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AN INTEGRAL INEQUALITY IN TWO INDEPENDENT VARIABLES

BY

ADRIAN CORDUNEANU

1. Introduction. There exists an extensive literature concerning various generalizations of the Gronwall inequality, in the case of two or more independent variables (see, for example, the recent works [1]–[6]). In [4] we proved that if the continuous function u satisfies the inequality

$$u(x, y) \leq f(x, y) + \int_0^x\int_0^y b(s, t)u(s, t)ds\, dt, \quad x, y \geq 0, \tag{1}$$

where f is continuous and monotone nondecreasing with respect to each variable, b is continuous and nonnegative, then it follows for $x, y \geq 0$

$$u(x, y) \leq f(x, y)\left[1 + \int_0^x\int_0^y b(s, t)\exp\left(\int_s^x\int_t^y b(\zeta, \eta)d\zeta d\eta\right) ds\, dt\right]. \tag{2}$$

This result was obtained using the notion of resolvent kernel of the theory of Volterra linear integral equations. A more general inequality that (1) is the following

$$u(x, y) \leq f(x, y) + a(x, y)\int_0^x\int_0^y b(s, t)u(s, t)ds\, dt, \quad x, y \geq 0, \tag{3}$$

where u, f, a, b are continuous and a, b are nonnegative. To obtain from (3) a bound for the function u, we will follow the same way as in the case of one independent variable. It is well known [3], that if a continuous function $u = u(x)$ satisfies

$$u(x) \leq f(x) + \int_0^x k(x, s)u(s)ds, \quad x \geq 0, \tag{4}$$

where f and k are continuous, $k \geq 0$, then it follows

$$u(x) \leq f(x) + \int^x r(x, s)f(s)ds, \quad x \geq 0, \tag{5}$$

where $r=r(x,s)$ is the resolvent kernel associated with $k=k(x,s)$. Now, if we take $k(x,s)=a(x)b(s)$, we deduce

$$r(x,s)=a(x)b(s)\exp\left(\int_s^x a(\tau)b(\tau)d\tau\right),\quad x\geqslant s\geqslant 0, \tag{6}$$

and consequently, from the inequality

$$u(x)\leqslant f(x)+a(x)\int_0^x b(s)u(s)ds,\quad x\geqslant 0, \tag{7}$$

where u, f, a, b are continuous and a, b are nonnegative, we obtain the bound

$$u(x)\leqslant f(x)+a(x)\int_0^x b(s)\exp\left(\int_s^x a(\tau)b(\tau)d\tau\right)f(s)ds,\quad x\geqslant 0. \tag{8}$$

This inequality is known [2] and, of course, may be also easily deduced using a linear differential equation of the first order. We will apply the same method in the case of two independent variables.

2. Main result. Proposition 1. *If the continuous function $u=u(x,y)$ satisfies the inequality* (3), *where*

(a) $f=f(x,y)$ *is continuous for* $x,y\geqslant 0$ *and monotone nondecreasing with respect to each variable.*

(b) $a=a(x,y)$ *and* $b=b(x,y)$ *are continuous and nonnegative for* $x,y\geqslant 0$. *then it follows for* $x,y\geqslant 0$ *that*

$$u(x,y)\leqslant f(x,y)\left[1+a(x,y)\iint_{00}^{xy} b(s,t)\exp\left(\iint_{st}^{xy} a(\xi,\eta)b(\xi,\eta)d\xi d\eta\right)ds\,dt\right]. \tag{9}$$

Proof. As we pointed out, from the inequality

$$u(x,y)\leqslant f(x,y)+\iint_{00}^{xy} k(x,y,s,t)u(s,t)ds\,dt,\quad x,y\geqslant 0, \tag{10}$$

in which k is continuous, $k\geqslant 0$, we deduce that

$$u(x,y)\leqslant f(x,y)+\iint_{00}^{xy} r(x,y,s,t)f(s,t)ds\,dt,\quad x,y\geqslant 0, \tag{11}$$

where

$$r(x,y,s,t)=\sum_{n=0}^{\infty} k_n(x,y,s,t), \tag{12}$$

the iterated kernels k_n being given by

$$k_0=k$$
$$k_n(x,y,s,t)=\iint_{st}^{xy} k_{n-1}(x,y,\xi,\eta)k(\xi,\eta,s,t)d\xi\,d\eta,\quad n\geqslant 1. \tag{13}$$

