k/\rho)}{e^{\rho(\sigma+k/\rho)}}<O(e^{-\rho\sigma})+\frac{\alpha+\varepsilon}{\rho}(1-o(1))+k/\rho\frac{V(\sigma+k/\rho)}{e^{\rho(\sigma+k/\rho)}}e^k.$$

Taking inferior limits, as $\sigma\to\infty$, of both sides, we get

$$\delta te^k\leqslant\frac{\alpha}{\rho}+k/\rho\beta e^k,$$

which, on taking $k=\log(\alpha/\beta)$, gives,

$$\delta\rho t\leqslant\beta\{1+\log(\alpha/\beta)\}\leqslant\alpha,$$

since $1+\log(\alpha/\beta\leqslant\exp\{\log(\alpha/\beta)\}$. This proves (2.5).

To prove (2.6), we note that

$$V(\sigma)<\rho\int_{\sigma}^{\sigma+\frac{1}{\rho}}V(x)dx,$$

since $V(\sigma)$ is a positive increasing function of σ. Adding $\rho\log I_\delta(\sigma)$ on both sides of the above relation and using (1.3), we get

$$\rho\log I_\delta(\sigma)+V(\sigma)\leqslant\rho\log I_\delta(\sigma+1/\rho).$$

Dividing throughout by $e^{\rho\sigma}$ and taking the superior limit we get (2.6). This completes the proof of theorem 1.

Remark. Since the function $e^x - ex$ attains its minimum for $x=1$, in the relation (2.4) actually $\beta < (\alpha/e)e^{\beta/\alpha}$ if $\alpha \neq \beta$, and in relation (2.5), $\beta\{1+\log(\alpha/\beta)\} < \alpha$ if $\alpha \neq \beta$. Thus the equality in the relations (2.4) and (2.5) will occur only if $\alpha = \beta$. Moreover, from (2.4),

$$(\alpha/e)e^{\beta/\alpha} \leqslant \delta\rho T,$$

or,

$$\alpha \leqslant e\delta\rho T. \tag{2.13}$$

A comparison of (2.6) and (2.13) shows that (2.6) is a refinement of (2.13).

Theorem 2. *If $f(s)$ is an entire function of non-zero finite Ritt-order ρ, then $\lim\limits_{\sigma\to\infty} V(\sigma)/e^{\rho\sigma}$ exists, if and only if, $\lim\limits_{\sigma\to\infty} \log I_\delta(\sigma)/e^{\rho\sigma}$ exists, in which case,*

$$\lim_{\sigma\to\infty} \frac{V(\sigma)}{e^{\rho\sigma}} = \rho\delta \lim_{\sigma\to\infty} \frac{\log I_\delta(\sigma)}{e^{\rho\sigma}}. \tag{2.14}$$

Proof. If $\lim\limits_{\sigma\to\infty} V(\sigma)/e^{\rho\sigma}$ exists, then $\lim\limits_{\sigma\to\infty} \log I_\delta(\sigma)/e^{\rho\sigma}$ exists on account of (2.3). We next suppose that

$$\lim_{\sigma\to\infty} \log I_\delta(\sigma)/e^{\rho\sigma} = \delta T. \tag{2.15}$$

First let $0 < T < \infty$; then for any $\varepsilon < 0$ and $\sigma > \sigma_0$,

$$\delta(T-\varepsilon)e^{\rho\sigma} < \log I_\delta(\sigma) < \delta(T+\varepsilon)e^{\rho\sigma}. \tag{2.16}$$

Hence, for any $0 < \Delta < 1$, we have

$$\int_\sigma^{\sigma+\Delta} V(x)dx = \int_{\sigma_0}^{\sigma+\Delta} V(x)dx - \int_{\sigma_0}^{\sigma} V(x)dx = \log I_\delta(\sigma+\Delta) - \log I_\delta(\sigma) < \delta(T+\varepsilon)e^{\rho(\sigma+\Delta)} -$$

$$-\delta(T-\varepsilon)e^{\rho\sigma} = \delta T(e^{\Delta\rho}-1)e^{\rho\sigma} + \delta\varepsilon(e^{\Delta\rho}+1)e^{\rho\sigma} = \delta T(\Delta\rho+\ldots)e^{\rho\sigma} +$$

$$\delta\varepsilon(2+\Delta\rho+\ldots)e^{\rho\sigma}.$$

Therefore, since $V(\sigma)$ is an increasing function of σ,

$$\Delta V(\sigma) \int_\sigma^{\sigma+\Delta} V(x)dx < \delta T(\Delta\rho+\ldots)e^{\rho\sigma} + \delta\varepsilon(2+\Delta\rho+\ldots)e^{\rho\sigma}.$$

Since ε and Δ are arbitrary, it follows that

$$\limsup_{\sigma\to\infty} V(\sigma)/e^{\rho\sigma} \leqslant \delta\rho T. \tag{2.17}$$

Proceeding in a similar way, we easily get

$$\int_{\sigma-\Delta}^{\sigma} V(x)dx > \delta T(\Delta\rho - \ldots)e^{\rho\sigma} - \delta\varepsilon(2-\Delta\rho+\ldots)e^{\rho\sigma}$$

for any $\varepsilon > 0$, $\sigma > \sigma_0$ and $0 < \Delta < 1$. Since $V(\sigma)$ is an increasing function of σ,

$$\Delta V(\sigma) \geqslant \int_{\sigma-\Delta}^{\sigma} V(x)dx > \delta T(\Delta\rho - \ldots)e^{\rho\sigma} - \delta\varepsilon(2-\Delta\rho+\ldots)e^{\rho\sigma},$$

whence, since ε and Δ are arbitrary,

$$\liminf_{\sigma\to\infty} [V(\sigma)/e^{\rho\sigma}] \geqslant \delta\rho T. \tag{2.18}$$

Thus for $0 < T < \infty$, the relations (2.17) and (2.18) gives

$$\lim_{\sigma\to\infty} V(\sigma)/e^{\rho\sigma}) = \delta\rho T. \tag{2.19}$$

If $T=0$, then (2.17) given

$$\lim_{\sigma\to\infty} [V(\sigma)/e^{\rho\sigma}] = 0,$$

and if $T=\infty$, then taking an arbitrary large number M instead $(T-\varepsilon)$ and proceeding as above, we get

$$\lim_{\sigma\to\infty} [V(\sigma)/e^{\rho\sigma}] = \infty.$$

Thus also in this case $\lim\limits_{\sigma\to\infty} [V(\sigma)/e^{\rho\sigma}]$ exists.

The equality (2.14) follows from (2.15) and (2.19).

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