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ASYMPTOTIC ANALYSIS OF NONLINEAR SECOND ORDER DIFFERENCE EQUATION

BY

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1. Introduction. In this paper we study the oscillation and asymptotic behavior of solutions of second-order nonlinear difference equation of the form

$$\Delta(r(n)\Delta y(n))+p(n+1)f(y(n+1))=0,\ n=0,\ 1,\ \ldots, \tag{1.1}$$

where the forward difference operator Δ is defined by the equation $\Delta y(n)= =y(n+1)-y(n)$ and the sequence $\{r(n)\}_0^\infty$ and function f satisfy the following conditions

$$r(n)>0\ (n\geqslant 0) \text{ and } f \text{ is a nondecreasing function such that} \tag{1.2}$$
$$uf(u)>0 \text{ for } u\neq 0.$$

In (1.1) $\{p(n)\}_0^\infty$ is a given real sequence.

It is well known (see [1] and [3]) that the general linear difference operator of second order can be presented as the first term of (1.1).

By a *solution* of (1.1) we mean a real sequence $\{y(n)\}_0^\infty$ satisfying (1.1). Obviously, a solution $\{y(n)\}_0^\infty$ of (1.1) is uniquely determined by the initial values $y(0)$ and $y(1)$ or equivalently by any two successive values $y(k)$ and $y(k+1)$ and can be defined for all $n=0,\ 1,\ \ldots$.

Now, we introduce some notions which will be used in the sequel.

A real sequence $\{g(n)\}_0^\infty$ eventually has some property if there exists $N\geqslant 0$ such that $g(n)$ has this property for $n=N,\ N+1,\ \ldots$.

Throughout this paper, we usually refer to a solution $\{y(n)\}_0^\infty$ of (1.1) simply as a solution y and consider only nontrivial solutions y i.e. the solutions such that $y(k)=y(k+1)=0$ do not hold for any $k=0,\ 1,\ \ldots$.

A nontrivial solution y of (1.1) is said to be *oscillatory* if $y(n)$ changes sign infinitely many times. Otherwise, y is said to be nonoscillatory.

Equation (1.1) is called oscillatory if every its solution is oscillatory. Otherwise, it is called *nonoscillatory*.

In Section 2. we list some well known results which will be used in the sequel and we obtain an useful *estimate* for nonoscillatory solutions of (1.1). In Section 3. we assume that $p(n)\geqslant 0$ and obtain the characterization of nonoscillatory solutions of minimal and maximal type mentioned in the preceding section. Section 4. contains some oscillatory results for (1.1) in the case where $p(n)$ is not of constant sign. In Section 5. we apply the obtained results to the discrete Emden-Fowler equation and get the complete information on the oscillatory and nonoscillatory behavior of this equation. For instance, we show that, under appropriate conditions, Emden-Fowler equation has only nonoscillatory solutions of minimal and maximal type. In Section 6. we get some *comparison result* of Sturm type. Finally, Section 7. discusses some possible extensions of presented results.

The results presented in this paper include previous results contained in [4].

We remark, that detailed discussion on various problems of qualitative theory of second order difference equation (1.1) can be found in [1], while in the case of oscillation and asymptotic behavior of nonoscillatory solution of (1.1), we refer to [3] for linear case and to [4] for nonlinear case.

The phrase "without loss of generality" is abbreviated as WLOG.

2. Preliminaries. In what follows, we will make use of a simple fixed point theorem due to K n a s t e r.

Theorem 2.1. [7]. *Let $\leqslant$ be a partial ordering with field A, and suppose that every $B\subseteq A$ has a least upper bound. Suppose that F maps A into A in such a way that for all x, y in A, $x\leqslant y$ implies that $Fx\leqslant Fy$. Then $Fx=x$ for some $x\in A$.*

Several of the arguments in subsequent sections make use of the following *summation by parts formula:*

$$\sum_{i=m}^{n-1} u_i\Delta v_i=u_n v_n-u_m v_m-\sum_{i=m}^{n-1} v_{i+1}\Delta u_i. \tag{2.1}$$

The following lemma provides an useful information about the asymptotic behavior of nonoscillatory solutions of (1.1).

Lemma 2.1. *Consider* (1.1) *subject to the conditions* (C_1) $p(n)\geqslant 0$ *for* $n=0, 1, \ldots$ *and* $p(n)$ *is not eventually zero, and*

$$(C_2)\qquad R(n)=\sum_{i=0}^{n-1}\frac{1}{r(i)}\to\infty \text{ for } n\to\infty.$$

Then, every nonoscillatory solution y of (1.1) *satisfies eventually* "a priori" *estimate*

$$a\leqslant|y(n)|\leqslant AR(n), \tag{2.2}$$

for some positive constants a and A (depending on y).

Proof. WLOG we suppose that $y>0$ eventually, which implies that $\Delta(r(n)\Delta y(n))\leqslant 0$ eventually, which means that $\Delta y(n)$ is eventually of constant sign. First, we show that $\Delta y(n)<0$ eventually, does not hold. For if, we have $r(n)\Delta y(n)\leqslant -M<0$ eventually, which yields

$$y(n)-y(n_1)\leqslant -M\sum_{i=n_1}^{n-1}\frac{1}{r(i)}\to-\infty(n\to\infty),$$

which is an immediate contradiction. Thus, we conclude that $\Delta y(n)\geqslant 0$ eventually, which immediately gives $y(n)\geqslant a>0$ eventually and $r(n)\Delta y(n)\leqslant A_1$, eventually from which we obtain

$$y(n)\leqslant y(n_1)+A_1\sum_{i=n_1}^{n-1}\frac{1}{r(i)}\leqslant AR(n)$$

for all large n.