In (3) we have $k(x,y,s,t)=a(x,y)b(s,t)\geqslant 0$ and then, after a simple computation, it follows for every n = natural that

$$k_n(x,y,s,t)\leqslant (1/n!)a(x,y)b(s,t)\left(\iint_{st}^{xy} a(\xi,\eta)b(\xi,\eta)d\xi\,d\eta\right)^n \tag{14}$$

and consequently, in this case

$$r(x,y,s,t)\leqslant a(x,y)b(s,t)\exp\left(\iint_{st}^{xy} a(\xi,\eta)\,b(\xi,\eta)d\xi\,d\eta\right). \tag{15}$$

The condition (*a*) implies that $f(s,t)\leqslant f(x,y)$ for $0\leqslant s\leqslant x$, $0\leqslant t\leqslant y$ and then, from (11) and (15), we obtain the desired inequality (9).

Remark 1. If we assume that $a=a(x,y)$, $b=b(x,y)$ are continuous and nonnegative for $x,y\geqslant 0$ and $a=a(x,y)$ is monotone nondecreasing with respect to each variable, then for $x,y\geqslant 0$ we have

$$1+a(x,y)\iint_{00}^{xy} b(s,t)\exp\left(\iint_{st}^{xy} a(\xi,\eta)\,b(\xi,\eta)d\xi\,d\eta\right)ds\,dt\leqslant$$
$$\leqslant\exp\left(a(x,y)\iint_{00}^{xy} b(\xi,\eta)\,d\xi\,d\eta\right), \tag{16}$$

the inequality being strict for $x,y>0$, if $a(x,y)>0$ and $b(x,y)>0$. To prove (16), we take for fixed $x,y>0$

$$F(s,t)=\exp\left(a(x,y)\iint_{st}^{xy} b(\xi,\eta)d\xi\,d\eta\right), \tag{17}$$

and we remark that

$$F''_{st}(s,t)=a(x,y)b(s,t)\exp\left(a(x,y)\iint_{st}^{xy} b(\xi,\eta)d\xi\,d\eta\right)+\text{nonnegative term}. \tag{18}$$

Integrating (18) on the rectangle $0\leqslant s\leqslant x$, $0\leqslant t\leqslant y$ we get that (16) is true. From Proposition 1 and the above Remark 1, it follows

Corollary 1. *Assume that all the functions involved in* (3) *are continuous and that*

(i) f, a, b, *are nonnegative for* $x,y\geqslant 0$

(ii) f *and* a *are monotone nondecreasing with respect to each their variables*;

Then, we have for $x, y \geq 0$ the following inequality:

$$u(x, y) \leq f(x, y) \exp\left(a(x, y) \iint_{00}^{xy} b(s, t)\, ds\, dt\right). \tag{19}$$

Remark 2. Proposition 1 and Corollary 1 may be carried over to the n-dimensional case, i.e. we can state similar results when the considered functions depending of n independent variables $x_1, x_2, \ldots, x_n$.

Remark 3. Proposition 1 and Corollary 1 may be formulated in a more general version, using the notions of measurable function and Lebesgue integral. For example, the following assertion is true: if u, f, a, b, are L_2-functions on every finite rectangle $0 \leq x \leq A$, $0 \leq y \leq B$; a, b, f are nonnegative for $x, y \geq 0$; a and f are monotone nondecreasing with respect to each variable x and y, then from the inequality

$$u(x, y) \leq f(x, y) + a(x, y) \iint_{00}^{xy} b(s, t) u(s, t)\, ds\, dt \tag{20}$$

almost everywhere for $x, y \geq 0$, we deduce that

$$u(x, y) \leq f(x, y) \exp\left(a(x, y) \iint_{00}^{xy} b(s, t)\, ds\, dt\right) \tag{21}$$

almost everywhere for $x, y \geq 0$. The proof is analogous consisting in the possibility to use the method of the resolvent kernel described in the Introduction.

