$$\varphi(B^A s)=\varphi(B_f s)=\varphi(\bigvee_{n\in N} R(s\wedge nf))=\bigvee_{n\in N}\varphi(R(s\wedge nf))\geq\bigvee_{n\in N} R(\varphi(s)\wedge n\varphi(f))=$$

$$=B_{\varphi(f)}(\varphi(s))=B^{\varphi^{-1}(A)}\ (\varphi(s)).$$

If A is naturally open, then there exists an increasing sequence (A_n) of naturally open sets of the form $[u_n<v_n]$, such that $A=\bigcup A_n$. Indeed, if $u\leq v$ and $u_1\leq v_1$ everywhere, then: $[u<v]\cup[u_1<v_1]=[u+u_1<v+v_1]$. If follows that 3) holds for any A naturally open. Let now A be arbitrary. Then [4, prop. 3.2.4] shows that $B^A s=\wedge B^U s$, with $A\subseteq U$, and U naturally open. Using the hypothese, we get:

$$\varphi(B^A s)=\varphi(\wedge B^U s)=\wedge\varphi(B^U s)\geq\wedge B^{\varphi^{-1}(U)}\ (\varphi(s))\geq B^{\varphi^{-1}(A)}\ (\varphi(s)).$$

Remark. The hypothesis "$\varphi(1)\in T$" is used to prove that the elements from $D(\varphi)$ define the natural topology. Hence it can be replaced by "φ is semi-finite" (morphism in the sense of [4]). We end with a result which shows that the morphisms considered above are not too different from H-maps.

Proposition 9. *Let S, T be H-cones of functions on X, Y. If* $\varphi\in\mathrm{Hom}(S, T)$ *then the following are equivalent:*

1) $\varphi(1)<\infty$ *on* Y *and* $\forall s,\ t\in D(\varphi):\ \varphi(s\wedge t)=\varphi(s)\wedge\varphi(t)$.

2) *There exists the maps*: $\psi: Y\to X_1$ *and* $\alpha: Y\to\mathbf{R}_+$ *such that*: $\forall s\in S$, $\forall y\in Y:\varphi(s)\,(y)=\alpha(y)\,.\,s(\psi(y)))$.

Remark. If we know that $\varphi^*(\mu)\in S^*$, for any $\mu\in T^*$ from an extreme ray (for exemple when φ^* is finite), then 1) is equivalent with the fact that: $(\mu\in T^*$ is an extreme ray$)\Rightarrow(\varphi^*(\mu)$ is on extreme ray). In this situation, φ^* induces a correspondence $Ex(T)\to Ex(S)$.

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ASYMPTOTIC PROPERTIES OF SOLUTIONS OF A VOLTERRA INTEGRAL EQUATION WITH DELAY

BY

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§1. Introduction. In this paper we consider the Volterra integral equation with delay of the form:

$$u(t)=h(t)+\int_0^t K(t,s)f\,(s,u(s),\ u(\sigma(s)))\,ds \qquad (1)$$

where $t\in[0,\infty)$ and the following hold:

$$(H_1)\quad\begin{cases}\text{(i) } f(t,u,v) \text{ is continuous on } [0,\infty)\times R\times R,\\ \text{(ii) } h(t) \text{ is continuous on } [0,\infty).\\ \text{(iii) } \sigma(t) \text{ is continuous, } \sigma(t)\to\infty \text{ as } t\to\infty \text{ and } 0\leq\sigma(t)\leq t.\end{cases}$$

For the equation (1) the asymptotic properties have been given by Ize [2] for $\sigma(t)=t$ under certain hypothesis on $f(t,u,v)$. Recently Werbowski [3] obtained results on asymptotic properties of (1) under hypotheses different from those considered by Ize in [2]. The basic tool in Werbowski's investigation was a non-linear version of the fundamental inequality of Gronwall-Bellman type. The results here are extensions of Werbowski's result to equation (1), with hypotheses on f which are much more general than those considered in [3]. We first establish a new integral inequality of the Gronwall-Bihari type and then use our inequality to obtain results on asymptotic behaviour of solutions of (1) which generalize results of [3].

§2. Main Results. We use the following assumptions in our study:

(H_2) The functions $\Phi_j:[0,\infty)\to[0,\infty)$, $j=1, 2$ are positive, nondecreasing, continuous and satisfy

(i) $\frac{1}{v}\Phi_j(u)\leq\Phi_j\left(\frac{u}{v}\right)$ for $u\geq0$, $v\geq1$, $j=1, 2$.

(ii) $\Phi_j(u)$ are submultiplicative for $u\geq0$.

