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ON SOME CLASSES OF VOLTERRA DIFFERENTIAL-INTEGRAL EQUATIONS

BY

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1. **Introduction.** By (PC) we shall denote the problem of finding the solution of the functional differential-integral equation

$$x'(t) = f(t, x(h(t)), \int_0^t k(s, x(g(s))ds),$$

satisfying the initial condition

$$x(0) = \eta,$$

where η is real and f, k, h, g are given functions.

In this note we are interested in the study of the problem (PC) on the half-axis. Many papers related to the problem (PC) have been published. The complete bibliography is given by Vainberg [13] and Myškis [7]. See also [1]–[5], [8], [10] and [12]. The method used here is based on the concept of fixed-point theorems in a Fréchet space, and the concept of $\mathcal{L}^*$-space (see e.g. [6]).

2. **Fixed point theorems.** Suppose that E is a Fréchet space [14] with a nonempty subset X and a nonempty convex closed subset K. Let T be an one-to-one transformation of E into itself for which $T[X]$ is a closed set. Moreover, denote by $\mathbf{P} = \{p_n : n = 1, 2, \ldots\}$ a saturated family of semi-norms which generates the topology of E.

Under these hypotheses we have the following basic fixed-point results (see [9], [11]):

Theorem I. *Let $\{F_n\}$ $(n = 1, 2, \ldots)$ and F_0 be mapping from X into E satisfying the following conditions:* (1) $F_n[X] \subset T[X]$ *for* $n \geq 1$, (2) $\lim\limits_{n\to\infty} F_n x = F_0 x$ *for x in X, and* (3) $p(F_n x - F_n y) \leqslant k, p(Tx - Ty)$ *for $n \geq 1$, p in* $\mathbf{P}$ *and x, y in X, where $k < 1$. Then there exists a unique point $x_m (m = 0, 1, \ldots)$ in X such that $F_m x_m = T x_m$ and $T x_n \to T x_0$ as $n \to \infty$.*

Theorem II. *Let F be a mapping from $X \times K$ into E, and let Q be a continuous mapping from K into a compact subset of E. Assume, moreover, that:* (1) $F[X \times K] \subset T[X] \subset K$, (2) $p(F(x, y_1) - F(x, y_2)) \leqslant p(Qy_1 - Qy_2)$ *for p in* $\mathbf{P}$ *x in X, and y_1, y_2 in K, and* (3) $p(F(x_1, y) - F(x_2, y)) \leqslant k, p(Tx_1 - Tx_2)$ *for p in* $\mathbf{P}$, *x_1 and x_2 in X, y in K, and with $k < 1$. Then there exists x in X such that $F(x, Tx) = Tx$.*

3. Assumptions and notations. Throughout this paper, $J=[0,\infty)$, $\mathbf{R}$ is the Euclidean space, and all functions are real. $C(J)$ denotes the set of all continuous functions defined on J.

Let us put : $p_n(x)=\sup\{|x(t)|:0\leqslant t\leqslant n\}$ $(n=1,2,\ldots)$ for x in $C(J)$. The set $C(J)$ will be considered as vector space with the topology of almost uniform convergence (i.e., uniform convergence on compact subset of J). This topology is determined by the above sequence (p_n) of seminorms, and therefore $C(J)$ is a Fréchet space.

Let M and N be locally integrable functions of J into itself. We use the following notations :

$\mathcal{F}_1$ — the set of continuous functions f defined on $J\times\mathbf{R}\times\mathbf{R}$ such that $|f(t,x_1,y_1)-f(t,x_2,y_2)|\leqslant M(t)[|x_1-x_2|+|y_1-y_2|]$ for $t\geqslant 0$ and x_1, x_2, y_1, y_2 in $\mathbf{R}$;

$\mathcal{K}_1$ — the set of continuous functions k defined on $J\times\mathbf{R}$ such that $|k(t,x)-k(t,y)|\leqslant N(t)|x-y|$ for $t\geqslant 0$ and x, y in $\mathbf{R}$;

$\mathcal{H}_1$ — the set of continuous functions h of J into itself such that $h(t)\leqslant\leqslant t$ for $t\geqslant 0$;

$\mathcal{F}_2$ (resp. $\mathcal{K}_2$) — the subset in $\mathcal{F}_1$ (resp. $\mathcal{K}_1$) consisting of uniformly bounded functions.

