ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII „AL. I. CUZA" DIN IAŞI
Tomul XXX, s.I a. Matematică, 1984—3

MORPHISMS OF H-CONES AND SEMI-POLAR SETS

BY

EUGEN POPA

§0. **Introduction and Preliminaries.** The main result of this paper is th. 1, in which a number of equivalent properties with the fact that the semi-polarity is preserved by inverse image of a H-map, are given. Th. 2 contains equivalent formulations for the fact that the polarity is preserved by inverse image of a H-map. These results are proved in §1 In §2 we present some remarks and complementary results. Th. 1 and 2 are completed with other equivalent or sufficient conditions, which hold in some special instances. §3 extends, to the case of general morphisms of H-cones, the notion of inverse image. After proving the main properties of the inverse image function, we show (th. 6) that some of the equivalences from th. 1 and 2 hold true in this more general setting. This fact allows to give (th. 7) some equivalent formulations for the axiom of convergence in standard H-cones. The paper ends by considering a certain class of morphisms of H-cones, which includes the H-maps, and for which the equivalent properties from th. 1 still hold true.

The notions, notations and results about H-cones are as in [4]. The notions of morphism of H-cones and H-map, as well as results related to them, are as in [5], [6], [7].

We shall use the notion of quasi-base in a standard H-cone of functions, defined as follows (this notion was introduced in [1], in a particular setting). *Let S be a standard H-cone of functions on X. For $A\subseteq X$, wede note $\rho(A)$ and call quasi-base of A, the smallest finely closed set $B\subseteq X$, such that $A\setminus B$ semi-polar*. Clearly $A\setminus\rho(A)$ is semi polar, and : $\rho(A)=\emptyset\Leftrightarrow A$ is semi-polar. We shall need also the following two facts about the quasi-base : a) $b(\rho(A))=\rho(A)$; b) $\beta(A)=\rho(b(A))$.

§1. **The main results.** Let S, T be standard H-cones of functions on X, Y $\varphi: Y\to X$ is a H-map. We use the same letter for the corresponding morphism of H-cones : $\varphi\in\mathrm{Hom}\ (S, T)$.

Theorem 1. *The following conditions are equivalent* :

1) $\forall A\subseteq X$ *semi polar* $\Rightarrow\varphi^{-1}(A)$ *is semi-polar*;

2) $\forall F\subseteq D(\varphi)$ *decreasing* $\Rightarrow\varphi(\wedge F)=\bigwedge_{s\in F}\varphi(s)$;

3) $\forall s\in D(\varphi)$, $A\subseteq X : B^{\varphi^{-1}(A)}(\varphi(s))\leqslant\varphi(B^A s)$;

4) $\forall A\subseteq X$, $y\in Y$: A *is thin at* $\varphi(y)\Rightarrow\varphi^{-1}(A)$ *is thin at* y,

5) $\forall A \subseteq X : b(\varphi^{-1}(A)) \subseteq (\varphi^{-1}(b(A))$;

6) $A \subseteq X$ *totally thin* $\Rightarrow \varphi^{-1}(A)$ *is totally thin*;

7) $\forall A \subseteq X : \rho(\varphi^{-1}(A)) \subseteq \varphi^{-1}(\rho(A))$;

8) $\forall s \in D(\varphi),\ A \subseteq X : B^{\rho(\varphi^{-1}(A))}\varphi(s) \leqslant \varphi(B^{\rho(A)}s)$;

9) $\forall A \subseteq X : \beta(\varphi^{-1}(A)) \subseteq \varphi^{-1}(\beta(A))$;