(H_3) There exist continuous, real-valued positive functions $a(t)$ and $b(t)$ defined for $t\geq0$ such that

$$|f(t,u,v)|\leq a(t)\,\Phi_1(|u|)+b(t)\,\Phi_2(|v|),$$

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(H_4) $K(t, s)$ is continuous on $0 \leq s \leq t \leq T$ and there exists a non-negative continuous function $Q(t, s)$ such that

$$|K(t, s)| \leq Q(t, s) \text{ for } t \geq s \geq 0.$$

In order to establish our results on the asymptotic behaviour of (1) we need the following integral inequality which is new to the literature.

Theorem 1. *Let $\sigma(t)$ be as in (H_1) (iii) and Φ_j, $j=1, 2$. as in (H_2). Let $u(t)$, $p(t)$, $q(t)$, $a(t)$ and $b(t)$ be non-negative continuous functions on $[0, \infty)$, such that the following inequality holds:*

$$(2) \qquad u(t) \leq p(t) + q(t) \int_0^t [a(s)\, \Phi_1(u(s)) + b(s)\, \Phi_2(u(\sigma(s)))]\, ds.$$

Then for all $t > 0$, $u(t)$ satisfies the inequality,

$$(3) \qquad u(t) \leq p(t) + q(t)\, G^{-1}\Big[\int_0^t \{a(s)\, \Phi_1(g(s)) + b(s)\, \Phi_2(g(\sigma(s)))\}\, ds\Big],$$

where $g(t) = \max\{p(t), q(t)\}$ and G^{-1} is the inverse function such that

$$(4) \qquad G(\omega) = \int_0^\omega \frac{ds}{\Phi(1+s)}, \text{ with } \Phi(u) = \max\{\Phi_1(u), \Phi_2(u)\}.$$

Proof.

$$\text{Set } R(t) = \int_0^t \{a(s)\, \Phi_1(u(s)) + b(s)\, \Phi_2(u(\sigma(s)))\}\, ds.$$

then (2) becomes

$$(5) \qquad u(t) \leq p(t) + q(t)\, R(t) \leq g(t)\, \{1 + R(t)\}$$

Differentiating $R(t)$, and using (5), we have

$$R'(t) = a(t)\, \Phi_1(u(t)) + b(t)\, \Phi_2(u(\sigma(t)))$$

$$\leq a(t)\, \Phi_1(g(t))\, \Phi_1(1+R(t)) + b(t)\, \Phi_2(g(\sigma(t)))\, \Phi_2(1+R(\sigma(t))).$$

Φ_2 is non-decreasing and $\sigma(t) \leq t$, hence

$$R'(t) \leq \{a(t)\, \Phi_1(g(t)) + b(t)\, \Phi_2(g(\sigma(t)))\}\, \Phi(1+R(t))$$

Thus, for $\Phi(1+R(t)) \neq 0$,

$$(6) \qquad \frac{R'(t)}{\Phi(1+R(t))} \leq a(t)\, \Phi_1(g(t)) + b(t)\, \Phi_2(g(\sigma(t))).$$

Integrating (6) from 0 to t and using (4),

$$G(R(t)) - G(R(0)) \leq \int_0^t \{a(s)\, \Phi_1(g(s)) + b(s)\, \Phi_2(g(\sigma(s)))\}\, ds.$$

Thus, since $R(0) = 0$,

$$(7) \qquad R(t) \leq G^{-1}\Big[\int_0^t \{a(s)\, \Phi_1(g(s)) + b(s)\, \Phi_2(g(\sigma(s)))\}\, ds\Big].$$

Putting (7) into (5) we have (3). The case $\Phi(1+R(t))$ leads to $R'(t) \leq 0$. Hence $R(t) \equiv 0$ and $\Phi_1 \equiv 0$, $\Phi_2 \equiv 0$ so that (2) becomes $u(t) \leq P(t)$ and the Theorem reduces to a triviality.

Remark. If $\Phi_1(u) = u^\alpha$ and $\Phi_2(u) = u^\beta$ with $\alpha, \beta \in (0, 1]$, Theorem 1 reduces to a result of Bobrowski, Popenda and Werbrowski [1].

We now use our result to establish conditions for the asymptotic behaviour of (1). The following Lemma was proved in [3].