Moreover, we shall be using the notion of $\mathcal{L}^*$-spaces, $\mathcal{L}^*$-product of $\mathcal{L}^*$-spaces and continuous mapping of one $\mathcal{L}^*$-space into another, see e.g. [6].

4. Main results. In the sequel we shall deal with the sets $\mathcal{F}_1$, $\mathcal{K}_1$, $\mathcal{H}_1$ as $\mathcal{L}^*$-spaces endowed with the almost uniform convergence on $J\times\mathbf{R}\times\mathbf{R}$, $J\times\mathbf{R}$ and J, respectively. The sets $\mathcal{F}_2$, $\mathcal{K}_2$ are endowed with the $\mathcal{L}^*$-space structures by the pointwise convergence on $J\times\mathbf{R}\times\mathbf{R}$ and $J\times\mathbf{R}$, respectively. Furthermore, denote by $\mathcal{H}_2$ the set $\mathcal{H}_1$ considered as $\mathcal{L}^*$-space with the pointwise convergence.

Theorem 1. *Let Q be a nonempty subset of* R *and let $\mathcal{X}$ be a nonempty closed subset of $C(J)$ such that for $f\in\mathcal{F}_1$, $k\in\mathcal{K}_1$, h and g in $\mathcal{H}_1$, $r_i\in Q$ and $x\in\mathcal{X}$ the functions*

$$t\to r_i+\int_0^t f\Big(s,x(h(s)),\int_0^s k(\sigma,x(\sigma))d\sigma\Big)ds \ \textit{ are in } \mathcal{X}.$$

Then for an arbitrary $\varphi=(f,k,h,g,r_i)$ in $\mathcal{F}_1\times\mathcal{K}_1\times\mathcal{H}_1\times\mathcal{H}_1\times Q$ there exists a unique function x_φ in $\mathcal{X}$ satisfying the problem (PC) on J. Moreover, the transformation $\varphi\to x_\varphi$ maps continuously the $\mathcal{L}^*$-product $\mathcal{F}_1\times\mathcal{K}_i\times\mathcal{H}_i\times\times\mathcal{H}\times Q(i=1,2)$ into $C(J)$.

Proof. Let us put $L(t)=M(t)+N(t)$ for $t\geqslant 0$. Let $\varphi=(f,k,h,g,r_i)\in\in\mathcal{F}_1\times\mathcal{K}_1\times\mathcal{H}_1\times\mathcal{H}_1\times Q$. We define :

$$(Tx)(t)=\exp\Big(-r\int_0^t L(s)ds\Big).x(t),$$

$$(Fx)(t)=\exp\Big(-r\int_0^t L(s)ds\Big)\cdot\Big(r_i+\int_0^t f\Big(s,x(h(s)),\int_0^s k(\sigma,x(g(\sigma)))d\sigma\Big)ds\Big).$$

for x in $\mathcal{X}$, where $r>2$ is a constant. Then, T is one-to-one continuous mapping with $F[\mathcal{X}]\subset T[\mathcal{X}]$ and $T[\mathcal{X}]$ is a closed subset of $C(J)$. For a positive integer n, and $0\leqslant t\leqslant n$, and u, v in $\mathcal{H}$, we have

$$\int_0^t\Big|f\Big((s,u(h(s)),\int_0^s k(\sigma,u(g(\sigma)))d\sigma\Big)-f\Big(s,v(h(s)),\int_0^s k(\sigma,v(g(\sigma)))d\sigma\Big)\Big|\,ds\leqslant$$

$$\leqslant\int_0^t M(s)\Big(\Big|(|u(h(s))-v(h(s))\Big|+\int_0^t N(\sigma)\Big|u(g(\sigma))-v(g(\sigma))\Big|d\sigma\Big)ds=$$

$$=\int_0^t M(s)\,|(Tu)(h(s))-(Tv)(h(s))|\,.\exp\Big(r\int_0^{h(s)}L(\tau)d\tau\Big)+\int_0^s N(\sigma)|\,(Tu)(g(\sigma))-$$

