$$\leqslant \frac{1}{w(t)}\int_0^t q(s)\,[a(s)\,\Phi_1(C_7)\,\Phi_1(w(s)\,p(s))+b(s)\,\Phi_2(C_7)\,\Phi_2\,w(\sigma(s))\,p(\sigma(s)))]\,ds$$

$$\leqslant \Phi(C_7)\,\frac{1}{w(t)}\int_0^t q(s)\,[a(s)\,\Phi_1(w(s))\,\Phi_1(p(s))+b(s)\,\Phi_2(w(\sigma(s)))\,\Phi_2\,(p(\sigma(s)))]\,ds.$$

Setting $F(s)=q(s)\,[a(s)\,\Phi_1(w(s))\,\Phi_1(p(s))+b(s)\,\Phi_2(w(\sigma(s)))\,\Phi_2(p(\sigma(s)))]$ and applying Lemma 2, we obtain using (iii) and (iv), $\lim_{t\to\infty}[1/w(t)]\int_0^t F(s)\,ds=0$, Hence $\lim_{t\to\infty} 1/[w(t)\,p(t)]\int_0^t K(t,s)\,f(s,\,u(s),\,u(\sigma(s)))\,ds=0$ and the result follows from (10).

Remark. If $\Phi_1(u)=u^\alpha$ and $\Phi_2(u)=u^\beta$ with $0<\alpha,\ \beta\leqslant 1$ then $\Phi(u)=u^\gamma$ where $\gamma=\max\{\alpha,\beta\}$ and we have as special cases of our Theorems 3 and 4 results of Werbrowski [3].

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ON A NONLINEAR FUNCTIONAL INTEGRAL EQUATION IN TWO INDEPENDENT VARIABLES

BY

B. G. PACHPATTE

1. Introduction. In the present paper we consider the nonlinear functional integral equation of the form

$$z(x,y)=F\Big(x,y,\int_0^{a(x,y)}\int_0^{b(x,y)} f_1(x,y,s,t,z(s,t))\,ds\,dt, \tag{1}$$

$$\int_0^{c(x,y)} f_2(x,y,s,z(s,p(x,y)))\,ds,$$

$$\int_0^{d(x,y)} f_3(x,y,t,z(q(x,y),t))\,dt,\ z(g(x,y),\,h(x,y)),\,u\Big)=(T_u z)\,(x,y)$$

where $z(x,y)$ is an unknown function. Equation (1) is a generalization of the integral equation recently studied by Kwapisz [4] and contain as special cases the functional equations studied by many authors (see, [2, 3, 9]). The Darboux problem for the functional differential equation

$$z_{xy}(x,y)=F(x,y,z(.,.),\ z_x(.,.),\ z_y(.,.),\ z_{xy}(g(x,y),\,h(x,y))), \tag{2}$$

with $z(x,0)=\sigma(x)$, $0\leqslant x\leqslant\alpha$, $z(0,y)=\tau(y)$, $0\leqslant y\leqslant\beta$ can be reduced to a particular case of equation (1). Recently, Kwapisz and Turo [5, 6, 7] have applied the general idea of Wazewski [10] to study the existence and uniqueness of solutions of equation (2) and its more general form. The main concern of this paper is to study the existence, uniqueness and continuous dependence on parameter of solutions of the functional integral equation (1). Our results in this paper are obtained by using the well known Banach contraction mapping principle [8], monotone iterative method and the notion of upper and lower solutions.

2. Existence and uniqueness. In this section we shall establish our results on the existence of a unique solution and continuous dependence on parameter of the solutions of equation (1). For convenience we first list the following hypotheses.

Hypothesis 1. (i) Let $(E,\|\cdot\|)$ be a Banach space, $\Delta=[0,\alpha]\times\times[0,\beta]$, $\Delta_1=\{(x,y,s,t):\ 0\leqslant s\leqslant x\leqslant\alpha,\ 0\leqslant t\leqslant y\leqslant\beta\}$, $\Delta_2=\{(x,y,s):\ 0\leqslant s\leqslant\leqslant x\leqslant\alpha,\ 0\leqslant y\leqslant\beta\}$, $\Delta_3=\{(x,y,t):0\leqslant x\leqslant\alpha,\ 0\leqslant t\leqslant y\leqslant\beta\}$. The functions a, b, c, d, p, q, g, h are defined and continuous on Δ and $a(x,y)$, $c(x,y)$, $q(x,y)$, $g(x,y)\in[0,\alpha]$ and $b(x,y)$, $d(x,y)$, $p(x,y)$, $h(x,y)\in[0,\beta]$, $f_1\in C[\Delta_1\times E,\,E]$,

$f_2 \in C[\Delta_2 \times E, E]$, $f_3 \in C[\Delta_3 \times E, E]$, $F \in C[\Delta \times E^4 \times R, E]$, where $R = (-\infty, \infty)$.