Lemma 2.1. naturally implies that solution asymptoticaly equivalent to nonzero constant will be called the *solution of minimal type*, while the solution which is asymptotic to $AR(n)$ will be called a *solution of maximal type.*

3. Characterization of minimal and maximal type solutions. Theorem 3.1. *Consider the difference equation* (1.1) *subject to the conditions* (C_1) *and* (C_2). *Then,* (1.1) *has a nonoscillatory solution of minimal type iff the following condition holds*

$$(C_3)\qquad \sum_{i=0}^{\infty} R(i)p(i)<\infty.$$

Proof. First, suppose that (C_3) holds. Then, there exists $n_1\geqslant 0$ such that $f(2K)\sum_{i=n_1}^{\infty} R(i)p(i)\leqslant K$, for some arbitrary constant $K>0$. Let S be the set of all nondecreasing sequences x with the property $K\leqslant x(n)\leqslant 2K$ for $n\geqslant n_1$. The set S is considered endowed with the usual point-wise ordering $\leqslant$: $x_1\leqslant x_2 \Leftrightarrow (\forall n\geqslant n_1)\, x_1(n)\leqslant x_2(n)$.

Obviously, for every $B\subseteq S$, inf $B\in S$ and sup $B\in S$. Hence, we consider the mapping F on S defined as follows:

$$(Fx)(n)=K+R(n)\sum_{i=n}^{\infty} p(i)f(x(i))+\sum_{i=n_1}^{n-1} R(i)p(i)f(x(i)),\quad n\geqslant n_1.$$

It is clear that all conditions of Theorem 2.1 are satisfied which implies that there exists $z\in S$ such that $Fz=z$ i.e.

$$z(n)=K+R(n)\sum_{i=n}^{\infty} p(i)f(x(i))+\sum_{i=n_1}^{n-1} R(i)p(i)f(x(i)),\quad n\geqslant n_1.$$

Obviously, the sequence z is the required nonoscillatory solution of (1.1).

Now, we suppose that (1.1) has a nonoscillatory solution y such that $y(n)\to a>0$ for $n\to\infty$, which by Lemma 2.1 implies that $\Delta y(n)\geqslant 0$ eventually. Thus, there exists $n_1\geqslant 0$ such that for $n\geqslant n_1$ $y(n)\geqslant a/2$ and $\Delta y(n)\geqslant 0$.

Denoting $U(i)=R(i)r(i)\Delta y(i)$ and using (1.1) we obtain

$$\Delta U(i)=\Delta y(i)+R(i+1)\Delta(r(i)\Delta y(i))=\Delta y(i)-R(i+1)p(i+1)f(y(i+1)),$$

which, by summation from n_1 to $n-1$, yields

$$0\leqslant U(n)=U(n_1)+y(n)-y(n_1)-\sum_{i=n_1}^{n-1} R(i+1)p(i+1)f(y(i+1))\leqslant$$

$$\leqslant U(n_1)+a-y(n_1)-f\left(\frac{a}{2}\right)\sum_{i=n_1}^{n-1} R(i+1)p(i+1).$$

Hence, $f(a/2)\sum_{i=n_1}^{n-1} R(i+1)p(i+1)\leqslant U(n_1)+a-y(n_1)$, which completes the proof of theorem.

Theorem 3.2. *Consider the difference equation* (1.1) *subject to the conditions* (C_1) *and* (C_2). *Then,* (1.1) *has a nonoscillatory solution of maximal type iff the following condition holds*

$$(C_4) \qquad \sum_{n=0}^{\infty} p(n)|f(\alpha R(n))|<\infty,$$

for some constant $\alpha \neq 0$.

Proof. First, suppose that (C_4) holds. Then, there exists $n_1 \geq 0$ such that $\sum_{i=n_1}^{\infty} p(i) f(\alpha R(i)) \leq \alpha/2$. Let S be the set of all nondecreasing sequences x such that $(\alpha/2)R(n) \leq x(n) \leq \alpha R(n)$ for $n \geq n_1$. The set S is considered endowed with the point-wise ordering $\leq$ as in the proof of preceding theorem. Furthermore, we consider the mapping G on S defined in the following way

$$(Gx)(n)=\frac{m}{2} R(n)+\sum_{i=n_1}^{n-1} \frac{1}{r(i)} \sum_{j=i+1}^{\infty} p(j) f(x(j)), \quad n \geq n_1.$$

Obviously, all conditions of Theorem 2.1 are satisfied, which implies that G has a fixed point z.i.e. $Gz=z$. Hence, we get

$$(3.1) \qquad \Delta z(n)=\frac{\alpha}{2} \frac{1}{r(n)}+\sum_{j=n+1}^{\infty} p(j) f(z(j))/r(n) \quad \text{for } n \geq n_1,$$

which implies that z is the required nonoscillatory solution of (1.1).

Next, we assume that (1.1) has a nonoscillatory solution of maximal type y such that $y(n) \rightarrow AR(n)$ for $n \rightarrow \infty$, where $A>0$ is constant. Lemma 2.1 implies that $\Delta y(n) \geq 0$ eventually. So, there exists $n_1 \geq 0$ such that for $n \geq n_1$

$$y(n) \geq (A/2) R(n) \text{ and } \Delta y(n) \geq 0.$$

Now, from (1.1) we obtain by summation from n_1 to $n-1$

$$(3.2) \qquad r(n_1) \Delta y(n_1)-r(n) \Delta y(n)=\sum_{i=n_1}^{n-1} p(i+1) f(y(i+1)),$$

which yields

$$\sum_{i=n_1}^{\infty} p(i+1) f(A/2) R(i+1)) \leq \sum_{i=n_1}^{\infty} p(i+1) f(y(i+1)) \leq r(n_1) \Delta y(n_1),$$

and the proof is complete.

Remark 3.1. Theorem 3.1 is related to the result of Hooker and Patula [4, Theorem 4.2], which is originated by the continuous analogue of a result of Atkinson [1]. Moreover, Theorem 3.2 is an improvement and generalization of a result of Hooker and Patula [4, Theorem 4.4], since obviously the solution which has *asymptotically positively bounded differences* in the sense defined in [4] is a solution of maximal type.