Remark 4. Obviously, the method used here in the case of the inequality (3) may be applied without modifications to the inequalities of the form

$$u(x, y) \leq f(x, y) + a(x, y) \iint_{x_0 y_0}^{xy} b(s, t) u(s, t)\, ds\, dt, \quad x \geq x_0,\ y \geq y_0, \tag{22}$$

where x_0 and y_0 are fixed.

3. Application. First, we consider the nonlinear Volterra equation

$$u(x, y) = f(x, y) + \iint_{00}^{xy} k(x, y, s, t) H(s, t, u(s, t))\, ds\, dt, \quad x, y \geq 0, \tag{23}$$

where $f = f(x, y)$ is continuous for $x, y \geq 0$, $k = k(x, y, s, t)$ is continuous for $x \geq s \geq 0$, $y \geq t \geq 0$ and $H = H(x, y, z)$ is continuous on the set

$$D = \{(x, y, z) ;\ x, y \geq 0,\ |z| < b \leq \infty\}. \tag{24}$$

Suppose that the kernel k satisfies

$$|k(x, y, s, t)| \leq a(x, y) b(s, t), \quad x \geq s \geq 0,\ y \geq t \geq 0, \tag{25}$$

where a, b are continuous nonnegative functions. Relatively to the function H, assume that $H(x, y, 0) = 0$ and that for every (x, y, z), $(x, y, \bar{z})$ from D, we have

$$|H(x, y, z) - H(x, y, \bar{z})| \leq h(x, y) |z - \bar{z}|, \tag{26}$$

where $h \geq 0$ is continuous for $x, y \geq 0$. Also assume that

$$\sup_{x, y \geq 0} a(x, y) \iint_{00}^{xy} b_1(s, t) \exp\left(\iint_{st}^{xy} a(\xi, \eta) b_1(\xi, \eta)\, d\xi\, d\eta\right) ds\, dt = K < \infty, \tag{27}$$

where $b_1(s, t) = b(s, t) h(s, t)$ for $s, t \geq 0$.

Denote by C the Banach space of all continuous bounded (for $x, y \geq 0$) functions $f = f(x, y)$, with the norm

$$\|f\| = \sup_{x, y \geq 0} |f(x, y)| < \infty. \tag{28}$$

We remark that if $b < \infty$, to be sure that the unique solution of the equation (23) is defined on the whole quadrant $x, y \geq 0$, we must suppose that $\|f\|$ is small enough. Then from Proposition 1, we can deduce

Corollary 2. *Assume that the conditions* (25) – (27) *are fulfilled. If $u_i = u_i(x, y)$ $(i = 1, 2)$ are two solutions of* (23) *corresponding to the free terms $f_i = f_i(x, y)$ $(i = 1, 2)$, then it follows that*

$$\|u_1 - u_2\| \leq (1 + K) \|f_1 - f_2\|. \tag{29}$$

Now, denote by C_g the Banach space consisting of all continuous (for $x, y \geq 0$) functions $f = f(x, y)$, endowed with the norm

$$\|f\|_g = \sup_{x, y \geq 0} (|f(x, y)| / g(x, y)), \tag{30}$$

where $g = g(x, y)$ is a given continuous positive function. Assume that the resolvent kernel $r = r(x, y, s, t)$ associated with $k = k(x, y, s, t)$ satisfies the inequality

$$|r(x, y, s, t)| \leq \alpha(x, y) \beta(s, t) (g(x, y) / g(s, t)) \tag{31}$$

for $x \geq s \geq 0$, $y \geq t \geq 0$, where $\alpha = \alpha(x, y)$ and $\beta = \beta(s, t)$ are continuous nonnegative functions, such that

$$\sup_{x, y \geq 0} \alpha(x, y) \iint_{00}^{xy} \beta(s, t)\, ds\, dt = A < \infty. \tag{32}$$

Finally, suppose that $H = H(x, y, z)$ satisfies (26) and that a similar condition with (27) is satisfied, more precisely assume that

$$\sup_{x, y \geq 0} \alpha(x, y) \iint_{00}^{xy} \beta_1(s, t) \exp\left(\iint_{st}^{xy} \alpha(\xi, \eta) \beta_1(\xi, \eta)\, d\xi\, d\eta\right) ds\, dt = B < \infty, \tag{33}$$

where $\beta_1(s, t) = \beta(s, t) h(s, t)$ for $s, t \geq 0$. Then we can state the following stability result