(ii) For $u \in R$, $(x, y, z_1, z_2, z_3, z_4, u)$, $(x, y, \bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4, u) \in \Delta \times E^4 \times R$,

$$\| F(x, y, z_1, z_2, z_3, z_4, u) - F(x, y, \bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4, u) \| \leq L_1 \| z_1 - \bar{z}_1 \| + L_2 \| z_2 - \bar{z}_2 \| + L_3 \| z_3 - \bar{z}_3 \| + M \| z_4 - \bar{z}_4 \|,$$

and

$$\| f_1(x, y, s, t, z) - f_1(x, y, s, t, \bar{z}) \| \leq M_1 \| z - \bar{z} \|,$$

$$\| f_2(x, y, s, z) - f_2(x, y, s, \bar{z}) \| \leq M_2 \| z - \bar{z} \|, \quad \| f_3(x, y, t, z) - f_3(x, y, t, \bar{z}) \| \leq M_3 \| z - \bar{z} \|,$$

where L_i, M_i and M are nonnegative constants for $i = 1, 2, 3$.

(iii) Let L be a positive number. For every fixed $u \in R$ there exists a constant $P \geq 0$ and $(x, y, z_1, z_2, z_3, z_4, u) \in \Delta \times E^4 \times R$ such that

$$\| (T_u \theta)(x, y) \| \leq P \exp(L(x+y)),$$

where θ is the zero element in E.

(iv) There exist constants N_1, N_2 such that $0 \leq N_1 + N_2 < 1$ and

$$\bar{L}^2 L_1 M_1 \exp(L[a(x, y) + b(x, y)]) + \bar{L}^1 L_2 M_2 \exp(L[c(x, y) + p(x, y)]) + \bar{L}^1 L_3 M_3 \exp(L[q(x, y) + d(x, y)]) \leq N_1 \exp(L(x+y)),$$

$$M \exp(L[g(x, y) + h(x, y)]) \leq N_2 \exp(L(x+y)).$$

Hypothesis 2. There exist a constant $K \geq 0$ and a function $A(x, y) : \Delta \to R_+ = [0, \infty)$ such that for $i = 1, 2$, $(x, y, z_1, z_2, z_3, z_4, u_i) \in \Delta \times E^4 \times R$,

$$\| F(x, y, z_1, z_2, z_3, z_4, u_1) - F(x, y, z_1, z_2, z_3, z_4, u_2) \| \leq A(x, y) \mid u_1 - u_2 \|,$$

and

$$\sup_{(x, y) \in \Delta} [A(x, y) \exp(-L(x+y))] \leq K,$$

where $\mid \eta \|$ denotes the absolute value of η.

Let G be a space of those functions $\Phi : \Delta \to E$ which are continuous in Δ and fulfil the condition

$$\| \Phi(x, y) \| = 0(\exp(L(x+y))). \tag{3}$$

In the space G we define the norm (see, [1])

$$\mid \Phi \mid = \sup_{(x, y) \in \Delta} [\| \Phi(x, y) \| \exp(-L(x+y))]. \tag{4}$$

It is easily seen that G with norm defined in (4) is a Banach space (see, [2]).