Remark 3.2. It is easy to prove that in a sufficient part of Theorem 3.1 condition (C_3) can be replaced by the following one:

$$(C_3') \qquad \sum_{n=0}^{\infty} \frac{1}{r(n)} \sum_{i=0}^{n} p(i)<\infty.$$

4. **Oscillation results.** In this section we obtain some oscillation results, which complement the nonoscillation results of preceding section.

Theorem 4.1. *Consider the difference equation* (1.1) *subject to the conditions*

$$(C_5) \quad \sum_{n=0}^{\infty} p(n)=\infty \quad \text{and} \quad (C_6) \quad \sum_{n=0}^{\infty} \frac{1}{r(n)} \sum_{i=0}^{n} p(i)=\infty.$$

If the following condition holds

$$(C_7) \qquad \int_0^{\pm c} \frac{du}{f(u)}<\infty, \quad c>0,$$

then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Then x is of constant sign eventually, say for $n \geq n_1$. Let

$$w(n)=\frac{r(n) \Delta x(n)}{f(x(n))}, \quad n \geq n_1.$$

Now, we have

$$\Delta w(n)=\frac{f(x(n)) \Delta(r(n) \Delta x(n)-r(n) \Delta x(n) \Delta f(x(n))}{f(x(n)) f(x(n+1))}=\frac{\Delta(r(n) \Delta x(n))}{f(x(n+1))}-\frac{r(n) \Delta x(n) \Delta f(x(n))}{f(x(n)) f(x(n+1))}$$

and

$$\Delta w(n)=-p(n+1)-\frac{r(n) \Delta x(n) \Delta f(x) n))}{f(x(n)) f(x(n+1))}, \quad n \geq n_1$$

which yields

$$\Delta w(n) \leq \frac{\Delta(r(n) \Delta x(n))}{f(x(n+1))}=-p(n+1), \quad n \geq n_0$$

and

$$(4.1) \qquad w(m+1)-w(n_0) \leq-\sum_{n=n_0}^{m} p(n+1).$$

Thus $\lim_{m \rightarrow \infty} w(m)=-\infty$ and consequently there exists $n_2 \geq n_1$ such that

$$x(n) \Delta x(n)<0 \text{ for every } n_2 \geq n_1.$$

So, we obtain that $\lim_{n \rightarrow \infty}|x(n)|=d, \ d \geq 0$.

Hence, by (C_5), (4.1) implies

$$w(m+1) \leq-1/2 \sum_{n=n_0}^{m} p(n+1) \quad \text{for every } m>n_3,$$

where $n_3 \geq n_2$ is chosen appropriately. Moreover, for every $m>n_3$,

$$\frac{\Delta x(m+1)}{f(x(m+1))} \leq-\frac{1}{2} \frac{1}{r(m+1)} \sum_{n=n_0}^{m} p(n+1),$$

which, by summation from n_3 to $n>n_3$ gives

$$\sum_{m=n_3}^{n} \frac{\Delta x(m+1)}{f(x(m+1))} \leq-\frac{1}{2} \sum_{m=n_3}^{n} \frac{1}{r(m+1)} \sum_{n=n_0+1}^{m+1} p(n).$$

Taking $n\to\infty$, by (C_6) we get

$$\int_{x(n_1)}^{d} \frac{du}{f(u)}=-\infty,$$

which is, by (C_7), an immediate contradiction.

Theorem 4.2. *Consider the difference equation* (1.1) *subject to the conditions* (C_2) *and*

$$(C_8) \qquad \sum_{n=0}^{\infty} R(n)p(n)=\infty \quad \textit{and} \quad (C_9) \qquad \int_{+c}^{+\infty} \frac{du}{f(u)}<\infty,\ c>0.$$

Then, the difference equation (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Then x is of constant sign eventually, say for $n\geqslant n_1$. If

$$v(n)=\frac{r(n)\Delta x(n)}{f(x(n))}R(n),\quad n\geqslant n_1,$$

then $$\Delta v(n)=R(n+1)\frac{f(x(n))\Delta(r(n)\Delta x(n))-r(n)(\Delta x(n))\Delta f(x(n))}{f(x(n))f(x(n+1))}+$$

$$+\frac{r(n)\Delta x(n)}{f(x(n))}\Delta R(n)=R(n+1)\frac{\Delta(r(n)\Delta x(n))}{f(x(n+1))}-R(n+1)\frac{r(n)\Delta x(n)(\Delta f(x(n)))}{f(x(n))f(x(n+1))}+$$

$$+\frac{r(n)\Delta x(n)}{f(x(n))}\Delta R(n)\leqslant -R(n+1)p(n+1)+\frac{\Delta x(n)}{f(x(n))},\quad n\geqslant n_1.$$

Summing the last inequality from n_1 to $m>n_1$, we get

$$v(m+1)\leqslant v(n_1)-\sum_{n=n_1}^{m} R(n+1)p(n+1)+\sum_{n=n_1}^{m}\frac{\Delta x(n)}{f(x(n))}\quad \text{for } m>n_1.$$

Taking into account the conditions (C_8) and (C_9) we conclude that $\lim_{m\to\infty}|x(m)|=d$, $d\geqslant 0$ and $x(n)\Delta x(n)<0$ for every $n_2\geqslant n_1$.

WLOG we suppose that $x(n)>0$, which implies that $\Delta x(n)\leqslant 0$.