$$-(Tv)(g(\sigma))|\,.\exp\Big(r\int_0^{g(\sigma)}L(\tau)d\tau\Big)d\sigma\Big)ds\leqslant p_n(Tu-Tv)\,.\int_0^t\Big(L(s).\exp\Big(r\int_0^s L(\tau)d\tau\Big)+$$

$$+M(s)\int_0^s L(\sigma)\exp\Big(r\int_0^\sigma L(\tau)d\tau\Big)d\sigma\Big)ds<2p_n(Tu-Tv)\,.\int_0^t L(s)\exp\Big(r\int_0^t L(\tau)d\tau\Big)ds<$$

$$<2.r^{-1}.p_n(Tu-Tv)\exp\Big(r\int_0^t L(s)ds\Big),$$

and its follows that $p_n(Fu-Fv)\leqslant 2r^{-1}p_n(Tu-Tv)$. Therefore, by Theorem I, we conclude that there exists a unique x_φ in $\mathcal{X}$ such that

$$x_\varphi(t)=r_i+\int_0^t f\Big(s,x_\varphi(h(s)),\int_0^s k(\sigma,x_\varphi(g(\sigma)))d\sigma\Big)ds$$

for $t\geqslant 0$.

Let $\varphi_m=(f_m,k_m,h_m,g_m,r_{im})\in\mathcal{F}_i\times\mathcal{K}_i\times\mathcal{H}_i\times\mathcal{H}_i\times Q$ $(i=1,2)$ for $m=0,1,\ldots$ Denote by F_m the mapping F whenever $\varphi=\varphi_m$.

Let us fix x in $\mathcal{X}$ and t in J. We have :

$$|T^{-1}(F_n x)(t)-T^{-1}(F_0 x)(t)|\leqslant|r_{in}-r_{i0}|+\int_0^t M(s)\,|x(h_n(s))-x(h_0(s))|\,ds+$$

$$+\int_0^t M(s)\Big(\int_0^s N(\sigma)\,|x(g_n(\sigma))-x(g_0(\sigma))|\,d\sigma\Big)ds+\int_0^t M(s)\Big|\int_0^s(k_n(\sigma,x(g_0(\sigma)))-$$

$$-k_0(\sigma, x(g_0(\sigma)))\Big)d\sigma\Big|ds + \Big|\int_0^t\Big(f_n\Big((s, x(h_0(s)), \int_0^s k_0(\sigma, x(g_0(\sigma)))\,d\sigma\Big) -$$

$$-f_0\Big((s, x(h_0(s)), \int_0^s k_0(\sigma, x(g_0(\sigma)))d\sigma\Big)\Big)ds\Big| \leqslant |\eta_n - \eta_0| + \sup_{0\leqslant s\leqslant t}|x(h_n(s)) -$$

$$-x(h_0(s))\Big|\cdot\int_0^t M(\tau)d\tau + \sup_{0\leqslant s\leqslant t}|x(g_n(s)) - x(g_0(s))|\cdot\int_0^t N(\tau)d\tau\cdot\int_0^t M(\tau)d\tau +$$

$$+t\cdot\sup_{0\leqslant s\leqslant t}|k_n(s, x(g_0(s))) - k_0(s, x(g_0(s)))|\cdot\int_0^s M(\tau)d\tau +$$

$$+t\cdot\sup_{0\leqslant s\leqslant t}\Big|f_n\Big((s, x(h_0(s)), \int_0^s k_0(\sigma, x(g_0(\sigma)))d\sigma\Big) - f_0(s, x(h_0(s)), \int_0^s k_0(\sigma, x(g_0(\sigma)))\,d\sigma\Big)\Big|$$

for $n\geqslant 1$. Assume that $\lim\limits_{n\to\infty}(f_n, k_n, h_n, g_n, \eta_n) = (f_0, k_0, h_0, g_0, \eta_0)$. Now, for $i = 1$, $\lim\limits_{n\to\infty} F_n x = F_0 x$ in $C(J)$. If $i = 2$, then using the Lebesgue bounded convergence theorem we obtain that $|T^{-1}(F_n x)(t) - T^{-1}(F_0 x)(t)|\to 0$ as $n\to\infty$. This implies, by the equicontinuity of sequence $(T^{-1}(F_n x))$ on each compact subset of J, that $(F_n x)$ converges almost uniformly on J to $F_0 x$. Finally, applying Theorem 1, we obtain that $\lim\limits_{n\to\infty} x_{\eta_n} = x_{\eta_0}$ in $C(J)$. This ends our proof.

