Proposition 2. *Assume that the conditions (26) and (31)–(33) are satisfied. If $u_i = u_i(x, y)$ $(i = 1, 2)$ are two solutions of*

$$u(x, y) = f(x, y) + \int_0^x\!\!\int_0^y k(x, y, s, t)(u(s, t) + H(s, t, u(s, t)))ds\,dt, \tag{34}$$

corresponding to the free terms $f_i = f_i(x, y)$ $(i = 1, 2)$, then it follows that

$$\|u_1 - u_2\|_g \leqslant (1 + A)(1 + B)\|f_1 - f_2\|_g. \tag{35}$$

Proof. We will use the representation formula

$$u_i(x, y) = v_i(x, y) + \int_0^x\!\!\int_0^y r(x, y, s, t)H(s, t, u_i(s, t))ds\,dt, \quad i = 1,2 \tag{36}$$

where

$$v_i(x, y) = f_i(x, y) + \int_0^x\!\!\int_0^y r(x, y, s, t)f_i(s, t)ds\,dt, \quad i = 1,2. \tag{37}$$

From (31), (32) and (37), we obtain that

$$\|v_1 - v_2\|_g \leqslant (1 + A)\|f_1 - f_2\|_g. \tag{38}$$

From (26), (31), (33) and (36), using also the result established in the Proposition 1, we obtain that (35) is true.

Remark 5. If $h(s, t) \geqslant h_0 > 0$ for $s, t \geqslant 0$, the restriction (32) is implied by (33) and we can take $A = Bh_0^{-1}$.

REFERENCES

1. B o n d g e B.K., P a c h p a t t e B.G. and W a l t e r W. — *On generalized Wendroff type inequalities and their applications*. Nonlinear Analysis, Theory, Methods & Applications, 4 (1980), 491–495
2. B e e s a c k P.R. — *Gronwall Inequalities-Carleton*. Math. Lecture Notes 11 (1975), p. 123
3. C h u S. C., M e t c a l f F.T. — *On Gronwall's inequality*, Proc. Amer. Math. Soc. Nr. 3, 18 (1967), 439–440
4. C o r d u n e a n u A. — *A note the Gronwall inequality in two independent variables* (to appear in Journal of Integral Equations, 1982)
5. F i n k A.M. — *Wendroff's inequalities*, Nonlinear Analysis. Theory, Methods & Applications, 8 (1981), 873–874
6. M a r t y n i u k A.A. and G u t o v s k i R. — *Integral Inequalities and the Stability of the Motion* (Russian) (Naukova Dumka, Kiev, 1979), pp. 51–78

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ON SOME APPLICATIONS OF WAŻEWSKI METHOD FOR MULTIPLE VOLTERRA INTEGRAL EQUATIONS

BY

B.G. PACHPATTE

1. Introduction. The general idea of the method of successive approximations, which is due to T. W a ż e w s k i, was introduced by him in 1960 in [10]. The idea of W a ż e w s k i, is very effective as well as versatile, and can be applied to investigate a sufficiently wide range of problems. In the present paper we apply W a ż e w s k i 's method to study the problems of existence and uniqueness of the solutions of the integral equation of the form

$$z(x, y) = F\left(x, y, \int_0^x\!\!\int_0^y f_1(x, y, s, t, z(s, t))ds\,dt, \int_0^x f_2(x, y, s, z(s, y))ds, \int_0^y (f_3(x, y, t, z(x, t))dt, z(x, y)\right), \tag{1}$$

where $z(x, y)$ is an unknown function. The Darboux problem for the partial differential equation of the form

$$u_{xy}(x, y) = f(x, y, u(x, y), u_x(x, y), u_y(x, y), u_{xy}(x, y)), \tag{2}$$

with boundary conditions

$$u(x, 0) = \sigma(x) \text{ for } 0 \leqslant x \leqslant a, \quad u(0, y) = \tau(y) \text{ for } 0 \leqslant y \leqslant b, \tag{3}$$

can be reduced to a particular case of equation (1). The fundamental theorems concerning the existence and uniqueness of the more general and special forms of Darboux problem (2)–(3) are established by many authors (see, [1]–[10]). In section 3, we establish our main results on the existence and uniqueness of the solutions of eqation (1) by using W a ż e w s k i 's method [10]. In section 4, we indicate further applications of this powerful method to study the existence and uniqueness of the solutions of more general equations involving more than two independent variables.