We note that the condition (3) implies that there exists a constant $M_0 \geq 0$ such that $\| \Phi(x, y) \| \leq M_0 \exp(L(x+y))$, $(x, y) \in \Delta$. Using this fact in (4) we observe that

$$\mid \Phi \mid \leq M_0. \tag{5}$$

Now we shall prove the following

Theorem 1. *Let Hypothesis 1 be fulfilled. Then for every $u \in R$ there exists exactly one solution $z \in G$ of equation (1), given as the limit of successiv approximations.*

Proof. Let $u \in R$ be fixed. We shall prove that T_u maps G into itself. Evidently $T_u z$ is continuous in Δ and $(T_u z)(x, y) \in E$ for $(x, y) \in \Delta$. We verify that (3) is fulfilled. Using (ii) — (iii) we obtain

$$\| (T_u z)(x, y) \| \leq \| (T_u z)(x, y) - (T_u 0)(x, y) \| + \| (T_u 0)(x, y) \| \leq$$

$$\leq L_1 M_1 \int_0^{a(x, y)} \int_0^{b(x, y)} \| z(s, t) \| \exp(-L(s+t)) \exp(L(s+t))\, ds dt +$$

$$+ L_2 M_2 \int_0^{c(x, y)} \| z(s, p(x, y)) \| \exp(-L[s + p(x, y)]) \exp(L[s + p(x, y)])\, ds +$$

$$+ L_3 M_3 \int_0^{d(x, y)} \| z(q(x, y), t) \| \exp(-L[q(x, y) + t]) \exp(L[q(x, y) + t])\, dt +$$

$$+ M \| z(g(x, y), h(x, y)) \| \exp(-L[g(x, y) + h(x, y)]) \exp(L[g(x, y) + h(x, y)]) + P \exp(L(x+y))$$

$$\leq \mid z \mid [\bar{L}^2 L_1 M_1 \exp(L[a(x, y) + b(x, y)]) + \bar{L}^1 L_2 M_2 \exp(L[p(x, y) + c(x, y)]) + \bar{L}^1 L_3 M_3 \exp(L[q(x, y) + d(x, y)]) + M \exp(L[g(x, y) + h(x, y)])] + P \exp(L(x+y)).$$

Using (iv) and (5) we obtain

$$\| (T_u z)(x, y) \| \leq [M_0 N_1 + M_0 N_2 + P)] \exp(L(x+y)).$$

From which it follows that $T_u z \in G$.

Now we verify that the transformation $T_u z$ is a contraction map. We assume that $z, \bar{z} \in G$. From Hypothesis 1, we have

$$\| (T_u z)(x, y) - (T_u \bar{z})(x, y) \| \leq L_1 M_1 \int_0^{a(x, y)} \int_0^{b(x, y)} \| z(s, t) - \bar{z}(s, t) \| \exp(-L(s+t)) \exp(L(s+t))\, ds dt + L_2 M_2 \int_0^{c(x, y)} \| z(s, p(x, y)) - \bar{z}(s, p(x, y)) \| \exp(-L[s + p(x, y)]) \exp(L[s + p(x, y)])\, ds + L_3 M_3 \int_0^{d(x, y)} \| z(q(x, y), t) - \bar{z}(q(x, y), t) \| \exp(-L[q(x, y) + t]) \exp(L[q(x, y) + t])\, dt +$$

$$+ M \| z(g(x, y), h(x, y)) - \bar{z}(g(x, y), h(x, y)) \| \exp(-L[g(x, y) + h(x, y)]) \cdot \exp(L[g(x, y) + h(x, y)])$$

$$\leq \mid z - \bar{z} \mid [\bar{L}^2 L_1 M_1 \exp(L[a(x, y) + b(x, y)]) + \bar{L}^1 L_2 M_2 \exp(L[c(x, y) + p(x, y)]) + \bar{L}^1 L_3 M_3 \exp(L[q(x, y) + d(x, y)]) + M \exp(L[g(x, y) + h(x, y)])] \leq \mid z - \bar{z} \mid [N_1 + N_2] \exp(L(x+y)).$$

Consequently, we have

(6) $$| T_u z - T_u \bar{z} | \leqslant (N_1 + N_2) | z - \bar{z} |.$$

By using the known Banach's fixed point principle for contraction map [8] for every fixed $u \in R$ there exists a unique fixed point of transformation T_u, i.e. there is a unique solution $z \in G$ of equation (1), and it is given as the limit of the successive approximations. This completes the proof the theorem.

Our next theorem deals with the problem of continuous dependence of solutions of equation (1) on a parameter u.