Now, introducing the function $U(n)=R(n)r(n)\Delta x(n)$ for $n>n_2$ we have

$$\Delta U(n)=R(n+1)\Delta(r(n)\Delta x(n))+r(n)(\Delta x(n))\Delta R(n)=$$
$$=-p(n+1)R(n+1)f(x(n+1))+\Delta x(n)$$

for $n>n_2$, which by summation from n_2 to $m>n_2$, yields

$$U(m+1)=U(n_2)+x(m+1)-x(n_2)-\sum_{n=n_2}^{m+1} p(n+1)R(n+1)f(x(n+1))=$$

$$=U(n_2)+x(m+1)-x(n_2)-\sum_{n=n_2}^{m} f(x(n+1))\Delta\left(\sum_{i=n_2}^{n} R(i)p(i)\right).$$

Now, applying the relation (2.1) we get

$$U(m+1)=U(n_2)+x(m+1)-x(n_2)+\sum_{n=n_2}^{m}\left(\sum_{i=n_2}^{n+1} R(i)p(i)\right)\Delta f(x(n+1))-$$

$$-f(x(m+2))\sum_{i=n_2}^{m+1} R(i)p(i)\leqslant U(n_2)<0 \quad\text{for } m>n_2.$$

Thus, we get

$$\Delta x(m+1)\leqslant U(n_2)\frac{1}{r(m+1)R(m+1)}=U(n_2)\frac{\Delta R(m+1)}{R(m+1)}\quad\text{for } m>n_2,$$

which implies

$$x(k+1)\leqslant x(n_0+1)+U(n_2)\sum_{m=n_0}^{k}\frac{\Delta R(m+1)}{R(m+1)}.$$

Now, taking into account the condition (C_2) we obtain $\lim_{k\to\infty} x(k)=-\infty$ which is an immediate contradiction.

Using similar methods one can prove the following result.

Theorem 4.3. *Consider the difference equation* (1.1) *subject to the conditions* (C_1), (C_2) *and*

$$(C_{10}) \qquad \sum_{n=0}^{\infty} p(n)\,|\,f(\alpha R(n))\,|=\infty,\ \text{for all } \alpha\neq 0.$$

If f is strongly sublinear i.e. if there exists a number $\nu<1$ such that $f(u)/|u|^{\nu}\operatorname{sgn} u$ is nonincreasing for $u>0$ and nondecreasing for $u<0$, then (1.1) *is oscillatory.*

Corollary 4.1. *Consider the difference equation* (1.1) *subject to the conditions* (C_1), (C_2) *and* (C_9). *Then,* (1.1) *is oscillatory iff* (C_8) *holds.*

Proof. It is an immediate consequence of Theorems 3.1 and 4.2.

Corollary 4.2. *Consider the difference equation* (1.1) *subject to the conditions* (C_1) *and* (C_2). *If f is strongly sublinear in the above mentioned sense, then* (1.1) *is oscillatory iff the condition* (C_{10}) *holds.*

Remark 4.1. Corollary 4.1 improves and generalizes in various directions the result of Hooker and Patula [4, Theorem 4.1.] Moreover, Corollary 4.2 is an improvement of a result of Hooker and Patula [4, Theorem 4.3].

We also note that Theorems 4.1 and 4.2 are discrete analogues of the corresponding results for differential equations and inequalities obtained by Kulenović and Grammatikopoulos [5]. Furthermore, Theorem 4.3 is a discrete analogue of a continuous result for Emden-Fowler differential equation, see [10] and references therein.

Finally Theorems 4.1 and 4.2 are, in point of contact, more general than the result of Szmanda [8].

5. Application to the Emden-Fowler equation. In this section we apply the obtained results to the special case of Emden-Fowler equation of the form

$$(5.1)\quad \Delta(r(n)\Delta y(n))+p(n+1)|y(n+1)|^{\nu}\operatorname{sgn} y(n+1)=0,\quad \nu>0 \text{ and } n=0,1,\ldots,$$

obtaining some further results on the asymptotic behavior of nonoscillatory solutions.

Theorem 5.1. *Consider the difference equation* (5.1) *subject to the conditions* (C_1) *and* (C_2). *If $\nu\geqslant 1$ and the condition* (C_4) *holds, then every nonoscillatory solution of* (5.1) *is either of minimal or of maximal type.*

Proof. Since $\nu\geqslant 1$, the condition (C_4) implies (C_3) and by Theorems 3.1 and 3.2 the equation (5.1) has both type of solutions. Now, we show that (5.1) has no other types of solutions.

Suppose that y is an unbounded solution of (5.1). WLOG we assume that $y>0$ eventually i.e. $y(n)>0$ for $n\geqslant n_1$. Then, by Lemma 2.1 we conclude that $\Delta y(n)\geqslant 0$ eventually and the estimate (2.2) holds. We may suppose that $\Delta y(n)\geqslant 0$ for $n\geqslant n_1$. Now, by discrete analogue of L'Hospital rule we get that

$$\lim_{n\to\infty}\frac{y(n)}{R(n)}=\lim_{n\to\infty} r(n)\Delta y(n)=L\in[0,\infty).$$

Multiplying (5.1) by $\sum_{k=n_1}^{n}[1/r(k)]$ and summing we obtain

$$\sum_{k=n_1}^{n-1}\Delta(r(k)\Delta y(k))\sum_{i=n_1}^{k}\frac{1}{r(i)}+\sum_{k=n_1}^{n-1}p(k+1)(y(k+1))^{\nu}\sum_{i=n_1}^{k}\frac{1}{r(i)}=0,$$

and

$$r(n)\Delta y(n)\sum_{k=n_1}^{n-1}\frac{1}{r(k)}-y(n)+y(n_1)+\sum_{k=n_1}^{n-1}p(k+1)(y(k+1))^{\nu}\sum_{i=n_1}^{k}\frac{1}{r(i)}=0, \tag{5.2}$$

which by $\Delta y(n)\geqslant 0$ for $n\geqslant n_1$ yields

$$y(n)\leqslant y(n_1)+\sum_{k=n_1}^{n-1}R(k+1)p(k+1)(y(k+1))^{\nu}+R(n)r(n)\Delta y(n),\quad n>n_1. \tag{5.3}$$