2. Preliminaries. For convenience we first introduce the following hypotheses used in our subsequent discussion.

Hypothesis A. Suppose that

1° E be a Banach space with norm $\|\cdot\|$, $\Delta = [0, a] \times [0, b]$,

$$\Delta_1 = \{(x, y, s, t) : 0 \leq s \leq x \leq a,\ 0 \leq t \leq y \leq b\},$$

$$\Delta_2 = \{(x, y, s) : 0 \leq s \leq x \leq a,\ 0 \leq y \leq b\},$$

$$\Delta_3 = \{(x, y, t) : 0 \leqslant x \leqslant a,\ 0 \leqslant t \leqslant y \leqslant b\},$$

the functions $f_1 \in C[\Delta_1 \times E, E]$, $f_2 \in C[\Delta_2 \times E, E]$, $f_3 \in C[\Delta_3 \times E, E]$ $F \in C[\Delta \times E^4, E]$; if $z \in C[\Delta, E]$ and

$$v(x, y) = F\left(x, y, \int_0^x\int_0^y f_1(x, y, s, t, z(s,t))dsdt, \int_0^x f_2(x, y, s, z(s, y))ds,\right.$$
$$\left.\int_0^y f_3(x, y, t, z(x, t))dt,\ z(x, y)\right),$$

then $v \in C[\Delta, E]$;

2° there exist functions $w_1(x, y, s, t, r)$, $w_2(x, y, s, r)$, $w_3(x, y, t, r)$ such that $w_1 \in C[\Delta_1 \times R^+, R^+]$, $w_2 \in C[\Delta_2 \times R^+, R^+]$, $w_3 \in C[\Delta_3 \times R^+, R^+]$ which are non-decreasing in r and fulfil the conditions

$$\|f_1(x, y, s, t, z) - f_1(x, y, s, t, \bar{z})\| \leqslant w_1(x, y, s, t\|z - \bar{z}\|),$$
$$\|f_2(x, y, s, z) - f_2(x, y, s, \bar{z})\| \leqslant w_2(x, y, s, \|z - \bar{z}\|),$$
$$\|f_3(x, y, t, z) - f_3(x, y, t, \bar{z})\| \leqslant w_3(x, y, t, \|z - \bar{z}\|);$$

3° there exists a function $H \in C[\Delta \times (R^+)^n, R^+]$ such that

(i) if $u \in C[\Delta, R^+]$ and

$$g(x, y) = H\left(x, y, \int_0^x\int_0^y w_1(x, y, s, t, u(s, t))dsdt, \int_0^x w_2(x, y, s, u(s, y))ds,\right.$$
$$\left.\int_0^y w_3(x, y, t, u(x, t))dt,\ u(x, y)\right),\ \text{then } g \in C[\Delta, R^+];$$

(ii) if $u, \bar{u} \in C[\Delta, R^+]$ and $u(x, y) \leqslant \bar{u}(x, y)$ for $(x, y) \in \Delta$, then

$$H\left(x, y, \int_0^x\int_0^y w_1(x, y, s, t, u(s, t))dsdt, \int_0^x w_2(x, y, s, u(s, y))ds.\right.$$
$$\left.\int_0^y w_3(x, y, t, u(x, t))dt,\ u(x, y)\right)$$
$$\leq H\left(x, y, \int_0^x\int_0^y w_1(x, y, s, t, \bar{u}(s, t))dsdt, \int_0^x w_2(x, y, s, \bar{u}(s, y))ds.\right.$$
$$\left.\int_0^y w_3(x, y, t, \bar{u}(x, t))dt,\ \bar{u}(x, y)\right),$$

for $(x, y) \in \Delta$;