Theorem 2. *If Hypotheses 1 and 2 are fulfilled, the solution $z(x, y, u)$ of equation (1) belonging to G is continuous with respect to the variables (x, y, u) in $\Delta \times R$.*

Proof. Let $z, \bar{z} \in G$, then from (6) we have

(7) $$| T_u(z) - T_u(\bar{z}) | \leqslant (N_1 + N_2) | z - \bar{z} |.$$

Next from Hypothesis 2, we obtain

(8) $$| T_u(z) - T_{u_0}(z) | \leqslant \sup_{(x, y) \in \Delta} [A(x, y) \| u - u_0 \| \exp(-L(x+y))] \leqslant K \| u - u_0 \|.$$

From Theorem 1 there exists a unique function $z(x, y, u)$ such that $T_u(z(x, y, u)) = z(x, y, u)$ and $T_{u_0}(z(x, y, u_0)) = z(x, y, u_0)$ for $(x, y) \in \Delta$. Therefore, from (7) and (8) we have

$$| z(x, y, u) - z(x, y, u_0) | \leqslant | T_u(z(x, y, u)) - T_u(z(x, y, u_0)) | + | T_u(z(x, y, u_0)) -$$
$$- T_{u_0}(z(x, y, u_0)) | \leqslant (N_1 + N_2) | z(x, y, u) - z(x, y, u_0) | + K \| u - u_0 \|.$$

Hence

$$| z(x, y, u) - z(x, y, u_0) | \leqslant (1 - N_1 - N_2)^{-1} K \| u - u_0 \|.$$

This shows that the function $z(x, y, u)$ is continuous with respect to the variable u in R uniformly with respect to the variables (x, y) in Δ, and consequently $z(x, y, u)$ is also continuous with respect to the variables (x, y, u) in $\Delta \times R$, which completes the proof.

3. Maximal and minimal solutions. In this section we employ the notion of upper and lower solutions to study the existence of maximal and minimal solutions of equation (1) when $u = 0$ and describe how these functions become upper and lower bounds of solutions of the equation.

$$z(x, y) = F\Big(x, y, \int_0^{a(x, y)} \int_0^{b(x, y)} f_1(x, y, s, t, z(s, t))\, ds dt, \int_0^{c(x, y)} f_2(x, y, s, z(s, p(x, y)))ds,$$
(9) $$\int_0^{d(x, y)} f_3(x, y, t, z(q(x, y), t))\, dt, z(g(x, y), h(x, y))\Big) = (T_0 z)(x, y)$$

where $z(x, y)$ is an unknown function. Hereafter, without further mention, we assume that $E = R^n$ the n dimensional Euclidean space and all inequalities between vectors are componentwise.

Definition 1. *A function $w \in C[\Delta, R^n]$ is said to be an upper solution of (9) if*

$$w(x, y) \geqslant (T_0 w)(x, y),$$

and similarly, a function $v \in C[\Delta, R^n]$ is said to be a lower solution of (9) if

$$v(x, y) \leqslant (T_0 v)(x, y).$$

Definition 2. *The functions $\bar{z}, z \in C[\Delta, R^n]$ are called maximal and minimal solutions of (9) respectively, if every other solution $z \in C[\Delta, R^n]$ of (9) satisfies the relation $z(x, y) \leqslant z(x, y) \leqslant \bar{z}(x, y)$ for all $(x, y) \in \Delta$.*

In our further discussion we need the following hypotheses.

Hypothesis 3. (i) The functions $v, w \in C[\Delta, R^n]$ with $v(x, y) \leqslant w(x, y)$ for $(x, y) \in \Delta$ are lower and upper solutions of equation (9).

(ii) The functions a, b, c, d, p, q, g, h are defined and continuous on Δ and $a(x, y)$, $c(x, y)$, $q(x, y)$, $g(x, y) \in [0, \alpha]$ and $b(x, y)$, $d(x, y)$, $p(x, y)$, $h(x, y) \in [0, \beta]$, $f_1 \in C[\Delta_1 \times R^n, R^n]$, $f_2 \in C[\Delta_2 \times R^n, R^n]$, $f_3 \in C[\Delta_3 \times R^n\ R^n]$ and $F \in C[\Delta \times (R^n)^4, R^n]$. For each i, $f_{1i}(x, y, s, t, \xi)$, $f_{2i}(x, y, s, \xi)$, $f_{3i}(x, y, t, \xi)$ are monotone increasing in ξ for fixed $(x, y, s, t) \in \Delta_1$, $(x, y, s) \in \Delta_2$, $(x, y, t) \in \Delta_3$, whenever $v(x, y) \leqslant \xi \leqslant w(x, y)$ for $(x, y) \in \Delta$ and $i = 1, \ldots, n$, where Δ, Δ_1, Δ_2 and Δ_3 are as defined in section 2.

