$+p(n)f(y(n))=0$, $n=0, 1, \ldots$ and $\Delta(r(n)\Delta y(n)+p(n+2)f(y(n+2))=0$, $n=0, 1, \ldots$, in which case, the conditions that we used, must be replaced by the related, appropriate conditions, while the methods are the same.

Also, the presented results of Section 2—4 and 6 hold if the condition (1.2) on function f is replaced with the condition that f is sublinear ($f(u)/u$ is nondecreasing for $u<0$ and nonincreasing for $u>0$) or superlinear ($f(u)/u$ is nonincreasing for $u<0$ and nondecreasing for $u>0$).

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ON THE CONVERGENCE OF THE METHOD OF FINITE DIFFERENCES APPLIED TO A BOUNDARY VALUE PROBLEM FOR FUNCTIONAL-DIFFERENTIAL NEUTRAL TYPE EQUATIONS

BY

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The convergence of the method of finite differences when applied to the problem of periodic solutions of ordinary differential equations without deviation of the argument is considered, e.g. in [1]. The convergence of this method in the case of differential equations with a transformed argument is considered by Saarniit I.R. in [2—4] and others. The convergence of this method in the case of an initial value problem for functional-differential neutral type equations is considered, e.g. in [5—7].

The present paper deals with the convergence of the method of finite differences applied to a boundary value problem for functional-differential neutral type equations.

1. **Formulation of the boundary value problem.** Let us consider the boundary value problem for a functional-differential neutral type equation with linear boundary conditions

$$(1) \qquad \dot{x}(t)=f(t, x_t, \dot{x}_t, \lambda), \quad t\in[0, T],$$

$$(2) \qquad x(t)=\varphi(t), \quad t\in[-t_0, 0], \qquad\qquad (3) \qquad \beta x(T)+\int_0^T [d\gamma(t)]\dot{x}(t)=l,$$

where the operator f: $[0, T]\times C([-t_0, 0], \mathbf{R}^m)\times C([-t_0, 0], \mathbf{R}^m)\times\mathbf{R}^m\to\mathbf{R}^m$ is continuous, $C^k([a, b], \mathbf{R}^m)$ $(k\geqslant 0)$ is the space of the continuous functions $x:[a, b]\to\mathbf{R}^m$ with continuous derivatives up to the k-th order endowed with the uniform norm $\|x\|_{C[a,b]}=\sum_{i=0}^{k}\max_{t\in[a,b]}|x^i(t)|$, $\mathbf{R}^m$ is the m-th dimensional space with a norm $|\cdot|$, $x\in\mathbf{R}^m$ is a parameter, $\varphi\in C^1([-t_0, 0], \mathbf{R}^m)$, x_t and $\dot{x}_t$ are elements of the space $C([-t_0, 0], \mathbf{R}^m)$ such that $x_t(s)=x(t+s)$ and $\dot{x}_t(s)=\dot{x}(t+s)$ for every $S\in[-t_0, 0]$ where $t_0>0$ is a fixed number, $\gamma(t)$ is $(m\times m)$-matrix with a bounded variation on $[0, T]$, is $(m\times m)$-matrix, $l\in\mathbf{R}^m$, $\dot{\varphi}(0)=f(0, \varphi, \dot{\varphi}, \lambda)$ for every $\lambda\in\mathbf{R}^m$.

The pair $(x, \lambda)\in C([-t_0, T], \mathbf{R}^m)\times\mathbf{R}^m$ satisfying the equations (1)—(3) will be considered as a solution of the problem (1)—(3).

Let

$$(J^0y)(t)=\begin{cases} 0, & t\in[-t_0, 0] \\ \int_0^t y(s)ds, & t\in[0, T]\end{cases} \qquad (Jy)(t)=\begin{cases}\varphi(t), & t\in[-t_0, 0] \\ \varphi(0)+\int_0^t y(s)ds, & t\in[0, T]\end{cases}$$

for $y\in C([0, T], \mathbf{R}^m)$.

By means of the substitution

$$y(t)=\begin{cases}\varphi(t), & t\in[-t_0, 0]\\ \dot{x}(t), & t\in[0, T]\end{cases}$$

we reduce the boundary value problem (1)—(3) to the problem

$$y(t)=f(t, (Jy)_t, y_t, \lambda),\ t\in[0, T],\ \beta(J^0y)(T)+\int_0^T[d\gamma(t)]y(t)=l-\varphi(0),$$

which may be written in the form of an operator equation

$$\mathfrak{U}u=w. \tag{4}$$

The following notation is used

$$E=\{y\in C([-t_0, T], \mathbf{R}^m)\,|\,y(t)=\dot{\varphi}(t),\ t\in[-t_0, 0]\},$$

$$F=\{y\in C([-t_0, T], \mathbf{R}^m)\,|\,y(t)=0,\ t\in[-t_0, 0]\}.$$

The operator $\mathfrak{U}: E\times\mathbf{R}^m\to F\times\mathbf{R}^m$ is defined by the formula

$$\mathfrak{U}=(\mathfrak{N}, \mathfrak{M}),\quad u=\begin{pmatrix}y\\ \lambda\end{pmatrix}\in E\times\mathbf{R}^m,\quad w=\begin{pmatrix}0\\ e-\beta\varphi(0)\end{pmatrix}\in F\times\mathbf{R}^m,$$

$$(\mathfrak{N}u)(t)=\mathfrak{N}\begin{pmatrix}y\\ \lambda\end{pmatrix}(t)=y(t)-f(t, (Jy)_t, y_t, \lambda),\ t\in[0, T]$$

$$\mathfrak{M}u=\mathfrak{M}\begin{pmatrix}y\\ \lambda\end{pmatrix}=\beta(J^0y)(T)+\int_0^T[d\gamma(t)]y(t).$$

The spaces $E\times\mathbf{R}^m$ and $F\times\mathbf{R}^m$ are endowed with the norm $\|u\|=\max\{\|y\|_C, |\lambda|\}$.

2. Finite-difference method. Let $h=T/n$ where n is an arbitrary large number; let $\mathbf{R}_L$ be a net space consisting of points of the type jh, $j=0, \pm1, \pm2, \ldots$, let $y_h: \mathbf{R}_h\to\mathbf{R}^m$ be the net function; let $\hat{y}_h=\pi_h y_h$ be a piecewise linear function with values $y_h(t)$ at the points $t\in\mathbf{R}_h$ (linear in the segments $[t, t+h]$, $t\in\mathbf{R}_h$); let $\tau_h y$ be the restriction of the function y on $\mathbf{R}_h$ and, finally, let ∂_h be the simplest difference quotient $(\partial_h x_h)(t)=h^{-1}[x_h(t+h)-x_h(t)]$, $t\in\mathbf{R}_h$.