Proposition 2. *Assume that the conditions (26) and (31)–(33) are satisfied. If $u_i = u_i(x, y)$ $(i = 1, 2)$ are two solutions of*

$$(34) \qquad u(x, y) = f(x, y) + \int_0^x\!\!\int_0^y k(x, y, s, t)(u(s, t) + H(s, t, u(s, t)))ds\,dt,$$

corresponding to the free terms $f_i = f_i(x, y)$ $(i = 1, 2)$, then it follows that

$$(35) \qquad \|u_1 - u_2\|_g \leqslant (1 + A)(1 + B)\|f_1 - f_2\|_g.$$

Proof. We will use the representation formula

$$(36) \qquad u_i(x, y) = v_i(x, y) + \int_0^x\!\!\int_0^y r(x, y, s, t)H(s, t, u_i(s, t))ds\,dt, \quad i = 1,2$$

where

$$(37) \qquad v_i(x, y) = f_i(x, y) + \int_0^x\!\!\int_0^y r(x, y, s, t)f_i(s, t)ds\,dt, \quad i = 1,2.$$

From (31), (32) and (37), we obtain that

$$(38) \qquad \|v_1 - v_2\|_g \leqslant (1 + A)\|f_1 - f_2\|_g.$$

From (26), (31), (33) and (36), using also the result established in the Proposition 1, we obtain that (35) is true.

Remark 5. If $h(s, t) \geqslant h_0 > 0$ for $s, t \geqslant 0$, the restriction (32) is implied by (33) and we can take $A = Bh_0^{-1}$.

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ON SOME APPLICATIONS OF WAŻEWSKI METHOD FOR MULTIPLE VOLTERRA INTEGRAL EQUATIONS

BY

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1. Introduction. The general idea of the method of successive approximations, which is due to T. Ważewski, was introduced by him in 1960 in [10]. The idea of Ważewski, is very effective as well as versatile, and can be applied to investigate a sufficiently wide range of problems. In the present paper we apply Ważewski's method to study the problems of existence and uniqueness of the solutions of the integral equation of the form

$$(1) \qquad z(x, y) = F\Big(x, y, \int_0^x\!\!\int_0^y f_1(x, y, s, t, z(s, t))ds\,dt, \int_0^x f_2(x, y, s, z(s, y))ds, \int_0^y (f_3(x, y, t, z(x, t))dt, z(x, y)\Big),$$

where $z(x, y)$ is an unknown function. The Darboux problem for the partial differential equation of the form

$$(2) \qquad u_{xy}(x, y) = f(x, y, u(x, y), u_x(x, y), u_y(x, y), u_{xy}(x, y)),$$

with boundary conditions

$$(3) \qquad u(x, 0) = \sigma(x) \text{ for } 0 \leqslant x \leqslant a, \quad u(0, y) = \tau(y) \text{ for } 0 \leqslant y \leqslant b,$$

can be reduced to a particular case of equation (1). The fundamental theorems concerning the existence and uniqueness of the more general and special forms of Darboux problem (2)–(3) are established by many authors (see, [1]–[10]). In section 3, we establish our main results on the existence and uniqueness of the solutions of eqation (1) by using Ważewski's method [10]. In section 4, we indicate further applications of this powerful method to study the existence and uniqueness of the solutions of more general equations involving more than two independent variables.

2. Preliminaries. For convenience we first introduce the following hypotheses used in our subsequent discussion.

Hypothesis A. Suppose that

1° E be a Banach space with norm $\|\cdot\|$, $\Delta = [0, a] \times [0, b]$,

$$\Delta_1 = \{(x, y, s, t) : 0 \leq s \leq x \leq a,\ 0 \leq t \leq y \leq b\},$$

$$\Delta_2 = \{(x, y, s) : 0 \leq s \leq x \leq a,\ 0 \leq y \leq b\},$$

$$\Delta_3 = \{(x, y, t) : 0 \leqslant x \leqslant a,\ 0 \leqslant t \leqslant y \leqslant b\},$$