(iii) For each i, $F_i(x, y, z_1, z_2, z_3, z_4)$ is increasing with respect to the variables z_1, z_2, z_3, fpor fixed $(x, y) \in \Delta$.

Hypothesis 4. For $(x, y, z_1, z_2, z_3, z_4)$, $(x, y, \bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4) \in \Delta \times (R^n)^4$,

$$\| F(x, y, z_1, z_2, z_3, z_4) - F(x, y, \bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4) \| \leqslant$$
$$\leqslant L_1 \| z_1 - \bar{z}_1 \| + L_2 \| z_2 - \bar{z}_2 \| + L_3 \| z_3 - \bar{z}_3 \| + L_4 \| z_4 - \bar{z}_4 \|.$$

and for $(x, y, s, t, z) \in \Delta_1 \times R^n$, $(x, y, s, z) \in \Delta_2 \times R^n$, $(x, y, t, z) \in \Delta_3 \times R^n$,

$$\| f_1(x, y, s, t, z) - f_1(x, y, s, t, \bar{z}) \| \leqslant M_1 \| z - \bar{z} \|,$$
$$\| f_2(x, y, s, z) - f_2(x, y, s, \bar{z}) \| \leqslant M_2 \| z - \bar{z} \|, \quad \| f_3(x, y, t, z) - f_3(x, y, t, \bar{z}) \| \leqslant$$
$$\leqslant M_3 \| z - \bar{z} \|,$$

where L_i, M_j are nonnegative constants for $i = 1, 2, 3, 4$, $j = 1, 2, 3$ such that

(10) $$L_1 M_1 \alpha\beta + L_2 M_2 \alpha + L_3 M_3 \beta + L_4 < 1,$$

and $\| . \|$ denote a convenient norm defined on R^n.

In order to prove the existence of maximal and minimal solutions of equation (9) we define the sequences $\{w_n\}$ and $\{v_n\}$ by the relations

(11) $$w_0(x, y) = w(x, y), \quad w_n(x, y) = (T_0 w_{n-1})(x, y)$$

(12) $$v_0(x, y) = v(x, y) = v_n(x, y) = (T_0 v_{n-1})(x, y), \text{ for } n = 1, 2,$$

We are now prepared to establish the following theorem on the existence of maximal and minimal solutions of equation (9).

Theorem 3. *Let Hypothesis 3 be fulfilled. Then the sequence $\{w_n\}$ defined by (11) converges uniformly from above to a maximal solution $\bar{z}$ of equation (9) while the sequence $\{v_n\}$ defined by (12) converges uniformly from* *below to a minimal solution $\underline{z}$ of equation* (9). *Furthermore, if $z(x, y)$ is any solution of equation* (9) *such that $v(x, y)\leqslant z(x, y)\leqslant w(x, y)$ on Δ, then*

$$v\leqslant v_1\leqslant v_2\leqslant \ldots \leqslant v_n\leqslant \ldots \leqslant \underline{z}\leqslant z\leqslant \bar{z}\leqslant \ldots \leqslant w_n\leqslant \ldots \leqslant w_2\leqslant w_1\leqslant w, \tag{13}$$

on Δ.

Proof. Define $p_i(x, y)=v_{1i}(x, y)-v_i(x, y)$ for $(x, y)\in\Delta$, then

$$p_i(x, y)\geqslant (T_{0i}v)(x, y)-(T_{0i}v)(x, y)=0,$$

which implies $v(x, y)\leqslant v_1(x, y)$ on Δ, where $T_{0i}z$ denotes the i-th component in the definition of the operator $(T_0 z)$ given in (9). Let us assume that for some integer $k>0$, $v_{k-1}(x, y)\leqslant v_k(x, y)$ on Δ. Then setting $p_i(x, y)=$ $=v_{(k+1)i}(x, y)-v_{ki}(x, y)$, we get

$$p_i(x, y)=(T_{0i}v_{k+1})(x, y)-(T_{0i}v_k)(x, y)\geqslant (T_{0i}v_k)(x, y)-(T_{0i}v_k)(x, y)=0,$$

which implies $v_k(x, y)\leqslant v_{k+1}(x, y)$ on Δ. Hence, it follows by induction $v_{n-1}(x, y)\leqslant v_n(x, y)$ on Δ. By following the above argument we can show that $w_n(x, y)\leqslant w_{n-1}(x, y)$ on Δ.