Proof. $1 \Rightarrow 2$. We have: $\varphi(\wedge F) \leqslant \bigwedge_{s \in F} \varphi(s)$ and: $[\varphi(\wedge F) < \bigwedge_{s \in F} \varphi(s)\varphi] \subseteq [(\wedge F)\circ] < (\inf F)\circ\varphi] = \varphi^{-1}([\wedge F < \inf F])$. Since $[\wedge F < <\inf F]$ is semi-polar, it follows that: $\varphi(\wedge F) = \bigwedge_{s \in F} \varphi(s)$, except for a semi-polar set, hence everywhere. $2 \Rightarrow 3$. Since $t \geqslant s$ on A implies $\varphi(t) \geqslant \varphi(s)$ on $\varphi^{-1}(A)$, it follows that: $\varphi(B^A s) = (\bigwedge_{t \geqslant s \text{ on } A} t)\circ\varphi = \bigwedge_{t \geqslant s \text{ on } A} (t\circ\varphi) \geqslant B^{\varphi^{-1}(A)}(\varphi(s))$. $3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 6 \Rightarrow 1$ are obvious. $1 \Rightarrow 7$. $\varphi^{-1}(\rho(A))$ is finely closed, and: $\varphi^{-1}(A) \setminus \varphi^{-1}(\rho)A)) = \varphi^{-1}(A \setminus \rho(A))$ is semi-polar. $7 \Rightarrow 8$. We have: $B\rho^{\varphi^{-1}(A)}(\varphi(s)) \leqslant B^{\varphi^{-1}(\rho(A))}(\varphi(s)) = B^{\varphi^{-1}(b\rho(A))}(\varphi(s)) \leqslant \varphi(B^{\rho(A)}s)$. The last inequality can be proved as follows: since $B^A s = s$ on $b(A)$, we have $\varphi(B^A s) = \varphi(s)$ on $\varphi^{-1}(b(A))$, hence: $\varphi(B^A s) \geqslant B^{\varphi^{-1}(b(A))}(\varphi(s))$. $8 \Rightarrow 1$. Since $\rho(A) = \varnothing$ it follows that $B^{\rho(\varphi^{-1}(A))}\,1 = 0$, hence $\rho(\varphi^{-1}(A))$ is polar. Then $\rho(\varphi^{-1}(A)) = \varnothing$ hence $\varphi^{-1}(A)$ is semipolar. $5+7 \Rightarrow 9$. We have: $\beta(\varphi^{-1}(A)) = \rho(b(\varphi^{-1}(A)) \subseteq \subseteq \rho(\varphi^{-1}(b(A))) \subseteq \varphi^{-1}(\rho(b(A)) = \varphi^{-1}(\beta(A))$. $9 \Rightarrow 1$. We can suppose that $A \subseteq X$ is totally thin. Then $b(A) = \varnothing$ hence $\beta(A) = \varnothing$ and it follows that $\beta(\varphi^{-1}(A)) = \varnothing$. Since $\varphi^{-1}(A) \setminus \beta(\varphi^{-1}(A))$ is semi-polar, we get $\varphi^{-1}(A)$ semi-polar.

Theorem 2. *The following conditions are equivalent:*

1) $A \subseteq X$ *polar* $\Rightarrow \varphi^{-1}(A)$ *is polar.*

2) φ *is a finite morphism.*

3) $\forall \mu \in T_0^*$, $\varphi^*(\mu)$ *is universally bounded.*

Proof. $1 \Rightarrow 2$. Suppose that φ is not finite. Then, there exists a $s \in S$, such that $\varphi(s)$ is not finite on a finely dense set. Hence: $[s\circ\varphi = =\infty] = \varphi^{-1}([s=\infty])$ has non-void fine interior. Hence, it is not a polar set, while $[s=\infty]$ is polar. $2 \Rightarrow 1$. If $A \subseteq X$ is polar, there exists $s \in S$ such that $A \subseteq [s=\infty]$. Then: $\varphi^{-1}(A) \subseteq \varphi^{-1}([s=\infty]) = [s\circ\varphi = \infty]$ and $\varphi^{-1}(A)$ is also polar. $2 \Rightarrow 3$. We need the following characterisation:

$$t \in S \text{ is universally bounded} \Leftrightarrow \forall \mu \in S^* : \mu(t) < \infty.$$

Indeed, let $t \in S$ be universally bounded and $\mu \in S^*$. There exists a weak unit $u \in S$, such that $\mu(u) < \infty$, hence $\mu(t) < \infty$. Conversely, if $t \in S$ is not universally bounded, then there exists a weak unit $u \in S$, such that the inequality $t \leqslant n.u$ does not hold for any $n \in N$. Hence, for each $n \in N$ there exists $x_n \in X_n$ such that $t(x_n) > n^3$. Let: $\mu = \sum_{n=1}^{\infty} n^{-2}\, \varepsilon_{x_n}$. Then $\mu \in S^*$ since $\mu(u) < \infty$. But $\mu(t) = \infty$. Let now φ be finite. For any $\mu \in T_0^*$ and any $s \in S$, we have: $\varphi((\mu)(s) = \mu(\varphi(s)) < \infty$, since $\varphi(s) \in T^{**}$. $3 \Rightarrow 2$. Let $s \in S$. For any $\mu \in T_0^*$ we have: $\mu(\varphi(s)) = \varphi^*(\mu)(s) < \infty$, since there exists a weak unit $\nu \in S^*$, such that $\nu(s) < \infty$ and, $\varphi^*(\mu)$ being universally bounded, there exists $\alpha \geqslant 0$ such that $\varphi^*(\mu) \leqslant \alpha.\nu$.