Thus, we have

$$y(n)\leqslant y(n_1)+y(n)\sum_{k=n_1}^{n-1}R(k+1)p(k+1)(y(k+1))^{\nu-1}+R(n)r(n)\Delta y(n),\quad n>n_1$$

and

$$1\leqslant\frac{y(n_1)}{y(n)}+\sum_{k=n_1}^{n-1}R(k+1)p(k+1)(y(k+1))^{\nu-1}+\frac{R(n)r(n)\Delta y(n)}{y(n)},\quad n>n_1. \tag{5.4}$$

Now, using the estimate (2.2) we obtain

$$\sum_{k=n_1+1}^{n}R(k)p(k)(y(k))^{\nu-1}\leqslant\sum_{k=n_1}^{n}R(k)p(k)(AR(k))^{\nu-1}\leqslant A^{\nu-1}\sum_{k=n_1}^{\infty}(R(k))^{\nu}\,p(k).$$

If n_1 is chosen large enough to provide $A^{\nu-1}\sum_{k=n_1}^{\infty}(R(k))^{\nu}p(k)\leqslant\varepsilon$, for arbitrary $\varepsilon>0$, then (5.4) implies

$$\frac{y(n_1)}{y(n)}+\frac{R(n)r(n)\Delta y(n)}{y(n)}\geqslant 1-\varepsilon,$$

which immediately gives

$$\liminf_{n\to\infty}\frac{R(n)r(n)\Delta y(n)}{y(n)}\geqslant 1-\varepsilon_1,$$

and

$$R(n)r(n)\Delta y(n)\geqslant(1-\varepsilon_1)y(n)\quad\text{for every } n\geqslant N\geqslant n_1. \tag{5.5}$$

Now, using (5.5) from (5.1) we obtain

$$(1-\varepsilon_1)[r(N)\Delta y(N)-r(n)\Delta y(n)]=(1-\varepsilon_1)\sum_{k=N}^{n-1}p(k+1)(y(k+1))^{\nu}\leqslant$$

$$\leqslant\sum_{k=N+1}^{n}p(k)(y(k))^{\nu-1}r(k)R(k)\Delta y(k)\leqslant A^{\nu-1}\sum_{k=N}^{n-1}p(k)R^{\nu-1}(k)r(k)R(k)\Delta y(k)\leqslant$$
$$\leqslant A^{\nu-1}r(N)\Delta y(N)\sum_{k=N}^{\infty}R^{\nu}(k)p(k)<\varepsilon_1 r(N)\Delta y(N),$$

if N is large enough to provide $A^{\nu-1}\sum_{k=N}^{\infty}R^{\nu}(k)p(k)<\varepsilon_1$. Thus, we get

$$(1-\varepsilon_1)[r(N)\Delta y(N)-r(n)\Delta y(n)]<\varepsilon_1 r(N)\Delta y(N) \tag{5.6}$$

and

$$r(n)\Delta y(n)>\frac{(1-2\varepsilon_1)r(N)\Delta y(N)}{1-\varepsilon_1}\quad\text{eventually,}$$

which implies that $\lim_{n\to\infty}r(n)\Delta y(n)=L>0$ and so $\lim_{n\to\infty}[y(n)/R(n)]=L>0$, which completes the proof of theorem.

The corresponding result for sublinear case is the following.

Theorem 5.2. *Consider the difference equation* (5.1) *subject to the conditions* (C_1) *and* (C_2). *If* $\nu\leqslant 1$ *and the condition* (C_3) *holds, then every nonoscillatory solution of* (5.1) *is either of minimal or of maximal type.*

Proof. Since $\nu\leqslant 1$, the condition (C_3) implies (C_4) and by Theorems 3.1 and 3.2, the equation (5.1) has both type of solutions.

Suppose that y is an unbounded solution of (5.1). WLOG we suppose that $y>0$ eventually which implies that $\Delta y(n)\geqslant 0$ eventually i.e. there exists $n_1\geqslant 0$ such that $y(n)>0$ and $\Delta y(n)\geqslant 0$ for $n\geqslant n_1$. Starting from the relation (5.2) we obtain (5.3) which by $\Delta y(n)\geqslant 0$ and the estimate (2.2) yields

$$y(n)\leqslant y(n_1)+\sum_{k=n_1}^{n-1}R(k+1)p(k+1)(y(k+1))^{\nu}+R(n)r(n)\Delta y(n)\leqslant$$
$$\leqslant y(n_1)+y(n)\sum_{k=n_1}^{n-1}R(k+1)p(k+1)(y(k+1))^{\nu-1}+R(n)r(n)\Delta y(n)\leqslant$$
$$\leqslant y(n_1)+a^{\nu-1}y(n)\sum_{k=n_1}^{n}R(k)p(k)+R(n)r(n)\Delta y(n).$$

Now, we obtain

$$1\leqslant\frac{y(n_1)}{y(n)}+a^{\nu-1}\sum_{k=n_1}^{n}R(k)p(k)+\frac{R(n)r(n)\Delta y(n)}{y(n)}\leqslant\frac{y(n_1)}{y(n)}+$$
$$+a^{\nu-1}\sum_{k=n_1}^{\infty}R(k)p(k)+\frac{R(n)r(n)\Delta y(n)}{y(n)}\leqslant\frac{y(n_1)}{y(n)}+\varepsilon+\frac{R(n)r(n)\Delta y(n)}{y(n)},\quad\text{for } n\geqslant n_1,$$

where n_1 is large enough to provide $a^{\nu-1}\sum_{k=n_1}^{\infty}R(k)p(k)\leqslant\varepsilon$, for an arbitrary $\varepsilon>0$. Using the same arguments as in the proof of preceding theorem we obtain the relation (5.5), which gives

$$(1-\varepsilon_1)[r(N)\Delta y(N)-r(n)\Delta y(n)]=(1-\varepsilon_1)\sum_{k=N+1}^{n}p(k)y^{\nu}(k)=$$
$$=(1-\varepsilon_1)\sum_{k=N+1}^{n}y(k)p(k)y^{\nu-1}(k)\leqslant\sum_{k=N+1}^{n}r(k)\Delta y(k)R(k)p(k)y^{\nu-1}(k)\leqslant$$

$$\leqslant a^{\nu-1}\sum_{k=N}^{n} R(k)p(k)r(k)\Delta y(k)\leqslant a^{\nu-1}r(N)\Delta y(N)\sum_{k=N}^{\infty} R(k)p(k)\leqslant \varepsilon_1 r(N)\Delta y(N),$$

where N is chosen such that $a^{\nu-1}\sum_{k=N}^{\infty} R(k)p(k)\leqslant \varepsilon_1$.