Now define $p_i(x, y)=w_{1i}(x, y)-v_{1i}(x, y)$, then using the fact that $v(x, y)\leqslant w(x, y)$ on Δ, we have

$$p_i(x, y)=T_{0i}w_1)(x, y)-(T_{0i}v_1)(x, y)\geqslant (T_{0i}v_1)(x, y)-(T_{0i}v_1)(x, y)=0,$$

which implies $v_1(x, y)\leqslant w_1(x, y)$ on Δ. By following an induction argument we have $v_n(x, y)\leqslant w_n(x, y)$ on Δ. Thus, the sequences $\{v_n\}$, $\{w_n\}$ are monotone nondecreasing, nonincreasing respectively, and $v(x, y)\leqslant v_n(x, y)\leqslant w_n(x, y)\leqslant$ $\leqslant w(x, y)$ on Δ. Furthermore, using the standard arguments, it follows that these sequences converge uniformly and monotonically to the solutions $\underline{z}(x, y)$ and $\bar{z}(x, y)$ of equation (9).

Let $z(x, y)$ be any solution of equation (9) on Δ such that $v(x, y)\leqslant$ $\leqslant z(x, y)\leqslant w(x, y)$ on Δ. Then, by the induction argument, it is easily seen that $z(x, y)\leqslant w_n(x, y)$ and $z(x, y)\geqslant v_n(x, y)$ on Δ for every $n=0, 1, 2, \ldots$ Hence, we have $\underline{z}(x, y)\leqslant z(x, y)\leqslant \bar{z}(x, y)$ on Δ. This shows that $\bar{z}(x, y)$ is a maximal solution and $\underline{z}(x, y)$ is a minimal solution of equation (9). This completes the proof of the theorem.

We next establish the following theorem which shows that under some additional conditions on the functions involved in (9) the maximal and minimal solutions obtained in Theorem 3 coincide on Δ

Theorem 4. *Let the Hypotheses 3 and 4 hold. Then the maximal solution $\bar{z}(x, y)$ and the minimal solution $\underline{z}(x, y)$ obtained in Theorem 3 coincide on Δ, that is $\bar{z}(x, y)=\underline{z}(x, y)$ on Δ.*

Proof. Since $\bar{z}(x, y)$ and $\underline{z}(x, y)$ are solutions of equation (9) we have

$$\| \bar{z}(x, y)-\underline{z}(x, y) \|=\|(T_0\bar{z})(x, y)-(T_0\underline{z})(x, y)\|$$

$$\leqslant L_1M_1 \int_0^{a(x, y)}\int_0^{b(x, y)} \|\bar{z}(s, t)-\underline{z}(s, t)\|\, dsdt+L_2M_2 . \int_0^{c(x, y)} \|\bar{z}(s, p(x, y))-$$

$$-\underline{z}(s, p(x, y))\|\, ds+L_3M_3 \int_0^{d(x, y)} \|\bar{z}(q(x, y), t)-\underline{z}(q(x, y), t)\|\, dt+$$

$$+L_4\|\bar{z}(g(x, y), h(x, y))-\underline{z}(g(x, y), h(x, y))\|\leqslant [L_1M_1\alpha\beta+L_2M_2\alpha+$$

$$+L_3M_3\beta+L_4]\sup_{\Delta}\|\bar{z}(x, y)-\underline{z}(x, y)\|$$

Then setting $\lambda=\sup_{\Delta}\|\bar{z}(x, y)-\underline{z}(x, y)\|$, we get

$$\lambda\leqslant [L_1M_1\alpha\beta+L_2M_2\alpha+L_3M_3\beta+L_4]\,\lambda,$$

which implies in virtue of the condition (10) that $\lambda=0$, i.e. $\bar{z}(x, y)=\underline{z}(x, y)$ on Δ. This completes the proof of the theorem.