§2. Remarks and Complementary Results. a) Conditions 2) form th. 1 and 2 show that the property that φ^{-1} preserves the semi-polarity resp. polarity, does not depend upon the base spaces X, Y. In this context we have:

Proposition 3. *Let S, T be H-cones and $\varphi \in$ Hom (S, T) a finite morphism. If S has a weak unit, then the following conditions are equivalent:*

1) $\forall u \in S$ *weak unit,* $\varphi(u)$ *is a weak unit.*

2) $\exists u \in S$ *weak unit, such that* $\varphi(u)$ *is a weak unit.*

3) $\forall t \in S$ *we have* : $t = \vee \{t' \in T \mid \exists s \in S$ *such that* : $t' \leqslant \varphi(s), t' \leqslant t\}$.

b) **Proposition 4.** *The condition:*
$\forall U \subseteq Y$ *naturally open,* $\varphi(U)$ *is finely open in* X, *implies the condition 2) from th. 1.*

Proof. For any function $f : X \to \bar{R}_+$ we have: $\hat{f}\circ\varphi(y) = \liminf_{x \to \varphi(y)} f(x) = = \sup_{V \in \mathcal{V}_f(\varphi(y))} (\inf_{x \in V} f(x)) = \sup_{W \in \mathcal{V}_n(y)} (\inf_{z \in W} f(\varphi(z)) = \widehat{f\circ\varphi}(y)$ hence: $\varphi(\wedge F) = \widehat{\inf F\circ\varphi} = =(\widehat{\inf F\circ\varphi} = \inf_{s \in F} \widehat{(s\circ\varphi)} = \bigwedge_{s \in F} \varphi(s)$, where $\mathcal{V}_f(y)$, resp. $\mathcal{V}_n(y)$ denote the filter of fine, resp. natural neighbourhoods of y.

Remark. It was proved in [5] that, if φ is finely open, then φ is finite.

c) **Proposition 5.** *If X is saturated, then: "$\forall \mu \in T_0^*$, $\varphi^*(\mu)$ is nearly continuous" is equivalent with condition 1) from th. 1.*

Proof. Indeed, let $A \subseteq X$ be semi-polar. For any $\mu \in T_0^*$, we have: $\mu(\varphi^{-1}(A)) = \varphi^*(\mu)(A) = 0$, hence $\varphi^{-1}(A)$ is semi-polar. Conversely, if X is saturated, then the above formula, together with [4, cor. 5.5.11] show that $\varphi^*(\mu)$ is nearly continuous, if $\mu \in T_0^*$.

d) Condition 3) from th. 1. implies condition 1) from th. 2. Indeed if we choose $s = 1$, then: $B^{\varphi^{-1}(A)}\,1 \leqslant B^A\,1$. If A is polar, then it follows that $\varphi^{-1}(A)$ is also polar. This remark shows that we can drop the assumption "$s \in D(\varphi)$" in conditions 2, 3 and 8 from th. 1. Also, it is known that it suffices that condition 2) holds only for countable F. One obtains another equivalent condition if one drops "F decreasing" from condition 2), th. 1, since φ is H-map.

§3. Some Extensions. We are going to extend th. 1 and 2 to more general morphisms of H-cones. In order to do that, we define an „inverse image" map, for a quite large class of morphisms. However, this correspondence between the subsets of X and the subsets of Y does not preserve all the properties of the usual inverse image, and so not all the equivalences stated in th. 1 will hold in this general setting.

For the rest of the paper, S and T will be standard H-cones of functions on X, Y and $\varphi \in$ Hom (S, T) will be a morphism with the properties: 1) φ has an adjoint; 2) $\varphi^*(Y) \subseteq \mathcal{M}_+(X)$ (positive measures on the natural Borel σ algebra).