Corollary. *Denote by* x_0 *the unique solution of the problem* (*PC*) *with* $f\in\mathcal{F}_1$, $k\in\mathcal{K}_1$, $h\in\mathcal{H}_1$, $g\in\mathcal{H}_1$ *and* $\eta\in\mathbf{R}$. *Let* $(x, y)\to(f(t, x, y)$ *and* $z\to k(t, z)$ *be monotonic nondecreasing for each* t *in* J. *Assume, moreover, that there exists functions* u_1, u_2 *such that*

$$u_1(t) \leqslant \eta + \int_0^t f(s, u_1(h(s)), \int_0^s k(\sigma, u_1(g(\sigma)))d\sigma)ds,$$

$$u_2(t) \geqslant \eta + \int_0^t f(s, u_2(h(s)), \int_0^s k(\sigma, u_2(g(\sigma)))d\sigma)ds$$

for $t\geqslant 0$. *Then* $u_1(t)\leqslant x_0(t)\leqslant u_2(t)$ *on* J.

Proof. Consider sets $\mathcal{X}_1$, $\mathcal{X}_2$ of all x in $C(J)$ such that $u_1(t)\leqslant x(t)$ and $u_2(t)\geqslant x(t)$ on J, respectively. Define the mapping W by

$$(Wx)(t) = \eta + \int_0^t f(s, x(h(s)), \int_0^s k(\sigma, x(g(\sigma)))d\sigma)ds$$

for x in $C(J)$. It can be easily seen that W maps $\mathcal{X}_i$ $(i = 1, 2)$ into itself. Since

each $\mathcal{X}_i$ is nonempty closed subset of $C(J)$, so by Theorem 1, W has a unique fixed point x_i in $\mathcal{X}_i$. This implies $u_1(t)\leqslant x_1(t) = x_0(t) = x_2(t)\leqslant u_2(t)$ on J, and completes the proof.

Now we begin to study the existence of solutions of the problem (*PC*) with $f = f_1 + f_2$ where $f_1\in\mathcal{F}_1$ and f_2 is a continuous function.

Theorem 2. *Let* $f_1\in\mathcal{F}_1$, $k_1\in\mathcal{K}_1$, $h_1\in\mathcal{H}_1$, $g_1\in\mathcal{H}_1$, *let* h_2, g_2 *be continuous functions of* J *into itself, and let* f_2, k_2 *be continuous functions defined on* $J\times\times\mathbf{R}\times\mathbf{R}$ *and* $J\times\mathbf{R}$, *respectively. Suppose that*

$$|f_1(t, x, y) + f_2(t, u, v)| \leqslant H(t, |x|, |y|, |u|, |v|) \text{ and}$$
$$|k_i(t, z)| \leqslant \alpha_i(t, |z|) \quad (i = 1, 2)$$

for $t\geqslant 0$ *and* x, y, u, v, z *in* $\mathbf{R}$. *Assume, moreover, that:*
(1) $(x, y, u, v)\to H(t, x, y, u, v)$ *is monotonic nondecreasing for each* $t\geqslant 0$ (i.e. *for* $t\geqslant 0$, $0\leqslant x_1\leqslant x_2$, $0\leqslant y_1\leqslant y_2$, $0\leqslant u_1\leqslant u_2$ *and* $0\leqslant v_1\leqslant v_2$ *implies* $H(t, x_1, y_1, u_1, v_1)\leqslant H(t, x_2, y_2, u_2, v_2)$).
(2) $x\to\alpha_i(t, x)$ $(i = 1, 2)$ *are monotonic nondecreasing for each* $t\geqslant 0$, *and* (3) *for each* $\zeta_0\geqslant 0$ *there exists a locally bounded function* ψ_0 *of* J *into itself such that* $t\to\int_0^t \alpha_2(s, \psi_0(g_2(s)))ds$ *is a locally bounded function on* J *and*

$$\psi_0(t)\geqslant\zeta_0 + \int_0^t H(s, \psi_0(h_1(s)), \int_0^s \alpha_1(\sigma, \psi_0(g_1(\sigma)))d\sigma, \psi_0(h_2(s)), \int_0^s \alpha_2(\sigma, \psi_0(g_2(\sigma)))d\sigma)ds$$

for $t\geqslant 0$.