We consider the following difference problem associated with the problem

$$(\partial_h x_h)(t)=f(t, (\hat{x}_h)_t, (\widehat{\partial_h x_h})_t, \lambda_h),\ t\in[0, T)\cap\mathbf{R}_h, \tag{5}$$

$$(x_h)(t)=\varphi(t),\ t\in[-t_0, 0]\cap\mathbf{R}_h, \tag{6}$$

$$\beta x_h(T)+\sum_{t_i\in[0,T)\cap\mathbf{R}_h}(\partial_h x_h)(t_i)[\gamma(t_{i+1})-\gamma(t_i)]=l, \tag{7}$$

where x_h is an element of the space $C_h([0, T], \mathbf{R}^m)$ of the net functions $x_h: [0, T]\cap\mathbf{R}_h\to R^m$ and λ_h is an element of the space $\mathbf{R}^m$ corresponding to the value h in the boundary value problem (5)—(7).

By means of the substitution

$$y_h(t)=\begin{cases}\partial_h\varphi(t), & t\in[-t_0, -h]\cap\mathbf{R}_h\\ (\partial_h x_h)(t), & t\in[0, T-h]\cap\mathbf{R}_h\end{cases}$$

the problem (5)—(7) can be reduced to the system

$$y_h(t)=f(t, (\widehat{J_h y_h})_t, (\hat{y}_h)_t, \lambda_h),\ t\in[0, T-h]\cap\mathbf{R}_h,$$

$$\beta(J_h y_h)(T)+\sum_{t_i\in[0,T)\cap\mathbf{R}_h}y_h(t_i)[\gamma(t_{i+1})-\gamma(t_i)]=l-\beta\varphi(0),$$

which can be written in the form of the operator equation

$$\mathfrak{U}_h u_h=w \tag{8}$$

where

$$[J_h^0 y_h](t)=\begin{cases}0, & t\in[-t_0+h, 0]\cap\mathbf{R}_h\\ \sum\limits_{s\in[0,t)\cap\mathbf{R}_h}hy_h(s), & t\in[h, T]\cap\mathbf{R}_h\end{cases}$$

$$[J_h y_h](t)=\begin{cases}\varphi(t), & t\in[-t_0+h, 0]\cap\mathbf{R}_h\\ \varphi(0)+\sum\limits_{s\in[0,t)\cap\mathbf{R}_h}hy_h(s), & t\in[h, T]\cap\mathbf{R}_h\end{cases}$$

$$u_h=\begin{pmatrix}y_h\\ \lambda_h\end{pmatrix}\in C_h([0, T], \mathbf{R}^m)\times\mathbf{R}^m.$$

The operator $\mathfrak{U}_h=(\mathfrak{N}_h, \mathfrak{M}_h)$ acts from $E_h\times\mathbf{R}^m$ into $F\times\mathbf{R}^m$ where

$$E_h=\{y_h\in C_h([-t_0, T], \mathbf{R}^m)\,|\,y_h(t)=\partial_h\varphi(t),\ t\in[-t_0, -h]\cap\mathbf{R}_h\},$$

$$F_h=\{y_h\in C_h([-t_0, T], \mathbf{R}^m)\,|\,y_h(t)=0,\ t\in[-t_0, -h]\cap\mathbf{R}_h\}$$

and it is defined by the formula

$$(\mathfrak{N}_h u_h)(t)=y_h(t)-f(t, (\widehat{J_h y_h})_t, (\hat{y}_h)_t, \lambda_h),\ t\in[0, T-h]\cap\mathbf{R}_h$$

$$\mathfrak{M}_h u_h=\beta(J_h^0 y_h)(T)+\sum_{t_i\in[0,T)\cap\mathbf{R}_h}y_h(t_i)[\gamma(t_{i+1})-\gamma(t_i)].$$

3. Main result. Theorem 1. *Let the boundary value problem (1)—(3) have a solution $\lambda^*\in R^m$, $x^*\in C^1([-t_0, T], \mathbf{R}^m)\cap(\{x\in C^2([-t_0, 0], \mathbf{R}^m),\ x\in C^2([0, T], \mathbf{R}^m)\}$ and let the operator $f(t, x, y, \lambda)$ be continuous with continuous derivatives with respect to x, y, λ in the domain*

$$t\in[0, T],\ |x(t)-x^*(t)|\leqslant\delta,\ |\dot{x}(t)-\dot{x}^*(t)|\leqslant\delta,\ |\lambda-\lambda^*|\leqslant\delta\ (\delta>0 \text{ const.}) \tag{9}$$

Besides, let $|\partial f(t, x_t^*, \dot{x}_t^*, \lambda^*)/\partial y|\leqslant k<1$ and let the boundary value problem

$$\dot{x}(t)=[\partial f(t, x_t^*, \dot{x}_t^*, \lambda^*)/\partial x]x_t+[\partial f(t, x_t^*, \dot{x}_t^*, \lambda^*)/\partial y]\dot{x}_t+$$

$$+[\partial f(t, x_t^*, \dot{x}_t^*, \lambda^*)/\partial\lambda]\lambda,\ t\in[0, T] \tag{10}$$

$$x(t)=0,\ t\in[-t_0, 0],\ \beta x(T)+\int_0^T[d\gamma(t)]\dot{x}(t)=0$$

have the zero solution only. Then there exist $h_0>0$ and $\delta_0>0$ such that:

1. For $h\leqslant h_0$ the difference problem (5)—(7) will have unique solution $\lambda_h^*\in\mathbf{R}^m$, $x_h^*\in C_h([-t_0, T-h], \mathbf{R}^m)$ in the domain $\max\{\|\hat{x}_h-\widehat{\tau_h x^*}\|_C, |\lambda_h-\lambda^*|\}\leqslant\delta_0$.

2. $$\hat{x}_h^*\xrightarrow[(h\to0)]{}x^* \text{ in } C([-t_0, T], \mathbf{R}^m),\ \lambda_h\to\lambda^* \text{ in } \mathbf{R}^m.$$

If we consider $d/dt\,\hat{x}_h^*$ as a multivalued function, that is, a two-valued one at the nodes and a single-valued one at the other points, then $d/dt\hat{x}_t^* \xrightarrow[(h\to 0)]{} \dot{x}^*$ uniformly with respect to t.