(iii) if $u_n \in C[\Delta, R^+]$, $u_{n+1} \leq u_n$, $n = 0, 1, 2, \ldots$, and $\lim_{n\to\infty} u_n(x, y) = u(x, y)$, then

$$\lim_{n\to\infty} H\left(x, y, \int_0^x\int_0^y w_1(x, y, s, t, u_n(s, t))dsdt, \int_0^x w_2(x, y, s, u_n(s, y))ds,\right.$$
$$\left.\int_0^y w_3(x, y, t, u_n(x, t))dt,\ u_n(x, y)\right)$$
$$= H\left(x, y, \int_0^x\int_0^y w_1(x, y, s, t, u(s, t))dsdt, \int_0^x w_2(x, y, s, u(s, y))ds,\right.$$
$$\left.\int_0^y w_3(x, y, t, u(x, t))dt,\ u(x, y)\right),$$

for $(x, y) \in \Delta$;

4° for $(x, y, u_1, u_2, u_3, u_4)$, $(x, y, \bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4) \in \Delta \times E^4$,

$$\|F(x, y, u_1, u_2, u_3, u_4) - F(x, y, \bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4)\|$$
$$\leq H(x, y, \|u_1 - \bar{u}_1\|, \|u_2 - \bar{u}_2\|, \|u_3 - \bar{u}_3\|, \|u_4 - \bar{u}_4\|).$$

Hypothesis B. Suppose that

1° there exists a nonnegative and continuous function $\bar{u}: \Delta \to R^+$ being the solution of the inequality

$$(4)\qquad H\left(x, y, \int_0^x\int_0^y w_1(x, y, s, t, u(s, t))dsdt, \int_0^x w_2(x, y, s, u(s, y))ds,\right.$$
$$\left.\int_0^y w_3(x, y, t, u(x, t))dt,\ u(x, y)\right) + h(x, y) \leq u(x, y)$$

where

$$h(x, y) = \sup_{0\leqslant\alpha\leqslant x}\ \sup_{0\leqslant\beta\leqslant y} \left\|F\left(\alpha, \beta, \int_0^\alpha\int_0^\beta f_1(\alpha, \beta, s, t, 0)dsdt, \int_0^\alpha f_2(\alpha, \beta, s, 0)ds,\right.\right.$$
$$\left.\left.\int_0^\beta f_3(\alpha, \beta, t, 0)dt, 0\right)\right\|;$$

2° in the class of functions satisfying the condition $0 \leqslant u(x, y) \leqslant \bar{u}(x, y)$, $(x, y) \in \Delta$, the function $u(x, y) \equiv 0$, $(x, y) \in \Delta$, is the only solution of the equation

$$(5)\qquad u(x, y) = H\left(x, y, \int_0^x\int_0^y w_1(x, y, s, t, u(s, t))dsdt, \int_0^x w_2(x, y, s, u(s, y))ds,\right.$$
$$\left.\int_0^y w_3(x, y, t, u(x, t))dt,\ u(x, y)\right).$$

In order to prove the existence of a solution of equation (1), we define the sequence $\{z_n\}$ by the relations

$$z_0(x,y)\equiv 0,$$

$$z_{n+1}(x,y)=F\Big(x,y,\int_0^x\int_0^y f_1(x,y,s,t,z_n(s,t))dsdt,\int_0^x f_2(x,y,s,z_n(s,y))ds, \tag{6}$$

$$\int_0^y f_3(x,y,t,z_n(x,t))dt,\ z_n(x,y)\Big),\ \text{for } n=0,1,2,\ldots .$$

To prove the convergence of the sequence $\{z_n\}$ to the solution $\bar z$ of equation (1), we define the sequence $\{u_n\}$ by the relations

$$u_0(x,y)=\bar u(x,y),$$

$$u_{n+1}(x,y)=H\Big(x,y,\int_0^x\int_0^y w_1(x,y,s,t,u_n(s,t))dsdt,\int_0^x w_2(x,y,s,u_n(s,y))ds, \tag{7}$$

$$\int_0^y w_3(x,y,t,u_n(x,t))dt,\ u_n(x,y)\Big),$$

for $n=0,1,2,\ldots$, where the function $\bar u(x,y)$ is from Hypothesis B.