Thus we obtain the relation (5.6) and the proof follows in the same way as the proof of the preceding result.

The following result gives a stronger result on the asymptotic behavior of nonoscillatory solutions of (5.1).

Theorem 5.3. *Consider the difference equation* (5.1) *subject to the conditions* (C_1) *and* (C_2). *If the following condition holds*

$$(C_{11}) \qquad \liminf_{n\to\infty} R(n))^{\lambda}\sum_{k=n}^{\infty} p(k)>0, \text{ for some } 0<\lambda\leqslant\nu$$

then, for every nonoscillatory solution the following estimate holds

$$(5.7) \qquad 0\leqslant |y(n)|\leqslant A(R(n))^{\lambda/\nu}, \text{ eventually},$$

where a and A are some appropriate positive constants depending on y.

Proof. Suppose that y is a nonoscillatory solution of (5.1). WLOG we suppose that $y>0$ eventually which, by Lemma 2.1 implies that $\Delta y(n)\geqslant 0$ eventually i.e. there exists $N\geqslant 0$ such that $y(n)>0$ and $\Delta y(n)\geqslant 0$ for $n\geqslant N$. Multiplying (5.1) by $\sum_{i=N}^{n} 1/r(i)$ and then by summation from N to $k-1$ we obtain

$$(5.8) \qquad \sum_{n=N}^{k-1}\Delta(r(n)\Delta y(n))\sum_{i=N}^{n}\frac{1}{r(i)}+\sum_{n=N}^{k-1}p(n+1)\,(y(n+1))^{\nu}\sum_{i=N}^{n}\frac{1}{r(i)}=0.$$

Applying the relation (2.1) to the first summand of the last relation we get

$$\sum_{n=N}^{k-1}\Delta(r(n)\Delta y(n))\sum_{i=N}^{n}\frac{1}{r(i)}=r(k)\Delta y(k)\frac{1}{r(i)}-\sum_{n=N}^{k-1}\left(\Delta\sum_{i=N}^{n}\frac{1}{r(i)}\right)r(n+1)\Delta y(n+$$

$$+1)-r(N)\Delta y(N)\frac{1}{r(N)}=r(k)\Delta y(k)\sum_{i=N}^{k}\frac{1}{r(i)}-\Delta y(N)-\sum_{n=N}^{k-1}\Delta y(n+1)=$$

$$=r(k)\Delta y(k)\sum_{i=N}^{k}\frac{1}{r(i)}-\sum_{n=N}^{k}\Delta y(n)=r(k)\Delta y(k)\sum_{i=N}^{k}\frac{1}{r(i)}-y(k+1)+y(N).$$

Thus, we obtain

$$(5.9) \qquad \sum_{n=N}^{k-1}\Delta(r(n)\Delta y(n))\sum_{i=N}^{n}\frac{1}{r(i)}=r(k)\Delta y(k)\sum_{i=N}^{k}\frac{1}{r(i)}-y(k+1)+y(N).$$

On the other hand from (5.1) we have

$$(5.10) \qquad r(M)\Delta y(M)-r(k)\Delta y(k)+\sum_{n=k}^{M-1}p(n+1)\Delta(y(n+1))^{\nu}=0.$$

Using (5.9) and (5.10), the relation (5.8) yields

$$(5.11) \qquad y(k+1)=y(N)+r(M)\Delta y(M)\sum_{i=N}^{k}\frac{1}{r(i)}+\left(\sum_{i=N}^{k}\frac{1}{r(i)}\right)\sum_{n=k}^{M-1}p(n+$$

$$+1)(y(n+1))^{\nu}+\sum_{n=N}^{k-1}p(n+1)(y(n+1))^{\nu}\sum_{i=N}^{n}\frac{1}{r(i)}, \text{ eventually},$$

which immediately implies

$$y(k+1)\geqslant\sum_{i=N}^{k}\frac{1}{r(i)}\sum_{n=k}^{M-1}p(n+1)(y(n+1))^{\nu}\geqslant CR(k)\sum_{n=k}^{M-1}p(n+1)(y(n+1))^{\nu},$$

eventually, for some appropriate constant $C\in(0,\ 1)$. Taking $M\to\infty$ we get

$$y(k+1)\geqslant CR(k)\sum_{n=k}^{\infty}p(n)(y(n))^{\nu}\geqslant CR(k)(y(k))^{\nu}\sum_{n=k}^{\infty}p(n), \text{ eventually}$$

which, by (C_{11}) and the estimate (2.2) yields :

$$y(n)\geqslant C\frac{(y(k))^{\nu}}{(R(k))^{\lambda}}(R(k))^{\lambda+1}\sum_{n=k}^{\infty}p(n)\geqslant\tilde{C}y(k)\frac{(y(k))^{\nu}}{(R(k))^{\lambda}}, \text{ eventually},$$

for some appropriate constant $\tilde{C}$, which completes the proof of theorem.

In a similar manner, we can prove the following result.