3. $\max\limits_{t\in[0,T)\cap R_h} |\partial_h\tau_h x_h^*(t)-\tau_h\dot{x}^*(t)| \leqslant \max\limits_{t\in[0,T)\cap R_h} |\partial_h\tau_h x^*(t)-\tau_h\dot{x}^*(t)| \leqslant ch,\ |\lambda_h^*-\lambda^*|\leqslant ch.$

We present a corollary of this theorem for the case, when the operator $f(t, x, y, \lambda)$ is linear that is, when the equation (1) has the form

$$(1^\circ)\qquad \dot{x}(t)=A(t)x_t+B(t)\dot{x}_t+C(t)\lambda+\varphi(t),\ t\in[0, T].$$

Here $A(t),\ B(t): C([-t_0, 0], \mathbf{R}^m)\to\mathbf{R}^m$; $C(t): \mathbf{R}^m\to\mathbf{R}^m$; $\varphi\in C([0, T], R^m)$. To this end we use the sufficient and necessary condition for solvability of the boundary value problem $(1)^\circ$, (2), (3) given in [8]. It enables us to reduce the conditions, under which the finite-difference method is convergent for the linear boundary value problem :

Corollary 1. *Let the linear operators $A(t), B(t)$ and $C(t)$ are bounded, and $\|B(t)\|\leqslant k<1$, $t\in[0, T]$. Moreover, let the boundary value problem $(1)^\circ$, (2), (3) have unique solution $\lambda^*\in\mathbf{R}^m$, $X^*\in C^1([-t_0, T], \mathbf{R}^m)\cap\{C^2([-t_0, 0], \mathbf{R}^m)\cup \cup C^2([0, T], \mathbf{R}^m)$. Then there exist $h_0>0$ and $\delta_0>0$ such that :*

1. *For $h\leqslant h_0$ the discrete boundary value problem associated with $(1)^\circ$, (2), (3) has unique solution $x_h^*\in C_h([-t_0, T], \mathbf{R}^m)$, $\lambda_h^*\in\mathbf{R}^m$ in the domain $\max\{\|\hat{x}_h-\widehat{\tau_h x^*}\|_{C^1},\ \|\lambda_h-\lambda^*\|\}\leqslant\delta_0$.*

2. $\hat{x}_h^* \xrightarrow[(h\to 0)]{} x^*$ in $C([-t_0, T], \mathbf{R}^m)$, $\lambda_h^* \xrightarrow[(h\to 0)]{} \lambda^*$ in $\mathbf{R}^m$.

If we consider $d/dt\,\hat{x}_h^*$ as a multivalued function — two-valued at the nodes and single-valued at the others, then $d/dt\,\hat{x}_h^* \xrightarrow[(t\to 0)]{} \dot{x}^*$ uniformly with respect to t.

3. $$\max_{t\in[0,T)\cap R_h} |(\partial_h\tau_h x_t^*)(t)-(\tau_h\dot{x}^*)(t)| \leqslant \max_{t\in[0,T)\cap R_h} |\partial_h\tau_h x^*)(t)-(\tau_h\dot{x}^*)(t)| \leqslant$$
$$\leqslant ch,\ |\lambda_h-\lambda^*|\leqslant ch.$$

4. *Proof.* In order to prove Theorem 1 it is sufficient to apply the following theorem proved in [9] :

Let E and $E_n (n\in N=\{1, 2, \ldots\})$ be real Banach spaces and let $P= =(p_n)_{n\in N}$ be a system of operators $p_n: E\to E_n (n\in N)$ such that

$$\|p_n x\|_{E_n}\to\|x\|_E\ \forall x\in E \text{ as } n\to+\infty.$$

Definition 1. *A sequence $(x_n)_{n\in N}$ with $x_n\in E_n$ P-converges [9] (or converges discretely) to $x\in E$ if $\|x_n-p_n x\|_{E_n} \xrightarrow[(n\to+\infty)]{} 0$.* We simply write $x_n\to x$.

Definition 2. *A sequence $(x_n)_{n\in N}$ with $x_n\in E_n$ is P-compact [9] (or discretely compact) whenever each of its subsequences $(x_n)_{n\in N'\subset N}$ contains a P-convergent subsequence $(x_n)_{n\in N''\subset N'\subset N}$.*

Definition 3. *A sequence of operators $\mathcal{A}_n: E_n\to E_n$ $(n\in N)$ converges regularly [9] to an operator $\mathcal{A}: E\to E$ if:*

(i) $x_n\to x\Rightarrow\mathcal{A}_n x_n\to\mathcal{A}x$;

(ii) $\|x_n\|_{E_n}\leqslant\text{const.}, (\mathcal{A}_n x_n)_{n\in N}$ *is discretely compact* $\Rightarrow (x_n)_{n\in N}$ is discretely compact.

For a detailed account of these concepts, see [9].

We consider the equations $\mathcal{A}x=y$ and $\mathcal{A}_n x_n=y_n$ where $\mathcal{A}: \Omega\subset E\to E$, $\mathcal{A}_n: \Omega_n\subset E_n\to E_n$ are operators defined on open sets $\Omega_n\subset E_n$, $\Omega\subset E$ respectively.

$$\begin{array}{ccc} E\supset\Omega & \xrightarrow{\mathcal{A}} & E \\ {\scriptstyle P_n}\downarrow & & \downarrow{\scriptstyle P_n} \\ E_n\supset\Omega_n & \xrightarrow{\mathcal{A}_n} & E_n \end{array}$$

Theorem 2. *Let the following conditions be satisfied :*

1. *The equation $\mathcal{A}x=y$ has a solution $x^*\in\Omega\subset E$, the operator $\mathcal{A}: \Omega\to E$ is Fréchet differentiable at the point x^* and the relation $[\mathcal{A}'(x^*)]x= =0\Rightarrow x=0$ holds.*

2. *$x_n^0\to x^*$, $A_n x_n^0\to Ax^*(n\in N)$ where x_n^0 are some elements of the space E_n.*

3. *The operators $A_n: \Omega_n\to E_n(\Omega_n\subset E_n$; E, E_n are Banach spaces) are Fréchet differentiable for $\|x_n-x_n^0\|\leqslant\delta(\delta>0$, δ does not depend on n).*

4. *For every $\varepsilon>0$ there exists a number δ' (which does not depend on n) such that $0<\delta'\leqslant\delta$ and $\|x_n-x_n^0\|_{E_n}\leqslant\delta'\Rightarrow\|\mathcal{A}_n'(x_n)-\mathcal{A}_n'(x_n^0)\|_{E_n}\leqslant\delta(n\in N)$.*

5. *The sequence of operators $\mathcal{A}_n'(x_n^0)\in\mathcal{L}(E_n, E_n)$ converges regularly to the operator $\mathcal{A}'(x^*)\in\mathcal{L}(E, E)$.*

6. *The operators $\mathcal{A}_n'(x_n^0)$ are Fredholm ones.*

Then there exists n_0 and δ_0: $0<\delta_0\leqslant\delta$ such that for $n\geqslant n_0$ the equation $\mathcal{A}_n x_n=y_n(y_n\to y, n\in N)$ has unique solution x_n^* in the domain $\|x_n-x_n^0\|\leqslant\delta$. The sequence $\{x_n^*\}_{n\in N}$ converges discretely to x^* (cf. [9]).