Under the above hypotheses, for every η *in* $\mathbf{R}$ *with* $|\eta|\leqslant\zeta_0$, *there exists a function* x *defined on* J *such that* $x(0) = \eta$,

$$x'(t) = f_1\Big((t, x(h_1(t)), \int_0^t k_1(s, x(g_1(s)))ds\Big) + f_2\Big(t, x(h_2(t)), \int_0^t k_2(s, x(g_2(s)))ds\Big)$$

and $|x(t)|\leqslant\psi_0(t)$ *for all* $t\geqslant 0$.

Proof. Let $|\eta|\leqslant\zeta_0$. We write $L(t) = M(t) + N(t)$ on J. Define mappings T, Q and F as follows:

$$(Tx)(t) = \exp\Big(-r\int_0^t L(s)ds\Big)\cdot x(t),$$

$$(Qy)(t) = \int_0^t f_2\Big(s, \exp\Big(r\cdot\int_0^{h_2(s)} L(\tau)d\tau\Big)y(h_2(s))\Big), \int_0^s k_2\Big(\sigma, \exp\Big(r\cdot\int_0^{g_2(\sigma)} L(\tau)d\tau\Big)y(g_2(\sigma))\Big)d\sigma\Big)ds,$$

$$F(x, y)(t) = \exp\Big(-r\int_0^t L(s)ds\Big)\cdot\Big(\eta +$$

$$+\int_0^t f_1\Big(s,\, x(h_1(s)),\, \int_0^s k_1(\sigma,\, x(g_1(\sigma)))\, d\sigma\Big) ds + (Qy)(t)\Big)$$

for x and y in $C(J)$, where $r > 2$ is a constant. Moreover, let us put :

$$X = \{x \in C(J) : |x(t)| \leqslant \psi_0(t) \text{ for } t \geqslant 0\},$$
$$K = \{x \in C(J) : |x(t)| \leqslant (T\psi_0)(t) \text{ for } t \geqslant 0\}.$$

Obviously, $T[X] \subset K$, K is convex, and in the topology of $C(J)$, the sets K, $T[X]$ are closed and the operator Q is continuous. By Ascoli-Arzelà Theorem, $Q[K]$ is conditionally compact in $C(J)$. Arguing as in the proof of Theorem 1, we obtain that $p_n(F(u, y) - F(v, y)) \leqslant 2r^{-1} \cdot p_n(Tu - Tv)$ $(n = 1, 2, \ldots)$ for u, v in $C(J)$. If $x \in X$ and $y \in K$, then

$$\Big|\eta + \int_0^t f_1\Big(s,\, x(h_1(s)),\, \int_0^s k_1(\sigma,\, x(g_1(\sigma)))\, d\sigma\Big) ds + (Qy)(t)\Big| \leqslant$$

$$\leqslant |\eta| + \int_0^t H\Big(s,\, |x(h_1(s))|,\, \Big|\int_0^s k_1(\sigma,\, x(g_1(\sigma)))\, d\sigma\Big|,\, \exp\Big(r \cdot \int_0^{h_2(s)} L(\tau) d\tau\Big) |y(h_2(s))|,$$

$$\Big|\int_0^s k_2\Big(\sigma,\, \exp\Big(r \cdot \int_0^{g_2(\sigma)} L(\tau) d\tau\Big) y(g_2(\sigma))\Big) d\sigma\Big|\Big) ds \leqslant \zeta_0 +$$

$$+\int_0^t H\Big(s,\, \psi_0(h_1(s)),\, \int_0^s \alpha_1(\sigma,\, \psi_0(g_1(\sigma))) d\sigma,\ \psi_0(h_2(s)),\, \int_0^s \alpha_2(\sigma,\, \psi_0(g_2(\sigma)))\, d\sigma\Big) ds \leqslant \psi_0(t)$$

for $t \geqslant 0$. This implies $F[X \times K] \subset T[X]$, and consequently all assumptions of Theorem II are satisfied. Hence there exists x in X such that $F(x, Tx)(t) = (Tx)(t)$ on J, and we are done.