Now we establish the following lemma needed in our subsequent discussion.

Lemma 1. *If Hypothesis B and the condition 3° of Hypothesis A are satisfied, then*

$$0\le u_{n+1}(x,y)\le u_n(x,y)\le \bar u(x,y),\ (x,y)\in\Delta,\ n=0,1,2,\ldots, \tag{8}$$

$$\lim_{n\to\infty} u_n(x,y)=0,$$

and the convergence is uniform in each bounded set.

Proof. From relations (4) and (7) we have

$$u_1(x,y)=H\Big(x,y,\int_0^x\int_0^y w_1(x,y,s,t,u_0(s,t))dsdt,\int_0^x w_2(x,y,s,u_0(s,y))ds,$$

$$\int_0^y w_3(x,y,t,u_0(x,t))dt,\ u_0(x,y)\Big)$$

$$\le H\Big(x,y,\int_0^x\int_0^y w_1(x,y,s,t,\bar u(s,t))dsdt,\int_0^x w_2(x,y,s,\bar u(s,y))ds,$$

$$\int_0^y w_3(x,y,t,\bar u(x,t))dt,\ \bar u(x,y)\Big)+h(x,y)\le \bar u(x,y)=u_0(x,y),$$

for $(x,y)\in\Delta$. Further, we obtain (8) by induction. But (8) implies the convergence of the sequence $\{u_n(x,y)\}$ to some nonnegative function $\Phi(x,y)$ for $(x,y)\in\Delta$. By Lebesgue's theorem and the continuity of H it follows that the function $\Phi(x,y)$ satisfies equation (5). Now from Hypothesis B, we have $\Phi(x,y)\equiv 0$, $(x,y)\in\Delta$. The uniform convergence of the sequence $\{u_n\}$ in Δ follows from Dini's theorem. This completes the proof of Lemma 1.

3. Main Results. In this section we establish our main results on the existence and uniqueness of the solutions of equation (1).

Theorem 1. *If Hypotheses A and B are satisfied, then there exists a continuous solution $\bar z$ of equation (1). The sequence $\{z_n\}$ defined by (6) converges uniformly on Δ to $\bar z$, and the following estimations*

$$\|\bar z(x,y)-z_n(x,y)\|\le u_n(x,y),\ (x,y)\in\Delta,\quad n=0,1,2,\ldots, \tag{9}$$

$$\|\bar z(x,y)\|\le \bar u(x,y),\quad (x,y)\in\Delta, \tag{10}$$

hold. The solution $\bar z$ of equation (1) is unique in the class of functions satisfying the condition (10).

Proof. First we shall prove that the sequence $\{z_n(x,y)\}$, $(x,y)\in\Delta$, fulfils the condition

$$\|z_n(x,y)\|\le\bar u(x,y),\quad (x,y)\in\Delta,\quad n=0,1,2,\ldots, \tag{11}$$

Evidently, we see that $\|z_0(x,y)\|=0\le\bar u(x,y)$, $(x,y)\in\Delta$. Further, if we suppose that the inequality (11) is true for $n\ge 0$, then

$$\|z_{n+1}(x,y)\|=\Big\|F\Big(x,y,\int_0^x\int_0^y f_1(x,y,s,t,z_n(s,t))dsdt,\int_0^x f_2(x,y,s,z_n(s,y))ds,$$

$$\int_0^y f_3(x,y,t,z_n(x,t))dt,\ z_n(x,y)\Big)$$

$$-F\Big(x,y,\int_0^x\int_0^y f_1(x,y,s,t,0)dsdt,\int_0^x f_2(x,y,s,0)ds,\int_0^y f_3(x,y,t,0)dt,0\Big)$$

$$+F\Big(x,y,\int_0^x\int_0^y f_1(x,y,s,t,0)dsdt,\int_0^x f_2(x,y,s,0)ds,\int_0^y f_3(x,y,t,0)\,dt,0\Big)\Big\|$$