The following estimate holds :

$$C_1\|\mathcal{A}_n x_n^0-y_n\|_{E_n}\leqslant\|x_n^*-x_n^0\|_{E_n}\leqslant C_2\|\mathcal{A}_n x_n^0-y_n\|_{E_n},\ (n\geqslant n_0,\ n\in N)$$

where C_1 and C_2 are constants, $0<C_1\leqslant C_2<\infty$.

Let us define

$$\Omega=\{y\in E,\ \lambda\in\mathbf{R}^m/\|y-y^*\|_C<\delta,\ |\lambda-\lambda^*|<\delta\},$$

$$\Omega_h=\{y_h\in E_h,\ \lambda_h\in\mathbf{R}^m/\|\hat{y}_n-\widehat{\tau_n y}\|_C<\delta,\ |\lambda_h-\lambda^*|<\delta\}.$$

Next, consider the operators

$$\mathfrak{U}: \Omega\to F\times\mathbf{R}^m,\quad \mathfrak{U}_h: \Omega_h\to F_h\times\mathbf{R}^m.$$

Without loss of generality, we may consider $\dot{\varphi}(t)\equiv 0$, $t\in[-t_0, 0]$. Then $E\equiv F$ being a closed subspace of a Banach space, is also a Banach space.

If $\dot{\varphi}(t)\not\equiv 0$, $t\in[-t_0, 0]$ we make the shift $y(t)=z(t)+\dot{\varphi}^*(t)$, where

$$\varphi^*(t)=\begin{cases}\dot{\varphi}(t), & t\in[-t_0, 0]\\ \dot{\varphi}(0), & t\in[0, T]\end{cases}$$

and consider the space

$$\hat{E}=\{z\in C([-t_0, T], \mathbf{R}^m)/z(t)=y(t)-\dot{\varphi}^*(t),\ y\in E\}.$$

Obviously, $\widehat{E}\equiv E$ and hence it is a Banach space. The original boundary value problem (1)—(3) takes the form

$$z(t)=g(t, (Jz)_t, z_t, \lambda),\ t\in[0, T],\ \beta(J^0z)(T)+\int_0^T [d\gamma(t)]z(t)=l_1-\beta\varphi(0),$$

where

$$l_1=l-\beta\varphi(0)T-\dot{\varphi}(0)T_0\gamma(t),\ g(t, x, y, \lambda)=f(t, x+(J^0\dot{\varphi}^*)_t,\ y+\dot{\varphi}^*_t, \lambda)-\dot{\varphi}^*(t),$$

$$(J^0\varphi^*)(t)=\begin{cases}\varphi(t)-\varphi(0), & t\in[-t_0, 0]\\ \dot{\varphi}(0)t, & t\in[0, T].\end{cases}$$

After substituting $z(t)=\dot{v}(t)$, $v(0)=\varphi(0)$ we reduce (1)—(3) to the boundary value problem

$$\dot{v}(t)=g(t, v_t, \dot{v}_t, \lambda),\ t\in[0, T],$$

(11) $$v(t)=\varphi(0),\ t\in[-t_0, 0],\ \beta v(T)+\int_0^T [d\gamma(t)]\dot{v}(t)=l_1,$$

for which the derivative of the initial value function $v(t)=\varphi(0)$, $t\in[-t_0, 0]$ is identically zero on the interval $[-t_0, 0]$. One can immediately verify that the obtained boundary value problem has the pair (v^*, λ^*), $v^*\in E$, $\lambda^*\in\mathbf{R}^m$,

$$v^*(t)=\begin{cases}\varphi(0), & t\in[-t_0, 0]\\ x^*(t)-\dot{\varphi}(0)t, & t\in[0, T]\end{cases}$$

as a solution, for $g(t, \zeta, \eta, \lambda)$ and $f(t, x, y, \lambda)$ the identity $|\partial g(t, v^*_t, \dot{v}^*_t, \lambda^*)|\partial\eta|\equiv\equiv\partial f(t, x^*_t, \dot{x}^*_t, \lambda^*)|\partial y|\leqslant k<1$ holds, and then the corresponding to (11) linearized homogeneous boundary value problem coincides with (10) and therefore has zero solution only.

This is to show that it is sufficient to carry out the proof of Theorem 1 in the case $\dot{\varphi}(t)\equiv 0$, $t\in[-t_0, 0]$ since the rest of the cases, by a suitable shift, can be reduced to it.

The proof of Theorem 1 is based on Theorem 2. We shall verify its conditions.

Let us verify Condition 1. In virtue of Theorem 1 the boundary value problem (1)—(3) has a solution (x^*, λ^*). It is clear that $y^*=\dot{x}^*$, λ^* is a solution of the operator equation

$$\mathfrak{U}\begin{pmatrix}y\\ \lambda\end{pmatrix}=w.$$

The Fréchet differentiability of the operator $f(t, x, y, \lambda)$ with respect to x, y, λ in the domain (9) implies the differentiability of the operator $\mathfrak{U}$ at the point $u^*=\begin{pmatrix}y^*\\ \lambda^*\end{pmatrix}$. Furthermore, if we denote $\mathfrak{U}'(u^*)$ by $\mathfrak{B}=(\mathfrak{B}_1, \mathfrak{B}_2)$ the following equality will hold

$$\left[\mathfrak{B}_1\begin{pmatrix}y\\ \lambda\end{pmatrix}\right](t)=y(t)-a(t)(J^0y)_t-C(t)y_t-d(t)\lambda,\ t\in[0, T]$$

$$\mathfrak{B}_2\begin{pmatrix}y\\ \lambda\end{pmatrix}=\beta(J^0y)(T)+\int_0^T [d\gamma(t)]y(t)$$

where

$$a(t)=\partial f(t, (Jy^*)_t, y^*_t, \lambda^*)/\partial x,\ b(t)=\partial f(t, (Jy^*)_t, y^*_t, \lambda^*)/\partial y,$$

$$d(t)=\partial f(t, (Jy^*)_t, y^*_t, \lambda^*)/\partial\lambda$$

for $f(t, x, y, \lambda)$.