Lemma 2. *Let $w(t) > 0$ be a continuous non-decreasing function on $[0, \infty)$ such that $\lim_{t\to\infty} w(t) = \infty$* and $\lim_{t\to\infty} w(r(t))/w(t) = 0$ for some function $r(t)$ such that $r(t) < t$ and eventually $\lim_{t\to\infty} r(t) = \infty$. Suppose

$$\int_T^\infty \frac{F(s)}{w(s)}\, ds < \infty \text{ for } F(s) \geq 0, \text{ then } \lim_{t\to\infty} \frac{1}{w(t)} \int_T^t F(s)\, ds = 0.$$

Theorem 3. *Let $w(t) \geq 1$ satisfy the hypothesis of Lemma 2. Suppose $(H_1)-(H_3)$ are satisfied with $Q(t, s)$ a non-increasing function in t for each s. Assume that*

(i) $\displaystyle \lim_{t\to\infty} \frac{h(t)}{w(t)} < \infty.$

(ii) $\displaystyle \int^\infty Q(s, s)\, \{a(s) + b(s)\}\, ds < \infty,$

(iii) $\displaystyle \int^\infty \frac{Q(s, s)}{w(s)}\, a(s)\, \Phi_1(w(s))\, ds < \infty,$

(iv) $\displaystyle \int^\infty \frac{Q(s, s)}{w(s)}\, b(s)\, \Phi_2(w(\sigma(s)))\, ds < \infty.$

Then $\lim_{t\to\infty} u(t)/w(t)$ exists and $\displaystyle \lim_{t\to\infty} u(t)/w(t) = \lim_{t\to\infty} \frac{h(t)}{w(t)}.$

Proof. Let $\displaystyle \lim_{t\to\infty} \frac{h(t)}{w(t)} = C < \infty$, then $\exists\, C_1 \ni \left|\frac{h(t)}{w(t)}\right| \leq C_1$ for all t.

Dividing (1) by $w(t)$, we have

$$(8) \qquad \frac{u(t)}{w(t)} = \frac{h(t)}{w(t)} + \frac{1}{w(t)} \int_0^t K(t, s)\, f(s, u(s), u(\sigma(s)))\, ds$$

and by assumptions.

$$\left|\frac{u(t)}{w(t)}\right| \leq \left|\frac{h(t)}{w(t)}\right| + \frac{1}{w(t)} \int_0^t Q(t, s)\, [a(s)\, \Phi_1(|u(s)|) + b(s)\, \Phi_2(|u(\sigma(s))|)]\, ds \leq$$

$$\leq C_1 + \frac{1}{w(t)} \int_0^t Q(s, s)\, \{a(s)\, \Phi_1(|u(s)|) + b(s)\, \Phi_2(|u(\sigma(s))|)\}\, ds \leq$$

$$\leqslant C_1+\int_0^t\left\{Q(s,s)\,a(s)\,\Phi_1\left(\left|\frac{u(s)}{w(s)}\right|\right)Q(s,s)\,b(s)\,\Phi_2\left(\left|\frac{u(\sigma(s))}{w(\sigma(s))}\right|\right)\right\}ds.$$

Using Theorem 1, with $g(t)=\max\{C_1, 1\}=C_2$, we have

$$\left|\frac{u(t)}{w(t)}\right|\leqslant C_1+G^{-1}\left[\Phi(C_2)\int_0^t Q(s,s)\,\{a(s)+b(s)\}\,ds\right]$$

where $G(\omega)=\int_0^\omega ds/\Phi(1+s)$ and so G^{-1} exists and is increasing.

Hence,

$$\left|\frac{u(t)}{w(t)}\right|\leqslant C_1+G^{-1}\left[\Phi(C_2)\int_0^\infty Q(s,s)\,\{a(s)+b(s)\}ds\right.$$

and by (ii), $\exists C_3>0$, constant such that $C_3=C_1+G^{-1}\left[\Phi(C_2)\int_0^\infty Q(s,s)\,\{a(s)+ +b(s)\}\,ds\right]$, so that

$|u(t)|\leqslant C_3 w(t)$. for $t>0$. So $\lim_{t\to\infty}|u(t)/w(t)|$ exists.