Theorem 5.4. *Consider the difference equation* (5.1) *subject to the conditions* (C_1), (C_2) *and* (C_{11}). *Then, for every nonoscillatory solution y the following estimate holds*

$$a\leqslant|y(n)|\leqslant A(R(n))^{\lambda-1/\nu-1}, \text{ eventually, for } 1<\lambda\leqslant\nu$$

and

$$B(R(n))^{1-\lambda/1-\nu}\leqslant|y(n)|\leqslant bR(n), \text{ eventually, for } 0<\nu<\lambda<1.$$

Corollary 5.1. *Consider the difference equation* (5.1) *subject to the conditions* (C_1), (C_2) *and* (C_4).

(a) *If $\nu>1$, then every solution with bounded differences of* (5.1) *is either of minimal or of maximal type:*

(b) *If $0<\nu<1$ and $\Delta y(n)\to 0$ as $n\to\infty$, then every solution of* (5.1) *is either of minimal or of maximal type.*

Proof. It is an immediate consequence of Theorems 5.1 and 5.2 and of [4, Theorem 5.1].

6. Comparison result. In this section we get a comparison result of Sturm's type for the nonlinear equations. Consider the difference equation

$$(6.1) \qquad \Delta(R(n)\Delta y(n))+P(n+1)F(y(n+1))=0,\ n=0,\ 1,\ \ldots$$

where $R(n)>0$ and $P(n)\geqslant 0$ for $n=0,\ 1,\ \ldots$ and F is nondecreasing function such that $uF(u)>0$ for $u\neq 0$.

Theorem 6.1. *Consider the equation* (1.1) *along with the equation* (6.1) *subject to the conditions*

$$(6.2) \qquad \sum_{n=0}^{\infty}[1/R(n)]=\infty,$$

$$(6.3) \qquad r(n)\geqslant R(n) \text{ and } P(n)\geqslant p(n) \text{ for } n=0,\ 1,\ \ldots,$$

and

(6.4) $$(F-f)[0,\infty)\geq 0 \text{ and } (F-f)(-\infty,0]\leq 0.$$

If the difference equation (1.1) *is oscillatory, then* (6.1) *is oscillatory too.*

Proof. Let y be a nonoscillatory solution of (6.1). WLOG we suppose that $y>0$ eventually which, by Lemma 2.1, implies that $\Delta y(n)\geq 0$ eventually i.e. there exists $N\geq 0$ such that $y(n)>0$ and $\Delta y(n)\geq 0$ for $n\geq N$. Hence, by (6.3) and (6.4) we get $P(n)F(y(n))\geq p(n)f(y(n))$ for $n\geq N$ and $\Delta(R(n-1)\Delta y(n-1)+p(n)f(y(n))\leq 0$, $n\geq N$ which implies

$$R(m-1)\Delta y(m-1)\geq \sum_{n=m}^{M-1} p(n)f(y(n)),\ m\geq N$$

and by (6.3)

$$\Delta y(m-1)\geq \frac{1}{R(m-1)}\sum_{n=m}^{\infty} p(n)f(y(n))\geq \frac{1}{r(m-1)}\sum_{n=m}^{\infty} p(n)f(y(n)).$$

Summing the last inequality from N to k, we obtain

$$y(k)\geq y(N-1)+\sum_{m=N}^{k}\frac{1}{r(m-1)}\sum_{n=m}^{\infty} p(n)f(y(n)),\ k>N.$$

Let X be the set of all nondecreasing sequences x, such that

$$y(N-1)\leq x(k)\leq y(k) \text{ for every } k\geq N.$$

The set X is considered endoved with the usual point-wise ordering as in the proof of Theorem 3.1. We consider the mapping H on X defined by :

$$(Hx)(k)=y(N-1)+\sum_{m=N}^{k}\frac{1}{r(m-1)}\sum_{n=m}^{\infty} p(n)f(x(n)),\ k>N.$$

Taking into account the assumptions, it is easy to see that all conditions of Theorem 2.1 are satisfied, which shows that H has a fixed point z i.e. $(Hz)(k)=z(k)$ for every $k>N$. Obviously z is a nonoscillatory solution of (1.1), which is a contradiction.

7. Concluding remarks. *Remark* 7.1. In the case of Thomas-Fermi difference equation (1.1) i.e. in the case where the condition (C$_1$) is replaced by the following condition

(C$_1'$) $\quad p(n)\leq 0$ for $n=0, 1, \ldots$ and $p(n)$ is not eventually zero,

it is easy to see that all solutions are nonoscillatory (see [4]) and using the similar methods as in the proofs of Theorems 3.1 and 3.2 we can prove, under the appropriate conditions, the existence of solutions asymptotically equivalent to constant and to $R(n)$. The corresponding results can be formulated as follows.

Theorem 7.1. *Consider the difference equation* (1.1) *subject to the conditions* (C$_1$) *and* (C$_2$). *Then* (1.1) *has a nonoscillatory solution of* (a) *minimal type iff the following condition holds*

$$\sum_{k=0}^{\infty}\frac{1}{r(k)}\sum_{i=k}^{\infty} p(i)<\infty.$$

(b) *maximal type iff the condition* (C$_3$) *holds.*

Unfortunately, in the case of Thomas-Fermi equation "a priori" estimate of the form (2.2) fails. Thus, the terms maximal and minimal type of solutions makes no sense in this case.

We note that in this case some comparison results for nonoscillatory solutions can be obtained using the similar methods as in the preceding section or using some elementary methods as it was done in [4].

Remark 7.2. Most of the obtained results can be extended to the more general case of the *difference inequality*

(7.1) $$y(n)[\Delta(r(n)\Delta y(n))+p(n+1)f(y(n+1))]\leq 0,\ n=0, 1, \ldots,$$

where the sequences r and p and the function f satisfy the same conditions as in the case of difference equation (1.1). Actually, all results of Sections 2.4 and 6 can be extended, under the same conditions, to the case of inequality (7.1). Moreover, as in the case of the corresponding differential equations and inequalities we can prove the following result on the equivalence of (1.1) and (7.1), with the respect to the oscillation (see [9] for detailed study on this subject).