The implication $\mathfrak{B}u=0\Rightarrow u=o$ holds, i.e. the linearized homogeneous problem (10) has unique solution in virtue of the conditions of Theorem 1.

Let us verify Condition 2. The convergence $p_h\widehat{u^*}\to u^*$ in $E\times\mathbf{R}^m$ is obvious since $\widehat{\tau_h y^*}\to y^*$ in $C([-t_0, PJ, \mathbf{R}^m)$. In order to show that $\mathfrak{U}_hp_hu^*\to\to\mathfrak{U}u^*\left(p_h=\begin{pmatrix}\tau_h\\ I\end{pmatrix}: E\times\mathbf{R}^m\to E_h\times\mathbf{R}^m;\ p_h\begin{pmatrix}y\\ \lambda\end{pmatrix}=\begin{pmatrix}\tau_hy\\ \lambda\end{pmatrix}\right)$ it is sufficient to show that the second components of the operators $\mathfrak{U}_h$ and $\mathfrak{U}$ satisfy $\mathfrak{N}_hp_hu^*\xrightarrow[(h\to 0)]{}\mathfrak{N}u^*$ in the sense of discrete convergence. We shall prove the more general relation $\mathfrak{N}_hu_h\to\mathfrak{N}u$ for $u_h\to u$, $u_h\in\bar{\Omega}_h$.

Since f is a continuous operator by condition, then for the purpose it is sufficient to show that

(12) $$y_h\to y\Rightarrow J_hy_h\to Jy$$

i.e. $$\|\hat{y}_h-y\|_C\to 0\Rightarrow\|\widehat{J_hy_h}-Jy\|_C\to 0.$$

In virtue of the boundedness of the operator J the latter correlation will hold if we establish the estimate

(13) $$\|\widehat{J_hy_h}-J\hat{y}_h\|_C\leqslant 3h\|\hat{y}_h\|_C.$$

First we shall estimate $\|\widehat{J_hy_h}-J\hat{y}_h\|_C$ at the nodes $t\in\mathbf{R}_h$. We have

$$(\widehat{J_hy_h}-J\hat{y}_h)(t)=h\sum_{s\in[0,t]\cap\mathbf{R}_h}y_h(s)-(t/\omega)h\sum_{s\in(0,\omega]\cap\mathbf{R}_h}y_h(s)-\int_0^t\hat{y}_h(s)ds+(t/\omega)\int_0^\omega\hat{y}_h(s)ds.$$

Since

$$\int_0^t\hat{y}_h(s)ds=h\left\{\frac{1}{2}y_h(0)+y_h(h)+y_h(2h)+\ldots+y_h(t-h)+\frac{1}{2}y_h(t)\right\},$$

then

$$\int_0^t\hat{y}_h(s)ds-h\sum_{s\in[0,t]\cap\mathbf{R}_h}y_h(s)=\frac{1}{2}h\{y_h(0)-y_h(t)\}$$

and hence

(14) $$\max_{t\in[0,T]\cap\mathbf{R}_h}|(\widehat{J_hy_h})(t)-(J\hat{y}_h)(t)|\leqslant\|y_h\|_C.h.$$

In the case $t\notin\mathbf{R}_h$, i.e. $jh<j<(j+1)h$ $(j\in\{1, 2, \ldots, n\})$ in virtue of the definition of the function $\hat{y}_h$ we have

$$\hat{y}_h(t)=\lambda y_h(jh)+(1-\lambda)y_h((j+1)h),\ \lambda\in(0, 1)$$

and then

$$(\widehat{J_h y_h})(t)-(J\hat{y}_h)(t)=\lambda(J_h y_h)(jh)+(1-\lambda)(J_h y_h)((j+1)h)-\lambda(J\hat{y}_h)(jh)-$$
$$-(1-\lambda)(J\hat{y}_h)((j+1)h)+\lambda\{(J\hat{y}_h)(jh)-(J\hat{y}_h)(t)\}+(1-\lambda)\{(J\hat{y}_h)((j+1)h)-$$
$$-(J\hat{y}_h)(t)\}=\lambda\{(J_h y_h)(jh)-(J\hat{y}_h)(jh)\}+(1-\lambda)\{(J_h y_h)((j+1)h)-$$
$$-(J\hat{y}_h)((j+1)h)\}+\lambda\{(J\hat{y}_h)(jh)-(J\hat{y}_h)(t)\}+(1-\lambda)\{(J\hat{y}_h)((j+1)h)-(J\hat{y}_h)(t)\},$$

whence, in virtue of (14) and the equality

$$(J\hat{y}_h)(jh)-(J\hat{y}_h)(t)=\int_t^{jh}\hat{y}_h(s)ds-\frac{jh-t}{\omega}\int_0^{\omega}\hat{y}_h(s)ds$$

there follows (13).

Thus the relation (12) is also valid, since $p_h u^* \xrightarrow{p} u^*$ and so is the property 2.

Condition 3 is satisfied since the differentiability of the operator f in the domain (9) implies the differentiability of the operators $\mathfrak{U}_h:\Omega_h\to\to F_h\times\mathbf{R}^m$ in the domain $\|\hat{y}_h-y^*\|_C\leqslant\delta$, $|\lambda_h-\lambda^*|\leqslant\delta$.

Furthermore

$$\left[\mathcal{U}_h'\begin{pmatrix}\tilde{y}_h\\ \lambda_h\end{pmatrix}\right]\begin{pmatrix}y_h\\ \lambda_h\end{pmatrix}(t)=\left[\mathfrak{B}_h\begin{pmatrix}y_h\\ \lambda_h\end{pmatrix}\right](t)=(\mathfrak{B}_h^{(1)},\ \mathfrak{B}_h^{(2)})\begin{pmatrix}y_h\\ \lambda_h\end{pmatrix}(t),$$

$$\left[\mathfrak{B}_h^{(1)}\begin{pmatrix}y_h\\ \lambda_h\end{pmatrix}\right](t)=y_h(t)-[\partial f(t,(\widehat{J_h\tilde{y}_h})_t,(\pi_h\tilde{y}_h)_t,\tilde{\lambda}_h)/\partial x]\times(J_h^0 y_h)_t-$$

$$-[\partial f(t,(\widehat{J_h\tilde{y}_h})_t,(\pi_h\tilde{y}_h)_t,\tilde{\lambda}_h)/\partial y](y_h)_t-[\partial f(t,(\widehat{J_h\tilde{y}_h})_t,(\pi_h\tilde{y}_h)_t,\tilde{\lambda}_h)/\partial\lambda]\lambda h,$$

$$\mathfrak{B}_h^{(2)}\begin{pmatrix}y_h\\ \lambda_h\end{pmatrix}=\beta(J_h^0 y_h)(T)+\sum_{t_i\in[0,T)\cap\mathbf{R}_h}y_h(t_i)[\gamma(t_{i+1})-\gamma(t_i)].$$