Remarks. a) φ^* exists in more general situations than those considered in [5]. If there exists a weak unit $\mu_0 \in T^*$, such that $\mu_0\circ\varphi \in S^*$, or as a special case, if φ is semi-finite and there exists a weak unit $\mu_0 \in T^*$, taking real values, then φ has an adjoint. For H-cones of functions it suffices to suppose that $\varphi(1)$ is a real valued function. Then φ^* exists and $\varphi^*(\varepsilon_y)\in S^*$, $\forall y\in Y$. b) If X is saturated, then any morphism φ, such that $\varphi(1)$ is a real-valued function, fulfills the above conditions 1) and 2).

For any naturally measurable function $f: X\to\overline{\mathbf{R}}_+$, we define: $f\circ\varphi(y)=\int_X f d\varphi^*(\varepsilon_y)$. If we denote by $\overline{\varphi^*(\varepsilon_y)}$ the completion of the measure $\varphi^*(\varepsilon_y)$ then, for each $A\subseteq X$ we define: $\varphi_0^{-1}(A)=\{y\in Y \mid \overline{\varphi^*(\varepsilon_y)}\,(X\setminus A)=0\}$ and: $\varphi^{-1}(A)=Y\setminus\varphi_0^{-1}(X\setminus A)$. Hence, we have defined the correspondences φ_0^{-1}, φ^{-1} from $\mathscr{D}(X)$ to $\mathscr{D}(Y)$. The following properties will be used in the sequel:

(i) If $Y_0=\{y\in Y \mid \varphi^*(\varepsilon_y)=0\}$ then: $\varphi_0^{-1}(A)\setminus Y_0\subseteq\varphi^{-1}(A)$;
(ii) If $A\subseteq A_1$, then: $\varphi_0^{-1}(A)\subseteq\varphi_0^{-1}(A_1)$ and $\varphi^{-1}(A)\subseteq\varphi^{-1}(A_1)$;
(iii) For any countable family (A_n) from $\mathscr{D}(X)$: $\varphi_0^{-1}(\bigcap A_n)=\bigcap\varphi_0^{-1}(A_n)$ and $\varphi^{-1}(\bigcup A_n)=\bigcup\varphi^{-1}(A_n)$;
(iv) If $f, g: X\to\overline{\mathbf{R}}_+$ are naturally measurable functions, such that $f\leqslant g$ on X, then: $\varphi_0^{-1}([f=g])=[f\circ\varphi=g\circ\varphi]$; $\varphi^{-1}([f<g])=[f\circ\varphi<g\circ\varphi]$;
(v) If $f, g: X\to\overline{\mathbf{R}}_+$ are naturally measurable and $f\leqslant g$ on $A\subseteq X$, then: $f\circ\varphi\leqslant g\circ\varphi$ on $\varphi_0^{-1}(A)$;
(vi) We have: $\varphi_0^{-1}(A\setminus A_1)=\varphi_0^{-1}(A)\setminus\varphi^{-1}(A_1)$. If A, A_1 are measurable, then: $\varphi_0^{-1}(A)\setminus\varphi_0^{-1}(A_1\subseteq\varphi^{-1}(A\setminus A_1)$;
(vii) For any $A\subseteq X$: $\varphi^{-1}(d(A))$ is finely open and $\varphi_0^{-1}(b(A))$ is finely closed. If $\varphi(s)$ is naturally continuous, for any $s\in S$, which is naturally continuous and bounded, then (U naturally open)$\Rightarrow$($\varphi^{-1}(U)$ is naturally open and: (F naturally closed)$\Rightarrow$($\varphi_0^{-1}(F)$ is naturally closed).

Examples. a) If $\varphi=\Sigma\alpha_k\varphi_k$, with $\alpha_k>0$ and φ_k H-maps, then: $\varphi_0^{-1}(A)=\bigcap\varphi_k^{-1}(A)$ and: $\varphi^{-1}(A)=\bigcup\varphi_k^{-1}(A)$. b) If $\varphi=B$ is a balayage and X is semisaturated [3], then:

$$B_0^{-1}(b(B))=B^{-1}(b(B))=X \text{ and } B_0^{-1}(d(B))=B^{-1}(d(B))=\varnothing.$$