We note that in general the maximal and minimal solutions obtained in Theorem 3 are different. However, when (9) has a unique solution, $z(x, y)$, they coincide. In this case, the iteration scheme (13) has the advantage that $\{v_n\}$ and $\{w_n\}$ converge to z on Δ and $v_n\leqslant z\leqslant w_n$. That is, an approximate solution can be constructed to any degree of accuracy by making $\| v_n-w_n \|$ sufficiently small.

In concluding this paper, we note that the results obtained in this paper can be very easily extended to the Volterra integral functional equation of the form

$$z(x, y)=F\Big(x, y, \int_0^{a_1(x, y)}\int_0^{b_1(x, y)} f_1(x, y, s, t, z(s, t))\, dsdt, \ldots, \tag{14}$$

$$\int_0^{a_r(x, y)}\int_0^{b_r(x, y)} f_r(x, y, s, t, z(s, t))\, dsdt, \int_0^{c_1(x, y)} g_1(x, y, s, z(s, p_1(x, y)))\, ds, \ldots,$$

$$\int_0^{c_m(x, y)} g_m(x, y, s, z(s, p_m(x, y)))\, ds, \int_0^{d_1(x, y)} h_1(x, y, t, z(q_1(x, y), t))\, dt, \ldots,$$

$$\int_0^{d_n(x, y)} h_n(x, y, t, z(q_n(x, y), t))\, dt, \quad z(\alpha_1(x, y), \beta_1(x, y)), \ldots,$$

$$z(\alpha_l(x, y), \beta_l(x, y)), u\Big),$$

with suitable modifications. Similar results can be obtained for more general forms of equations (1) and (14) with an unknown function of n variables. In this case the notation will be much more complicated but the methods will be essentially the same. We omit the details.

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ASYMPTOTIC ANALYSIS OF NONLINEAR SECOND ORDER DIFFERENCE EQUATION

BY

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1. Introduction. In this paper we study the oscillation and asymptotic behavior of solutions of second-order nonlinear difference equation of the form

$$\Delta(r(n)\Delta y(n))+p(n+1)f(y(n+1))=0,\ n=0,\ 1,\ \ldots, \tag{1.1}$$

where the forward difference operator Δ is defined by the equation $\Delta y(n)=y(n+1)-y(n)$ and the sequence $\{r(n)\}_0^\infty$ and function f satisfy the following conditions

$$r(n)>0\ (n\geqslant 0) \text{ and } f \text{ is a nondecreasing function such that} \tag{1.2}$$
$$uf(u)>0 \text{ for } u\neq 0.$$

In (1.1) $\{p(n)\}_0^\infty$ is a given real sequence.

It is well known (see [1] and [3]) that the general linear difference operator of second order can be presented as the first term of (1.1).

By a *solution* of (1.1) we mean a real sequence $\{y(n)\}_0^\infty$ satisfying (1.1). Obviously, a solution $\{y(n)\}_0^\infty$ of (1.1) is uniquely determined by the initial values $y(0)$ and $y(1)$ or equivalently by any two successive values $y(k)$ and $y(k+1)$ and can be defined for all $n=0,\ 1,\ \ldots$.

Now, we introduce some notions which will be used in the sequel.

A real sequence $\{g(n)\}_0^\infty$ eventually has some property if there exists $N\geqslant 0$ such that $g(n)$ has this property for $n=N,\ N+1,\ \ldots$.

Throughout this paper, we usually refer to a solution $\{y(n)\}_0^\infty$ of (1.1) simply as a solution y and consider only nontrivial solutions y i.e. the solutions such that $y(k)=y(k+1)=0$ do not hold for any $k=0,\ 1,\ \ldots$.

A nontrivial solution y of (1.1) is said to be *oscillatory* if $y(n)$ changes sign infinitely many times. Otherwise, y is said to be nonoscillatory.

Equation (1.1) is called oscillatory if every its solution is oscillatory. Otherwise, it is called *nonoscillatory*.

In Section 2. we list some well known results which will be used in the sequel and we obtain an useful *estimate* for nonoscillatory solutions of (1.1). In Section 3. we assume that $p(n)\geqslant 0$ and obtain the characterization of nonoscillatory solutions of minimal and maximal type mentioned in the preceding section. Section 4. contains some oscillatory results for (1.1) in the case where $p(n)$ is not of constant sign. In Section 5. we apply the obtained results to the discrete Emden-Fowler equation and get the