Theorem 7.2. *Consider the difference equation* (1.1) *and the corresponding inequality* (7.1), *subject to the conditions* (C$_1$) *and* (C$_2$). *Then the following conclusion holds*

(1.1) *is oscillatory* $\Leftrightarrow$(7.1) *is oscillatory.*

Further, in the case of linear difference equations and inequalities, we can get stronger conclusion (see [6] for continuous analogue).

Theorem 7.3. *Consider the linear difference equation* (1.1) *and the linear inequality* (7.1). *Then*

(1.1) *is oscillatory* $\Leftrightarrow$ (7.1) *is oscillatory.*

Finally, we note that in the case of Thomas-Fermi equation (1.1) i.e. in the case where $p(n)\leq 0$, for $n=0, 1, \ldots$ the corresponding inequality has the form

(7.2) $$y(n)[\Delta(r(n)\Delta y(n))+p(n+1)f(y(n+1))]\geq 0,\ n=0, 1, \ldots,$$

and the result of Remark 7.1 as far as the results of [4], concerning this equation, can be extended to this more general case.

Remark 7.3. We note that in the case where (C$_2$) holds we can transform the equation (1.1) to the form where $r(i)=1$ for every $i=0, 1, \ldots$. In fact introducing the following discrete analogue of standard Liouville transformation : $m=R(n)$, $x(m)=y(n)$, we reduce (1.1) to the form

$$\Delta^2 x(m)+r(n)p(n+1)f(x(m+1))=0,\ m=0, 1, \ldots.$$

Thus, we could treat the last equation instead of (1.1), but we prefer our method for the sake of technical simplicity.

Finally, all obtained results can be extended to the other two types of second order difference equations namely to the equations $\Delta(r(n)\Delta y(n))+$

$+p(n)f(y(n))=0$, $n=0, 1, \ldots$ and $\Delta(r(n)\Delta y(n)+p(n+2)f(y(n+2))=0$, $n=0, 1, \ldots$, in which case, the conditions that we used, must be replaced by the related, appropriate conditions, while the methods are the same.

Also, the presented results of Section 2—4 and 6 hold if the condition (1.2) on function f is replaced with the condition that f is sublinear ($f(u)/u$ is nondecreasing for $u<0$ and nonincreasing for $u>0$) or superlinear ($f(u)/u$ is nonincreasing for $n<0$ and nondecreasing for $n>0$).

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ON THE CONVERGENCE OF THE METHOD OF FINITE DIFFERENCES APPLIED TO A BOUNDARY VALUE PROBLEM FOR FUNCTIONAL-DIFFERENTIAL NEUTRAL TYPE EQUATIONS

BY

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The convergence of the method of finite differences when applied to the problem of periodic solutions of ordinary differential equations without deviation of the argument is considered, e.g. in [1]. The convergence of this method in the case of differential equations with a transformed argument is considered by Saarniit I.R. in [2—4] and others. The convergence of this method in the case of an initial value problem for functional-differential neutral type equations is considered, e.g. in [5—7].

The present paper deals with the convergence of the method of finite differences applied to a boundary value problem for functional-differential neutral type equations.

1. **Formulation of the boundary value problem.** Let us consider the boundary value problem for a functional-differential neutral type equation with linear boundary conditions

$$\dot{x}(t)=f(t, x_t, \dot{x}_t, \lambda),\ t\in[0, T], \tag{1}$$

$$x(t)=\varphi(t),\ t\in[-t_0, 0], \tag{2}$$

$$\beta x(T)+\int_0^T [d\gamma(t)]\dot{x}(t)=l, \tag{3}$$

where the operator f: $[0, T]\times C([-t_0, 0], \mathbf{R}^m)\times C([-t_0, 0], \mathbf{R}^m)\times\mathbf{R}^m\to\mathbf{R}^m$ is continuous, $C^k([a, b], \mathbf{R}^m)$ $(k\geqslant 0)$ is the space of the continuous functions $x:[a, b]\to\mathbf{R}^m$ with continuous derivatives up to the k-th order endowed with the uniform norm $\|x\|_{C[a,b]}=\sum_{i=0}^{k}\max_{t\in[a,b]}|x^i(t)|$, $\mathbf{R}^m$ is the m-th dimensional space with a norm $|\cdot|$, $x\in\mathbf{R}^m$ is a parameter, $\varphi\in C^1([-t_0, 0], \mathbf{R}^m)$, x_t and $\dot{x}_t$ are elements of the space $C([-t_0, 0], \mathbf{R}^m)$ such that $x_t(s)=x(t+s)$ and $\dot{x}_t(s)=\dot{x}(t+s)$ for every $S\in[-t_0, 0]$ where $t_0>0$ is a fixed number, $\gamma(t)$ is $(m\times m)$-matrix with a bounded variation on $[0, T]$, is $(m\times m)$-matrix, $l\in\mathbf{R}^m$, $\dot{\varphi}(0)=f(0, \varphi, \dot{\varphi}, \lambda)$ for every $\lambda\in\mathbf{R}^m$.

The pair $(x, \lambda)\in C([-t_0, T], \mathbf{R}^m)\times\mathbf{R}^m$ satisfying the equations (1)—(3) will be considered as a solution of the problem (1)—(3).

Let

$$(J^0y)(t)=\begin{cases}0, & t\in[-t_0, 0]\\ \int_0^t y(s)ds, & t\in[0, T]\end{cases}\qquad (Jy)(t)=\begin{cases}\varphi(t), & t\in[-t_0, 0]\\ \varphi(0)+\int_0^t y(s)ds, & t\in[0, T]\end{cases}$$

for $y\in C([0, T], \mathbf{R}^m)$.