In this general situation, th. 2 holds, with the same proof, if we use as „inverse image" φ_0^{-1}. Particularly, for any standard H-cone of functions on a semi-saturated space X, the following holds: $\forall B$ balayage on S, $\forall A\subseteq X$ polar, the set: $\{x\in X \mid B^*(\varepsilon_x)$ is carried by $A\}$ is polar. The example of balayages shows that generally φ^{-1} does not preserve the polar sets. Also holds the following equivalence from th. 1

Theorem 6. *The following statements are equivalent:*
1) *$A\subseteq X$ semi-polar$\Rightarrow\varphi^{-1}(A)$ is semi-polar,*
2) *$F\subseteq D(\varphi)$ decreasing $\Rightarrow\varphi(\wedge F)=\bigwedge_{s\in F}\varphi(s)$.*

Proof. 1$\Rightarrow$2. The same proof holds, since by [4, cor. 4.2. 10.c)] there exists a countable family $(s_n)\subseteq F$ such that: $\wedge F=\bigwedge_{n\in N}s_n$. Then:

$$\bigwedge_{s\in F}\varphi(s)\geqslant\varphi(\wedge F)=\varphi(\bigwedge_{n\in N}s_n)=\bigwedge_{n\in N}\varphi(s_n)\geqslant\bigwedge_{s\in F}\varphi(s).$$

2$\Rightarrow$1. Let $A\subseteq X$ be totally thin. If p is a finite, continuous generator, then $B^A p<p$ on A. Hence $A\subseteq[\wedge F<\inf F]$, where: $F=\{s\in S \mid s\geqslant p$ on $A\}$ Since $\varphi^{-1}(A)\subseteq[\wedge(s\circ\varphi)<\inf(s\circ\varphi)]$, it follows that $\varphi^{-1}(A)$ is semi-polar. The general case follows now, using property (iii).

Condition 2) from the above theorem allows us to establish a remarkable connection with the axiom of convergence. Let us denote by $\mathfrak{D}$ the set of the morphisms $\varphi\in\operatorname{Hom}(S, T)$, such that: $\forall_\mu\in T_0^*$, $\exists\nu\in S_0^*$ such that $\varphi^*(\mu)\leqslant\nu$.

Theorem 7. *The following are equivalent:*
1) *S satisfies the axiom of convergence.*
2) *$\forall T$ standard H-cone of functions, $\forall\varphi\in\mathfrak{D}$ $\forall F\subseteq D(\varphi)$ decreasing $\varphi(\wedge F)=\bigwedge_{s\in F}\varphi(s)$.*
3) *$\exists T\neq\{0\}$ standard H-cone of functions, such that $\forall\varphi\in\mathfrak{D}$, for any decreasing sequence $s_n\in D(\varphi)$: $\varphi(\wedge s_n)=\wedge\varphi(s_n)$.*
4) *X being semi-saturated, $\forall$ B balayage on S, $\forall A\subseteq X$ measurable semi-polar set, the set: $\{x\in X \mid B^*(\varepsilon_x)$ charges $A\}$ is semi-polar.*

Proof. 1$\Rightarrow$2. It suffices to show that: $\mu(\varphi(\wedge F))=\mu(\bigwedge_{s\in F}\varphi(s))$, for any $\mu\in T_0^*$. But, using the fact that $\varphi\in\mathfrak{D}$ and S satisfy the axiom of convergence, we can write:

$$\mu(\varphi(\wedge F))=\varphi^*(\mu)(\wedge F)=\inf_{s\in F}\varphi^*(\mu)(s)=\inf_{s\in F}\mu(\varphi(s))=\mu(\bigwedge_{s\in F}\varphi(s))$$

2$\Rightarrow$3. is clear. 3$\Rightarrow$1. Let $t\in T$, $t\neq 0$ be fixed. If $\mu\in S^*$ is dominated by a universally continuous element, let us define $\varphi\in\operatorname{Hom}(S, T)$ by: $\varphi(s)=\mu(s).\,t$. This definition is correct, and $\varphi^*(\nu)=\nu(t).\,\mu$. It follows that $\varphi\in\mathfrak{D}$. Hence, for any decreasing sequence $s_n\in D(\varphi)$ we have: $\mu(\bigwedge_{n\in N}s_n).\,t=\bigwedge_{n\in N}\mu(s_n).t=\inf_{n\in N}\mu(s_n).\,t$, and this relation proves that $\mu(\wedge s_n)=\inf\mu(s_n)$. 4$\Leftrightarrow$1. Follows from th. 6, since $B\in\mathfrak{D}$ by [4, th. 5.6.3] and $B^{-1}(A)=\{x\in X \mid B^*(\varepsilon_x)$ charges $A\}$.