Now, let us verify the condition 4. From the continuity of the functions $\partial f(t,x,y,\lambda)/\partial x$, $\partial f(t,x,y,\lambda)/\partial y$ and $\partial f(t,x,y,\lambda)/\partial\lambda$ it follows that for every $\varepsilon>0$ there exists a number δ' which does not depend on h, such that $0<\delta'\leqslant\delta$ and

$$\|\hat{y}_h-y^*\|_c\leqslant\delta',\ |\lambda_h-\lambda^*|\leqslant\delta'\Rightarrow\|\mathfrak{B}_h(u_h)-\mathfrak{B}_h(p_h u^*)\|\leqslant\varepsilon$$

where

$$u_h=\begin{pmatrix}y_h\\ \lambda_h\end{pmatrix},\qquad u^*=\begin{pmatrix}y^*\\ \lambda^*\end{pmatrix}.$$

In order to verify Condition 5 we should know that the sequence of operators $\mathfrak{U}_h'(p_h u^*)\in\mathcal{L}(C_h\times\mathbf{R}^m,\ C_h\times\mathbf{R}^m)$ converges regularly [9] to the operator

$$\mathfrak{N}'(u^*)\in\mathcal{L}(E\times\mathbf{R}^m,\ F\times\mathbf{R}^m).$$

Let us denote the coefficients of the differential operator $\mathfrak{N}_h'(p_h u^*)$ by a_h, b_h and d_h where

$$a_h(t)=\partial f(t,(\pi_h J_h\tau_h y^*)_t,\ (\pi_h\tau_h y^*)_t,\ \lambda^*)/\partial x,\quad b_h(t)=\partial f(t,(\pi_h J_h\tau_h y^*)_t,$$

$$(\pi_h\tau_h y^*)_t,\ \lambda^*)/\partial y,\quad d_h(t)=\partial f(t,(\pi_h J_h\tau_h y^*)_t,\ (\pi_h\tau_h y^*)_t,\ \lambda^*)/\partial\lambda.$$

Since $\|\hat{a}_h-a\|_C\to0$, $\|\hat{b}_h-b\|_C\to0$, $\|\hat{d}_h-d\|_C\to0$ then in order to prove that the operators $\mathfrak{N}_h'(p_h u^*)$ converge regularly [9] to the operator $\mathfrak{N}'(u^*)$ it is sufficient to show that the sequence of the operators

$$(\mathfrak{F}_h u_h)(t)=y_h(t)-\tau_h a(t)(\widehat{J_h y_h})_t-\tau_h b(t)(\hat{y}_h)_t-\tau_h d(t)\lambda_h,$$
$$\mathfrak{T}_h u_h=\beta(J_h^0 y_h)(T)+\sum_{t_i\in[0,T)\cap\mathbf{R}_h}y_h(t_i)[\gamma(t_{i+1})-\gamma(t_i)]$$

converges regularly to the operator

$$\mathfrak{N}'[(u^*)]u(t)=y(t)-a(t)(J^0y)_t-b(t)y_t-d(t)\lambda,$$

$$[\mathfrak{M}'(u^*)]u(t)=\beta(J^0y)(T)+\int_0^T[d\gamma(t)]y(t)$$

when $\hat{y}_h\to y$, $\lambda_h\to\lambda$.

In order to prove the latter relation we must show (cf. Definition 3) that

1. $\hat{u}_h\to u\Rightarrow(\pi_h\mathfrak{F}_h u_h,\ \mathfrak{T}_h u_h)\to(\mathfrak{N}'(u^*)u,\ \mathfrak{M}'(u^*)u)$

where

$$u_h=\begin{pmatrix}y_h\\ \lambda_h\end{pmatrix},\qquad u=\begin{pmatrix}y\\ \lambda\end{pmatrix}.$$

2. $\|\hat{u}_h\|\leqslant$ const., $\{\mathfrak{F}_h u_h,\ \mathfrak{T}_h u_h\}$ is discretely compact $\Rightarrow(u_h)$ is discretely compact.

The first relation is obvious since $\widehat{\tau_h a}\to a$, $\widehat{\tau_h b}\to b$, $\widehat{\tau_h d}\to d$, $\pi_h J_h^0 u_h\to J^0 u$ if $\hat{u}_h\to u$.

Let us verify the second relation. Since the sequence $(\mathfrak{F}_h u_h,\ \mathfrak{T}_h u_h)$ is discretely compact, we have $\mu\{\mathfrak{F}_h u_h,\ \mathfrak{T}_h u_h)\}=0$, where μ is the discrete measure of noncompactness in (cf. [9]) defined by the formula

$$\mu\left\{\begin{pmatrix}y_h\\ \lambda_h\end{pmatrix}\right\}=\mu^*\{(y_h)\},$$

μ^* is the discrete measure of noncompactness in $C_h([-t_0,T],\mathbf{R}^m)$.

From the relation $\mu\{(\mathfrak{F}_h u_h,\ \mathfrak{T}_h u_h)\}=0$ and from the definition of the measure of noncompactness μ it follows that

$$\mu^*\{(y_h)\}\leqslant\mu^*\{\tau_h a(t)(J_h^0 y_h)_t+\tau_h b(t)(y_h)_t+\tau_h d(t)\lambda_h\}\leqslant k\mu^*\{(y_h)\}$$

since if $\|\hat{u}_h\|\leqslant$ const then (λ_h) is compact as a bounded sequence in $\mathbf{R}^m$ and by virtue of Arzelà's theorem the sequence $(J_h^0 y_h)$ is compact as well. By assumption we have $\|b(t)\|\leqslant k<1$ and therefore $\mu^*(y_h)=0$ i.e. $(\hat{y}_h)$ is discretely compact. Hence $u_h=\begin{pmatrix}y_h\\ \lambda_h\end{pmatrix}$ is discretely compact too, since $\mu(u_h)=\mu^*(y_h)=0$. Thus Condition 5 of Theorem 2 is verified.