$$\le H\Big(x,y,\int_0^x\int_0^y w_1(x,y,s,t,\|z_n(s,t)\|)\,dsdt,\int_0^x w_2(x,y,s,\|z_n(s,y)\|)ds,$$

$$\int_0^y w_3(x, y, t, \|z_n(x, t)\|)dt, \|z_n(x, y)\|\Big) + h(x, y)$$

$$\le H\Big(x, y, \int_0^x\int_0^y w_1(x, y, s, t, \bar{u}(s, t))dsdt, \int_0^y w_2(x, y, s, \bar{u}(s, y))ds,$$

$$\int_0^y w_3(x, y, t, \bar{u}(x, t))dt, \bar{u}(x, y)\Big) + h(x, y) \le \bar{u}(x, y),$$

for $(x, y) \in \Delta$. Now we obtain (11) by induction.

Next we prove that

(12) $\|z_{n+r}(x, y) - z_n(x, y)\| \le u_n(x, y)$, $(x, y) \in \Delta$, $n = 0, 1, 2, \ldots$, $r = 0, 1, 2, \ldots$

By (11) we have

$$\|z_r(x, y) - z_0(x, y)\| = \|z_r(x, y)\| \le \bar{u}(x, y) = u_0(x, y), \quad (x, y) \in \Delta, \quad r = 0, 1, 2, \ldots$$

Suppose that (12) is true for n, $r \ge 0$, then

$$\|z_{n+r+1}(x, y) - z_{n+1}(x, y)\| = \Big\| F\Big(x, y, \int_0^x\int_0^y f_1(x, y, s, t, z_{n+r}(s, t))dsdt,$$

$$\int_0^x f_2(x, y, s, z_{n+r}(s, y))ds, \int_0^y f_3(x, y, t, z_{n+r}(x, t))dt, z_{n+r}(x, y)\Big) -$$

$$- F\Big(x, y, \int_0^x\int_0^y f_1(x, y, s, t, z_n(s, t))dsdt, \int_0^x f_2(x, y, s, z_n(s, y))ds,$$

$$\int_0^y f_3(x, y, t, z_n(x, t))dt, z_n(x, y)\Big)\Big\|$$

$$\le H\Big(x, y, \int_0^x\int_0^y w_1(x, y, s, t, \|z_{n+r}(s, t) - z_n(s, t)\|)ds\,dt, \int_0^x w_2(x, y, s, \|z_{n+r}(s, y) -$$

$$- z_n(s, y)\|)ds, \int_0^y w_3(x, y, t, \|z_{n+r}(x, t) - z_n(x, t)\|)dt, \|z_{n+r}(x, y) - z_n(x, y)\|\Big)$$

$$\le H\Big(x, y, \int_0^x\int_0^y w_1(x, y, s, t, u_n(s, t))dsdt, \int_0^x w_2(x, y, s, u_n(s, y))ds,$$

$$\int_0^y w_3(x, y, t, u_n(x, t))dt, u_n(x, y)\Big) u_{n+1}(x, y),$$

for $(x, y) \in \Delta$. Now we obtain (12) by induction.

Because of Lemma 1, $\lim_{n\to\infty} u_n(x, y) = 0$ in Δ, we have from (12) $z_n \to \bar{z}$ in Δ. The continuity of $\bar{z}$ follows from the uniform convergence of the sequence $\{z_n\}$ and the continuity of all functions z_n. If $r \to \infty$, then (12) gives estimation (9). Estimation (10) is implied by (11). It is obvious that $\bar{z}$ is a solution of equation (1).

To prove that the solution $\bar{z}$ is a unique solution of equation (1) in the class of functions satisfying the condition (10), let us suppose that there exists another solution $\hat{z}$ defined in Δ and such that $\bar{z}(x, y) \not\equiv \hat{z}(x, y)$ for $(x, y) \in \Delta$, and $\|\hat{z}(x, y)\| \le \bar{u}(x, y)$ for $(x, y) \in \Delta$. From (9) we get

(13) $\|\hat{z}(x, y) - z_n(x, y)\| \le u_n(x, y)$, $(x, y) \in \Delta$, $n = 0, 1, 2, \ldots$

Since $\lim_{n\to\infty} u_n(x, y) = 0$ by Lemma 1 and $z_n \to \bar{z}$ in Δ, it follows from (13) that $\bar{z}(x, y) = \hat{z}(x, y)$ for $(x, y) \in \Delta$. This contradiction proves the uniqueness of $\bar{z}$ in the class of functions satisfying the relation (10). This completes the proof of the theorem.