5. An inequality of Gronwall type. We conclude this note by remarking that our Theorem 1 enables us to get a variant of the well-known Gronwall inequality. More precisely, we state the following result :

Proposition. *Let $c \geqslant 0$, $L \geqslant 0$, $h \in \mathscr{H}_1$, $g \in \mathscr{H}_1$ and let G, H be continuous functions of J into itself. Suppose that u_0 is a function such that*

$$u_0(t) \leqslant c + L\int_0^t \Big(H(s) u_0(h(s)) + \int_0^s G(\sigma) u_0(g(\sigma)) d\sigma\Big) ds$$

for $t \geqslant 0$. Then

$$u_0(t) \leqslant c \cdot \exp\Big(L \int_0^t \Big((H(s) + \int_0^s G(\sigma) d\sigma\Big) ds\Big)$$

or all t in J.

Proof. Let us put :

$f(t, x, y) = L(H(t)x + y)$, $\quad k(t, z) = G(t)z$, and

$$u_1(t) \equiv u_0(t), \quad u_2(t) = c \cdot \exp\Big(L \int_0^t \Big(H(s) + \int_0^s G(\sigma) d\sigma\Big) ds\Big)$$

for t in J and x, y, z in $\mathbf{R}$. We have

$$c + \int_0^t f\Big(s,\, u_2(h(s)),\, \int_0^s k(\sigma,\, u_2(g(\sigma)))\, d\sigma\Big) ds \leqslant c + \int_0^t L\Big((H(s) u_2(s) + u_2(s)\int_0^s G(\sigma) d\sigma\Big) ds =$$

$$= c + c\int_0^t L\Big(H(s) + \int_0^s G(\sigma) d\sigma\Big) \cdot \exp\Big(L \int_0^s \Big(H(\tau) + \int_0^\tau G(\sigma) d\sigma\Big) d\tau\Big) ds =$$

$$= c \cdot \exp\Big(L \int_0^t \Big(H(s) + \int_0^s G(\sigma) d\sigma\Big) ds\Big) = u_2(t)$$

for $t \geqslant 0$. Consequently, the Corollary given in Sec. 4 is applicable to the above mappings f, k, u_1, u_2 and $\eta = c$. Hence $u_1(t) \leqslant u_2(t)$ on J, and the proof is completed.

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AN INTEGRAL INEQUALITY IN TWO INDEPENDENT VARIABLES

BY

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1. Introduction. There exists an extensive literature concerning various generalizations of the Gronwall inequality, in the case of two or more independent variables (see, for example, the recent works [1]—[6]). In [4] we proved that if the continuous function u satisfies the inequality

$$u(x, y) \leqslant f(x, y) + \int_0^x\int_0^y b(s, t)u(s, t)ds\, dt, \quad x, y \geqslant 0, \tag{1}$$

where f is continuous and monotone nondecreasing with respect to each variable, b is continuous and nonnegative, then it follows for $x, y \geqslant 0$

$$u(x, y) \leqslant f(x, y)\left[1 + \int_0^x\int_0^y b(s, t)\exp\left(\int_s^x\int_t^y b(\zeta, \eta)d\zeta d\eta\right) ds\, dt\right]. \tag{2}$$

This result was obtained using the notion of resolvent kernel of the theory of Volterra linear integral equations. A more general inequality that (1) is the following

$$u(x, y) \leqslant f(x, y) + a(x, y)\int_0^x\int_0^y b(s, t)u(s, t)ds\, dt, \quad x, y \geqslant 0, \tag{3}$$

where u, f, a, b are continuous and a, b are nonnegative. To obtain from (3) a bound for the function u, we will follow the same way as in the case of one independent variable. It is well known [3], that if a continuous function $u = u(x)$ satisfies

$$u(x) \leqslant f(x) + \int_0^x k(x, s)u(s)ds, \quad x \geqslant 0, \tag{4}$$

where f and k are continuous, $k \geqslant 0$, then it follows

$$u(x) \leqslant f(x) + \int_0^x r(x, s)f(s)ds, \quad x \geqslant 0, \tag{5}$$