From the finite dimensionality of the space $C_h([0,T],\ \mathbf{R}^m)$ and $\mathbf{R}^m$ it follows that the operators $\mathfrak{U}_h'(\tau_h u^*)\in\mathcal{L}(C_h\times\mathbf{R}^m,\ C_h\times\mathbf{R}^m)$ are Fredholm ones.

According to Theorem 2 there exist $h_0>0$ and $\delta_0: 0<\delta_0\le\delta$ such that for $h\le h_0$ the equation $\mathfrak{U}_h u_h=0$ (and consequently the difference problem (5)—(7) as well) will have a unique solution $u_h^*=\binom{y_h^*}{\lambda_h^*}$, $\lambda_h^*\in\mathbf{R}^m$, $y_h^*\in C_h([-t_0,T],\mathbf{R}^m)$ in the domain $|\lambda_h^*-\lambda^*|\le\delta_0$, $\|\hat{y}_h^*-y^*\|_C\le\delta_0$. Furthermore $\hat{y}_h^*\to y^*$ in $C([-t_0,T],\mathbf{R}^m)$, $\lambda_h^*\to\lambda^*$ in $\mathbf{R}^m$ (and consequently $\widehat{\partial_h x_h^*}\to\dot{x}_h^*$, $\hat{x}_h^*\to x^*$, $\lambda_h^*\to\lambda^*$) with the same rate as

$$\max_{t\in[0,T]\cap\mathbf{R}_h}\|(u_h^*-\tau_h u^*)(t)\|\le c.\max_{t\in[0,T]\cap\mathbf{R}_h}\|\mathfrak{U}_h\tau_h u^*(t)-\tau_h\mathfrak{U}u^*(t)\|,$$

i.e.

$$\max\{\max_{t\in[0,T]\cap\mathbf{R}_h}|y_h^*(t)-\tau_h y^*(t)|,\ |\lambda_h^*-\lambda^*|\}\le$$

$$\le c.\max_{t\in[0,T]\cap\mathbf{R}_h}\{\max\|\mathfrak{N}_h\tau_h u^*(t)-\tau_h\mathfrak{N}u^*(t)\|,|\int_0^T[d\gamma(t)]y^*(t)+$$

$$+\beta(J_h^*\tau_h y^*-\tau_h J^0 y^*)(T)-\sum_{t_i\in[0,T)\cap\mathbf{R}_h}y^*(t_i)[\gamma(t_{i+1})-\gamma(t_i)]\}.$$

Since $y=\dot{x}$, $y_h=\partial_h x_h$ we obtain the estimate

$$\max\{\max_{\in[0,T]\cap\mathbf{R}_h}|\partial_h x_h^*(t)-\tau_h\dot{x}^*(t)|,\ |\lambda_h^*-\lambda^*|\}\le$$

$$\le c.\max\{\max_{t\in[0,T]\cap\mathbf{R}_h}|[\partial_h\tau_h x^*(t)-\tau_h\dot{x}^*(t)]+$$

$$+[f(t,(\pi_h\tau_h x^*)_t,(\pi_h\partial_h\tau_h x^*)_t,\lambda^*)-\tau_h f(t,x_t^*,\dot{x}_t^*,\lambda^*)]|,$$

$$\|\beta\|.|(J_h\tau_h y^*-\tau_h Jy^*)(T)|+|\int_0^T[d\gamma(t)]y^*(t)-\sum_{t_i\in[0,T)\cap\mathbf{R}_h}y^*(t_i)[\gamma(t_{i+1})-\gamma(t_i)]|\}.$$

It is easy to obtain the estimate

$$\eta_h\equiv\|\int_0^T[d\gamma(t)y^*(t)-\sum_{t_i\in[0,T)\cap\mathbf{R}_h}y^*(t_i)[\gamma(t_{i+1})-\gamma(t_i)]\|\le\|\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}[d\gamma(t)][y^*(t)-y^*(t_i)]\|$$

and therefore $\eta_h\le h.\|\dot{y}^*\|_C.\bigvee_0^T{}_0(\gamma)$. Since for $x\in C^2([0,T],\mathbf{R}^m)$

$$\|h^{-1}[x(jh+h)-x(jh)]-\dot{x}(jh)\|\approx 0(h),$$

i.e. for $\|J_h\tau_h y^*-\tau_h Jy^*\|_C\le 2h\|y^*\|_C$, $\eta_h=\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}[y^*(t)-y^*(t_i)]d\gamma(t)$ the following relation holds

(16) $$\|J_h\tau_h y^*-\tau_h Jy^*\|_C\le\text{const}.\ h,\quad \eta_h\le\text{const}.\ h.$$

On the other hand

$$\max_{t\in[0,T]\cap\mathbf{R}_h}|\partial_h\tau_h x^*(t)-\tau_h\dot{x}^*(t)|+$$

$$+\max_{\in[0,T]\cap\mathbf{R}_h}|f(t,(\pi_h\tau_h x^*)_t,(\pi_h\partial_h\tau_h x^*)_t,\lambda_h)-\tau_h f(t,x_t^*,\dot{x}_t^*,\lambda^*)|\le$$

(17) $$\le\max_{t\in[-t_0,T]\cap\mathbf{R}_h}|(\pi_h\partial_h\tau_h x^*)(t)-\tau_h\dot{x}^*(t)|+a.\max_{t\in[-t_0,T]\cap\mathbf{R}_h}|(\pi_h\tau_h x^*)(t)-x^*(t)|+$$

$$+b.\max_{t\in[-t_0,T]\cap\mathbf{R}_h}|(\pi_h\partial_h\tau_h x^*)(t)-\tau_h\dot{x}^*(t)|+d|\lambda^*-\lambda^*|\le$$

$$\le(b+1)\max_{t\in[-t_0,T]\cap\mathbf{R}_h}|\partial_h\tau_h x^*(t)-\tau_h\dot{x}^*(t)|\le\text{const}.\ h\quad(a,\ b-\text{const}.)$$

since

$$\|h^{-1}[x(jh+h)-x(jh)]-\dot{x}(jh)\|\approx 0(h),\quad x\in C^2([0,T],\mathbf{R}^m).$$

Taking into account (15), (16), (17) we obtain the estimate

$$\max_{t\in[0,T]\cap\mathbf{R}_h}|\partial_h x_h^*(t)-\tau_h\dot{x}^*(t)|\le\text{const}.\ h,\quad |\lambda_h^*-\lambda^*|\le\text{const}.\ h.$$

Thus Theorem 1 is proved.