We next establish a theorem which gives conditions under which equation (1) has at most one solution, these conditions do not guarantee existence.

Theorem 2. *If Hypothesis A is satisfied and the function $m(x, y) \equiv 0$, $(x, y) \in \Delta$, is the only nonnegative continuous solution of the inequality*

(14) $$m(x, y) \le H\Big(x, y, \int_0^x\int_0^y w_1(x, y, s, t, m(s, t))dsdt, \int_0^x w_2(x, y, s, m(s, y))ds,$$

$$\int_0^y w_3(x, y, t, m(x, t))dt, m(x, y)\Big)$$

for $(x, y) \in \Delta$, then equation (1) has at most one solution in the set Δ.

Proof. Let us suppose that there exist two solutions $\bar{z}$ and $\hat{z}$ of equation (1) such that $\bar{z}(x, y) \not\equiv \hat{z}(x, y)$, $(x, y) \in \Delta$. Put

$$m(x, y) = \|\bar{z}(x, y) - \hat{z}(x, y)\|, \quad (x, y) \in \Delta,$$

then

$$m(x, y) = \Big\| F\Big(x, y, \int_0^x\int_0^y f_1(x, y, s, t, \bar{z}(s, t))dsdt, \int_0^x f_2(x, y, s, \bar{z}(s, y))ds,$$

$$\int_0^y f_3(x, y, t, \bar{z}(x, t))dt, \bar{z}(x, y)\Big) - F\Big(x, y, \int_0^x\int_0^y f_1(x, y, s, t, \hat{z}(s, t))dsdt,$$

$$\int_0^x f_2(x, y, s, \hat{z}(s, y))ds, \int_0^y f_3(x, y, t, \hat{z}(x, t))dt, \hat{z}(x, y)\Big)\Big\|$$

$$\leq H\left(x, y, \int_0^x\int_0^y w_1(x, y, s, t, \|\bar{z}(s,t)-\hat{z}(s,t)\|)ds\,dt, \int_0^x w_2(x, y, s, \|\bar{z}(s,y)-\hat{z}(s,y)\|)\,ds,\right.$$

$$\left.\int_0^y w_3(x, y, t, \|\bar{z}(x,t)-\hat{z}(x,t)\|)dt, \|\bar{z}(x,y)-\hat{z}(x,y)\|\right)$$

$$= H\left(x, y, \int_0^x\int_0^y w_1(x, y, s, t, m(s,t))ds\,dt, \int_0^x w_2(x, y, s, m(s,y))ds,\right.$$

$$\left.\int_0^y w_3(x, y, t, m(x,t))dt,\ m(x,y)\right),$$

and by (14) we conclude that $m(x, y)\equiv 0$ for $(x, y)\in\Delta$, i.e. $\bar{z}(x, y)=\hat{z}(x, y)$, $(x, y)\in\Delta$. This contradiction proves theorem 2.

4. Further Applications. In this section, we indicate further applications of Ważewski method [10] to study the existence and uniqueness problems of more general partial differential and integral equations.

In [7], B. Palczewski has studied the existence and uniqueness of solutions of the Darboux problem for the equation

(15) $$u_{x_1x_2x_3}=f(x_1, x_2, x_3, u, u_{x_1}, u_{x_2}, u_{x_3}, u_{x_1x_2}, u_{x_1x_3}, u_{x_2x_3}),$$

under the conditions

(16) $$u(0, x_2, x_3)=\varphi_1(x_2, x_3),\ u(x_1, 0, x_3)=\varphi_2(x_1, x_3),\ u(x_1, x_2, 0)=\varphi_3(x_1, x_2),$$

by using the method of successive approximations and Schauder fixed point theorem. The problem (15)–(16) is a special case of the following more general integral equation