Now

$$\left|\frac{1}{w(t)}\int_0^t K(t,s)\,f(s,u(s),\ u(\sigma(s)))\,ds\right|$$

$$\leqslant\frac{1}{w(t)}\int_0^t Q(s,s)\,[a(s)\,\Phi_1(|u(s)|)+b(s)\,\Phi_2(|u(\sigma(s))|)]\,ds\leqslant$$

(9)
$$\leqslant\frac{1}{w(t)}\int Q(s,s)\,[a(s)\,\Phi_1(C_3)\,\Phi_1(w(s))+b(s)\,\Phi_2(C_3)\,\Phi_2(w(\sigma(s)))]\,ds\leqslant$$

$$\leqslant\Phi(C_3)\cdot\frac{1}{w(t)}\int_0^t Q(s,s)\,[a(s)\,\Phi_1(w(s)(+b(s))\,\Phi_2(w(\sigma(s)))]\,ds.$$

Set $F(s)=Q(s,s)\,[a(s)\,\Phi_1(w(s))+b(s)\,\Phi_2(w(\sigma(s)))]$ in Lemma 2, then $\int_T^\infty F(s)/w(s)]\,ds<\infty$ by (iii) and (iv). So Lemma 2 implies $\lim_{t\to\infty}(1/w(t)\int_0^t(s)\,ds=0$ and taking the limit as $t\to\infty$ in (9), $\lim_{t\to\infty}|1/w(t)\int_0^t K(t,s)\,f(s,u(s),u(\sigma(s)))\,ds|=0$. As $t\to\infty$ in (8) $\lim_{t\to\infty}u(t)/w(t)=\lim_{t\to\infty}h(t)/w(t)=C<\infty$ and the proof is complete.

Theorem 4. *Let $(H_1)-(H_3)$ be satisfied and $w(t)\geqslant 1$ be as in Lemma 2. Let $Q(t,s)=p(t)\,q(s)$ where $p(t)\geqslant 1$ and non-decreasing, with $q(s)\geqslant 0$. Assume that*

(i) $$\lim_{t\to\infty}[h(t)/(w(t)\,p(t))<\infty,$$

(ii) $$\int^\infty p(s)\,q(s)\,\{a(s)+b(s)\}\,ds<\infty.$$

(iii) $$\int^\infty\frac{q(s)}{w(s)}\{a(s)\,\Phi_1(w(s))\,\Phi_1(p(s))\}\,ds<\infty,$$

(iv) $$\int^\infty\frac{q(s)}{w(s)}\{b(s)\,\Phi_2(w(\sigma(s)))\,\Phi_3(p(\sigma(s)))\}\,ds<\infty.$$

Then $\lim_{t\to\infty}u(t)/[w(t)]\,p(t)$ exists and is equal to $\lim_{t\to\infty}h(t)/[w(t)\,p(t)]$.

Proof. Suppose $\lim_{t\to\infty}h(t)/[w(t)\,p(t)]=C_4<\infty$, then

(10) $$\frac{u(t)}{w(t)\,p(t)}=\frac{h(t)}{w(t)\,p(t)}+\frac{1}{w(t)\,p(t)}\int_0^t K(t,s)\,f(s,u(s),\ u(\sigma(s)))\,ds.$$

So $\exists\ C_5$ such that,

$$\left|\frac{u(t)}{w(t)\,p(t)}\right|\leqslant C_5+\frac{1}{w(t)}\int_0^t q(s)\,[a(s)\,\Phi_1(|u(s)|)+b(s)\,\Phi_2(|u(\sigma(s))|)]\,ds$$

$$\leqslant C_5+\int_0^t p(s)\,q(s)\left[a(s)\,\Phi_1\left(\left|\frac{u(s)}{w(s)\,p(s)}\right|\right)+b(s)\,\Phi_2\left(\left|\frac{u(\sigma(s))}{w(\sigma(s))\,p(\sigma(s))}\right|\right)\right]ds.$$

Theorem 1 with $g(t)=\max\{C_5, 1\}=C_6$, implies

$$\left|\frac{u(t)}{w(t)\,p(t)}\right|\leqslant C_5+G^{-1}\left[\Phi(C_6)\int_0^t p(s)\,q(s)\,[a(s)+b(s)]\,ds\right]$$

and as in the last theorem using hypothesis (ii), $\exists$ a positive constant

$C_7\ni C_7=C_5+G^{-1}[\Phi(C_6)\int^\infty p(s)\,q(s)\{a(s)+b(s)\}\,ds]$ and

$|u(t)/[w(t)\,p(t)]|\leqslant C_7$ for $t\geqslant 0$. Moreover $|u(\sigma(t))|\leqslant C_7 w(\sigma(t))\,p(\sigma(t))$ for $t\geqslant 0$, hence

$$\left|\frac{1}{w(t)\,p(t)}\int_0^t K(t,s)\,f(s,u(s),\ u(\sigma(s)))\,ds\right|$$

$$\leqslant\frac{1}{w(t)\,p(t)}\int_0^t p(t)\,q(s)\,[a(s)\,\Phi_1(|u(s)|)+b(s)\,\Phi_2(|u(\sigma(s))|)]\,ds.$$

$$\leqslant \frac{1}{w(t)}\int_0^t q(s)\,[a(s)\,\Phi_1(C_7)\,\Phi_1(w(s)\,p(s))+b(s)\,\Phi_2(C_7)\,\Phi_2\,w(\sigma(s))\,p(\sigma(s)))]\,ds$$

$$\leqslant \Phi(C_7)\,\frac{1}{w(t)}\int_0^t q(s)\,[a(s)\,\Phi_1(w(s))\,\Phi_1(p(s))+b(s)\,\Phi_2(w(\sigma(s)))\,\Phi_2\,(p(\sigma(s)))]\,ds.$$