5. Example. We consider the boundary value problem

$$\dot{x}(t)=x(t-1)+\frac{1}{2}\dot{x}(t-1)+\lambda,\quad t\in[0,2],$$

$$x(t)=1,\quad t\in[-1,0],\quad x(2)-x(1)-x(0)=2.$$

It has unique solution

$$\lambda=\frac{1}{2},\quad x(t)=\begin{cases}1, & t\in[-1,0]\\ \frac{3}{2}t+1, & t\in[0,1]\\ \frac{3}{4}t^2+\frac{3}{4}t+1, & t\in[1,2].\end{cases}$$

By means of the finite-difference method we get the following formulas for approximate deriving the solution and its derivative at the nodes jh (j is integer)

$$\lambda_h=\frac{2+h}{4-h},\quad x_h(jh)=1+\frac{6jh}{4-h}\left(0\le j\le\frac{n}{2}\right),$$

$$x_h(1+jh)=1+\frac{6}{4-h}\left[1+\frac{3}{2}jh+\frac{j(j-1)}{2}h^2\right]\left(0\le j\le\frac{n}{2}\right),$$

$$(\partial_h x_h)(jh)=\frac{6}{4-h}\left(0\le j\le\frac{n}{2}-1\right),\quad(\partial_h x_h)(1+jh)=3\frac{3+2jh}{4-h}\left(0\le j\le\frac{n}{2}-1\right).$$

For obtaining the error we use the expressions

$$\max_{t\in[0,2]\cap\mathbf{R}_h}|x(t)-x_h(t)|-\frac{3h}{2(4-h)},$$

$$\max_{t\in[0,2)\cap\mathbf{R}_h}|\dot{x}(t)-\partial_h x_h(t)|=\frac{3h}{2(4-h)}\cdot\frac{5-2h}{2},$$

$$|\lambda_h-\lambda|=\frac{3h}{2(4-h)}.$$

The convergence rate of the approximate method considered for steo sizes $h=1$; 0,1 ; 0,01 ; 0,001 ; 0,0001 (with accuracy to the fifth sigh) is presented in the following table.

a error	h				
	1	0,1	0,01	0,001	0,0001
$\max\limits_{[0,2]\cap R_h} \lvert x(t)-x_h(t)\rvert$	0,5	0,03846	0,00375	0,00037	0,00003
$\max\limits_{[0,2]\cap R_h} \lvert \dot{x}(t)-\partial_h x_h(t)\rvert$ $(t\neq 1)$	0,75	0,09230	0,00936	0,00092	0,00007
$\lambda_h-\lambda$	0,5	0,03846	0,00375	0,00037	0,00003

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SHARP ESTIMATES FOR THE WENDROFF DISCRETE INEQUALITY IN n-INDEPENDENT VARIABLES

BY

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1. Introduction. In our recent paper [1] we have defined the discrete analogue of Riemann's function and used it in the study of Gronwall type inequalities in n independent variables. Here, we shall use this function and obtain Wendroff type of estimates. As a consequence of this approach like in the continuous case [2, 7], we relax some of the conditions needed on the functions which appear in the inequalities and obtain sharper estimates than those obtained for the particular cases in [3—6].

2. Basic Lemmas. Let $N=\{0, 1, \dots\}$ and the product $N\times N\times \dots \times N$ (n times) be denoted by N^n. For all $a>b\in N$ and any $f(s)$ defined on N, $\sum\limits_{s=a}^{b} f(s)=0$ and $\prod\limits_{s=a}^{b} f(s)=1$. For all $s,\ x\in N^n$; $0\leqslant s<x$ that is $0\leqslant s_i<x_i$ and any $f(p)$ defined on N^n, $\sum\limits_{p=s}^{x-1} f(p)$ represents the n-fold sum $\sum\limits_{p_1=s_1}^{x_1-1}\dots\sum\limits_{p_n=s_n}^{x_n-1} f(p_1, \dots, p_n)$ and $\sum\limits_{p=s}^{x_i-1} f(s)$ the $(n-1)$-fold sum $\sum\limits_{p_1=s_1}^{x_1-1}\dots\sum\limits_{p_{i-1}=s_{i-1}}^{x_{i-1}-1}\ \sum\limits_{p_{i+1}=s_{i+1}}^{x_n-1}\ \sum\limits_{p_n=s_n}^{x_{n+1}-1} f(p_1, \dots, p_n)$. The $\Delta_p^n f(p)$ denotes $\Delta_{p_1}\dots\Delta_{p_n} f(p_1, \dots, p_n)$ where Δ is the forward difference.

Lemma 2.1. [1] *Let $g(x)$ and $h(x)$ be defined and nonnegative on N^n and the following inequality be satisfied*

$$\Delta_x^n u(x)\leqslant g(x)\, u(x)+h(x),\quad u(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)=0,\quad 1\leqslant i\leqslant n.$$

Then, $u(x)\leqslant \sum\limits_{s=0}^{x-1} h(s)\, v(s+1,\ x)$ where $v(s, x)$ is the solution of

$$(-1)^n\Delta_s^n v(s, x)=g(s)\, v(s+1, x),\quad v(s, x)=1 \text{ on } s_i=x_i,\quad 1\leqslant i\leqslant n.$$

Lemma 2.2. [1] *Let $g(x)$ and $v(s, x)$ be same as in lemma 2.1. Then,*

$$v(s, x)\leqslant \min_{1\leqslant i\leqslant n}\left\{\prod_{p_i=s_i}^{x_i-1}\left[1+\sum_{p=s}^{x_i+1} g(p)\right]\right\}.$$

Lemma 2.3. *Let $a_i,\ b_i$; $0\leqslant i\leqslant m$ be nonnegative constants. Then,*

$$\prod_{i=1}^{m}(1+a_i+b_i)\geqslant \prod_{i=0}^{m}(1+b_i)+\sum_{i=0}^{m} a_i\prod_{j=0}^{i-1}(1+a_j)\prod_{j=i+1}^{m}(1+b_j). \tag{2.1}$$

Proof. For $m=0$, (2.1) reduces to an identity. Let (2.1) be true for $m=k$, then

$$\prod_{i=0}^{k+1}(1+a_i+b_i)\geqslant \prod_{i=0}^{k+1}(1+b_i)+\sum_{i=0}^{k} a_i\prod_{j=0}^{i-1}(1+a_j)\prod_{j=i+1}^{k+1}(1+b_j)+$$

$$+a_{k+1}\sum_{i=0}^{k} a_i\prod_{j=0}^{i-1}(1+a_j)\prod_{j=i+1}^{k}(1+b_j)+a_{k+1}\prod_{i=0}^{k}(1+b_i)\geqslant \prod_{i=0}^{k+1}(1+b_i)+$$