(17) $$p(x, y, z)=F\left(x, y, z, \int_0^x\int_0^y\int_0^z f_1(x, y, z, s, t, r, p(s, t, r))\right)ds\,dt\,dr,$$

$$\int_0^y\int_0^z f_2(x, y, z, t, r, p(x, t, r))dt\,dr,$$

$$\int_0^x\int_0^z f_3(x, y, z, s, r, p(s, y, r))ds\,dr, \int_0^x\int_0^y f_4(x, y, z, s, t, p(s, t, z))ds\,dt,$$

$$\int_0^z f_5(x, y, z, r, p(x, y, r))dr, \int_0^y f_6(x, y, z, t, p(x, t, z))dt,$$

$$\int_0^x f_7(x, y, z, s, p(s, y, z))\,ds,\ p(x, y, z)).$$

We note that the method employed in Section 3 can be very easily extended to study the existence and uniqueness of the solutions of equation (17) under some suitable conditions on the functions involved in (17).

In [2] and [3] the authors have studied the existence and uniqueness of the solutions of hyperbolic partial differential equation of the form

(18) $$u_{x_1\ldots x_n}=f(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}, u_{x_1x_2}, \ldots, u_{x_1\ldots x_{n-1}}, \ldots, u_{x_2\ldots x_n})$$

where only the pure mixed derivatives of u appear, under the given boundary conditions, by using the Euler-Cauchy Polygon method. We note that one can very easily extend the results given in section 3 to study the existence and uniqueness of solutions of a more general integral form of equation (18) as explained in case of equation (17). We do not discuss the details of these results.

REFERENCES

1. Alexiewicz A. and Orlicz W. — *Some remarks on the existence and uniqueness of solutions of the hyperbolic equation* $\partial^2 z/\partial x\partial y=f(x, y, z, \partial z/\partial x, \partial z/\partial y)$. Studia Math. 15 (1956), 201–215
2. Conlan J. and Diaz J.B. — *Existence of solutions of an n-th order hyperbolic partial differential equation*. Conti. Diff. Equations 2 (1963), 277–289
3. Glick I.I. — *On an analog of the Euler-Cauchy palygon method for the partial differential equation* $u_{x_1\ldots x_n}=f$, Cont. Diff. Equations 3 (1964), 1–59
4. Kisynski J. and Pelczar A. — *Comparison of solutions and successive approximations in the theory of the equation* $\partial^2 z/\partial x\partial y=f(x, y, z, \partial z/\partial x, \partial z/\partial y)$. Dissertations Mathematicae. 76 (1970), 1–77
5. Kwapisz M. and Turo J. — *On the existence and uniqueness of solutions of Darboux problem for partial differential-functional equations*. Colloq. Math. 29 (1974), 295–318
6. Kwapisz M. and Turo J. — *On the existence and uniqueness of solutions of the Darboux problem for partial differential-functional equations in a Banach space*. Ann. Polon. Math. 29 (1974), 89–118
7. Palczewski B. — *Existence and uniqueness of solutions of the Darboux problem for the equation*

$$\frac{\partial^3 u}{\partial x_1\partial x_2\partial x_3}=f\left(x_1, x_2, x_3, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}, \frac{\partial^2 u}{\partial x_1\partial x_2}, \frac{\partial^2 u}{\partial x_1\partial x_3}, \frac{\partial^2 u}{\partial x_2\partial x_3}\right),$$

Ann. Polon. Math. 13 (1963), 267–277
8. Porath G. — *Über die Differentialgleichung* $z_{xy}=\varphi(x, y, z, z_{xy})$, Math. Nachr. 33 (1967), 73–89
9. Walter W. — *Differential and Integral Inequalities*, Springer-Verlag-Berlin, New York, 1970
10. Ważewski T. — *Sur un procédé de prouver la convergence des approximations successives sans utilisation des séries de comparison*, Bull. Acad. Sci., Ser. Sci. Math. Astr. et. Phys. 8 (1960), 45–52

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