Setting $F(s)=q(s)\,[a(s)\,\Phi_1(w(s))\,\Phi_1(p(s))+b(s)\,\Phi_2(w(\sigma(s)))\,\Phi_2(p(\sigma(s)))]$ and applying Lemma 2, we obtain using (iii) and (iv), $\lim_{t\to\infty}[1/w(t)]\int_0^t F(s)\,ds=0$, Hence $\lim_{t\to\infty} 1/[w(t)\,p(t)]\int_0^t K(t,s)\,f(s,\ u(s),\ u(\sigma(s)))\,ds=0$ and the result follows from (10).

Remark. If $\Phi_1(u)=u^\alpha$ and $\Phi_2(u)=u^\beta$ with $0<\alpha,\ \beta\leqslant 1$ then $\Phi(u)=u^\gamma$ where $\gamma=\max\{\alpha,\beta\}$ and we have as special cases of our Theorems 3 and 4 results of Werbrowski [3].

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ON A NONLINEAR FUNCTIONAL INTEGRAL EQUATION IN TWO INDEPENDENT VARIABLES

BY

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1. Introduction. In the present paper we consider the nonlinear functional integral equation of the form

$$z(x,y)=F\Big(x,y,\int_0^{a(x,y)}\int_0^{b(x,y)} f_1(x,y,s,t,z(s,t))\,ds\,dt,\ \int_0^{c(x,y)} f_2(x,y,s,z(s,p(x,y)))\,ds,\ \int_0^{d(x,y)} f_3(x,y,t,z(q(x,y),t))\,dt,\ z(g(x,y),\ h(x,y)),\ u\Big)=(T_u z)(x,y) \tag{1}$$

where $z(x,y)$ is an unknown function. Equation (1) is a generalization of the integral equation recently studied by Kwapisz [4] and contain as special cases the functional equations studied by many authors (see, [2, 3, 9]). The Darboux problem for the functional differential equation

$$z_{xy}(x,y)=F(x,y,z(.,.),\ z_x(.,.),\ z_y(.,.),\ z_{xy}(g(x,y),\ h(x,y))), \tag{2}$$

with $z(x,0)=\sigma(x)$, $0\leqslant x\leqslant\alpha$, $z(0,y)=\tau(y)$, $0\leqslant y\leqslant\beta$ can be reduced to a particular case of equation (1). Recently, Kwapisz and Turo [5, 6, 7] have applied the general idea of Wazewski [10] to study the existence and uniqueness of solutions of equation (2) and its more general form. The main concern of this paper is to study the existence, uniqueness and continuous dependence on parameter of solutions of the functional integral equation (1). Our results in this paper are obtained by using the well known Banach contraction mapping principle [8], monotone iterative method and the notion of upper and lower solutions.

2. Existence and uniqueness. In this section we shall establish our results on the existence of a unique solution and continuous dependence on parameter of the solutions of equation (1). For convenience we first list the following hypotheses.

Hypothesis 1. (i) Let $(E, \|\cdot\|)$ be a Banach space, $\Delta=[0,\alpha]\times[0,\beta]$, $\Delta_1=\{(x,y,s,t): 0\leqslant s\leqslant x\leqslant\alpha,\ 0\leqslant t\leqslant y\leqslant\beta\}$, $\Delta_2=\{(x,y,s): 0\leqslant s\leqslant x\leqslant\alpha,\ 0\leqslant y\leqslant\beta\}$, $\Delta_3=\{(x,y,t): 0\leqslant x\leqslant\alpha,\ 0\leqslant t\leqslant y\leqslant\beta\}$. The functions a, b, c, d, p, q, g, h are defined and continuous on Δ and $a(x,y)$, $c(x,y)$, $q(x,y)$, $g(x,y)\in[0,\alpha]$ and $b(x,y)$, $d(x,y)$, $p(x,y)$, $h(x,y)\in[0,\beta]$, $f_1\in C[\Delta_1\times E, E]$,