Remark. In this general situation, the following implications from th. 1 hold: 3$\Rightarrow$5$\Rightarrow$6$\Rightarrow$1. When φ^{-1} is replaced by φ_0^{-1}, we have the following implications: 2$\Rightarrow$3$\Rightarrow$5$\Rightarrow$6; 1$\Rightarrow$7$\Rightarrow$8; 1$\Rightarrow$9. These implications are equivalent iff: $\forall A$ semi-polar, $\varphi^{-1}(A)\setminus\varphi_0^{-1}(A)$ is semi-polar; or, what is the same: $\exists Y'\subseteq Y$ semi-polar, such that $\forall y\notin Y'$ we have: $\varphi^*(\varepsilon_y)=\mu+\nu$, with μ difuse, not charging the semipolar sets, and ν atomic, with point support.

For a more restrictive class of morphisms (still larger than the H-maps), some of the equivalences from th. 1 still hold. Let us denote by $Ex(S)$ the set of extreme rays of S^*, endowed with quotient (natural or fine) topology. For each weak unit $u\in S$, there is a continuous canonical injection: $X_u\to Ex(S)$. The inverse of this injection is given by: $\mu\to\mu/\mu(u)$, hence it is finely (and generally not naturally) continuous. If u, v are weak units, there exists polar sets $A_1\subseteq X_u$ and $A_2\subseteq X_v$, such that $X_u\setminus A_1$ and $X_v\setminus A_2$ are finely homeomorphic. $Ex(S)$ appears thus as the inductive limit of the spaces X_u, considered with the fine topology.

Theorem 8. *If φ has the properties: $\varphi(1)\in T$ and $\forall s$, $t\in D(\varphi)$: $\varphi(s\wedge t)=\varphi(s)\wedge\varphi(t)$, then the conditions 1, 2, 3, 5, 6, from th. 1 are equivalent.*

Proof. As remarked above, it is enough to prove the implication 2$\Rightarrow$3. Let us consider first the case $A=[u<v]$, with $u,v\in D(\varphi)$. If $f=u-v$:

$$\varphi(B^A s)=\varphi(B_f s)=\varphi(\bigvee_{n\in N} R(s\wedge nf))=\bigvee_{n\in N}\varphi(R(s\wedge nf))\geqslant\bigvee_{n\in N} R(\varphi(s)\wedge n\varphi(f))=$$
$$=B_{\varphi(f)}(\varphi(s))=B^{\varphi^{-1}(A)}\ (\varphi(s)).$$

If A is naturally open, then there exists an increasing sequence (A_n) of naturally open sets of the form $[u_n<v_n]$, such that $A=\bigcup A_n$. Indeed, if $u\leqslant v$ and $u_1\leqslant v_1$ everywhere, then: $[u<v]\cup[u_1<v_1]=[u+u_1<v+v_1]$. If follows that 3) holds for any A naturally open. Let now A be arbitrary. Then [4, prop. 3.2.4] shows that $B^A s=\wedge B^U s$, with $A\subseteq U$, and U naturally open. Using the hypothese, we get:

$$\varphi(B^A s)=\varphi(\wedge B^U s)=\wedge\varphi(B^U s)\geqslant\wedge B^{\varphi^{-1}(U)}\ (\varphi(s))\geqslant B^{\varphi^{-1}(A)}\ (\varphi(s)).$$

Remark. The hypothesis "$\varphi(1)\in T$" is used to prove that the elements from $D(\varphi)$ define the natural topology. Hence it can be replaced by "φ is semi-finite" (morphism in the sense of [4]). We end with a result which shows that the morphisms considered above are not too different from H-maps.

Proposition 9. *Let S, T be H-cones of functions on X, Y. If* $\varphi\in\mathrm{Hom}$ $(S,\ T)$ *then the following are equivalent:*

1) $\varphi(1)<\infty$ *on* Y *and* $\forall s,\ t\in D(\varphi):\ \varphi(s\wedge t)=\varphi(s)\wedge\varphi(t)$.

2) *There exists the maps:* $\psi: Y\to X_1$ *and* $\alpha: Y\to\mathbf{R}_+$ *such that:* $\forall s\in S$, $\forall y\in Y:\varphi(s)\ (y)=\alpha(y)\ .\ s(\psi(y))$.

Remark. If we know that $\varphi^*(\mu)\in S^*$, for any $\mu\in T^*$ from an extreme ray (for exemple when φ^* is finite), then 1) is equivalent with the fact that: $(\mu\in T^*$ is an extreme ray$)\Rightarrow(\varphi^*(\mu)$ is on extreme ray$)$. In this situation, φ^* induces a correspondence $Ex(T)\to Ex(S)$.

BIBLIOGRAPHY

1. Bauerman U. — *Balayage-Operatoren in der Potentialtheorie.* Math. Ann. 231, 181—186 (1977)
2. Bliedtner J., Hansen W. — *Bases in Standard Balayage Spaces. Potential Theory,* Copenhagen 1979, 55—63. In: L.N.M. vol. 787, Berlin, Heidelberg, New-York: Springer., 1980
3. Boboc N., Bucur Gh. — *Natural Sheaf Property in Standard H-cones of Functions.* Preprint INCREST 32/1982.
4. Boboc N., Bucur Gh., Cornea A. — *Order and Convexity in Potential Theory: H-cones,* L.N.M. vol. 853, Springer, 1981
5 Popa E. — *Morphisms of H-cones,* Preprint, INCREST, 48/1980
6. Popa E. — *Morphisms of H-cones I,* An. şt. Univ. Iaşi, t. XXIX, f. 2, 1983, pg. 53—62
7. Popa E. — *Morphisms of H-cones II,* An. şt. Univ., Iaşi, t. XXX, f. 1, 1984, p. 3—2

Received 14.IV.1983

Seminarul matematic
Universitatea „Al. I. Cuza"
6600 — Iaşi, România

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII „AL. I. CUZA" DIN IAŞI
Tomul XXX, s.I a. Matematică, 1984—3

ASYMPTOTIC PROPERTIES OF SOLUTIONS OF A VOLTERRA INTEGRAL EQUATION WITH DELAY

BY

OLUSOLA AKINYELE*

§1. Introduction. In this paper we consider the Volterra integral equation with delay of the form:

$$u(t)=h(t)+\int_0^t K(t,s)f\ (s,\ u(s),\ u(\sigma(s)))\ ds \tag{1}$$

where $t\in[0,\ \infty)$ and the following hold:

$$(H_1)\quad\begin{cases}\text{(i) } f(t,\ u,\ v) \text{ is continuous on } [0,\ \infty)\times R\times R,\\ \text{(ii) } h(t) \text{ is continuous on } [0,\ \infty).\\ \text{(iii) } \sigma(t) \text{ is continuous, } \sigma(t)\to\infty \text{ as } t\to\infty \text{ and } 0\leqslant\sigma(t)\leqslant t.\end{cases}$$

For the equation (1) the asymptotic properties have been given by Ize [2] for $\sigma(t)=t$ under certain hypothesis on $f(t,\ u,\ v)$. Recently Werbowski [3] obtained results on asymptotic properties of (1) under hypotheses different from those considered by Ize in [2]. The basic tool in Werbowski's investigation was a non-linear version of the fundamental inequality of Gronwall-Bellman type. The results here are extensions of Werbowski's result to equation (1), with hypotheses on f which are much more general than those considered in [3]. We first establish a new integral inequality of the Gronwall-Bihari type and then use our inequality to obtain results on asymptotic behaviour of solutions of (1) which generalize results of [3].

§2. Main Results. We use the following assumptions in our study:

(H$_2$) The functions $\Phi_j:[0,\ \infty)\to[0,\ \infty)$, $j=1,\ 2$ are positive, nondecreasing, continuous and satisfy

(i) $\frac{1}{v}\ \Phi_j(u)\leqslant\Phi_j\left(\frac{u}{v}\right)$ for $u\geqslant 0,\ v\geqslant 1,\ j=1,\ 2.$

(ii) $\Phi_j(u)$ are submultiplicative for $u\geqslant 0.$

(H$_3$) There exist continuous, real-valued positive functions $a(t)$ and $b(t)$ defined for $t\geqslant 0$ such that

$$|\,f(t,\ u,\ v)\ |\leqslant a(t)\ \Phi_1(|\ u\ |)+b(t)\ \Phi_2(|\ v\ |),$$

* This paper was written during the author's visit at Iowa State University, Ames, Iowa.