The convergence rate of the approximate method considered for steo sizes $h=1$; 0,1 ; 0,01 ; 0,001 ; 0,0001 (with accuracy to the fifth sigh) is presented in the following table.

a error	h				
	1	0,1	0,01	0,001	0,0001
$\max\limits_{[0,2]\cap R_h} \lvert x(t)-x_h(t)\rvert$	0,5	0,03846	0,00375	0,00037	0,00003
$\max\limits_{[0,2]\cap R_h} \lvert \dot{x}(t)-\partial_h x_h(t)\rvert$ $(t\neq 1)$	0,75	0,09230	0,00936	0,00092	0,00007
$\lambda_h-\lambda$	0,5	0,03846	0,00375	0,00037	0,00003

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SHARP ESTIMATES FOR THE WENDROFF DISCRETE INEQUALITY IN n-INDEPENDENT VARIABLES

BY

RAVI P. AGARWAL

1. Introduction. In our recent paper [1] we have defined the discrete analogue of Riemann's function and used it in the study of Gronwall type inequalities in n independent variables. Here, we shall use this function and obtain Wendroff type of estimates. As a consequence of this approach like in the continuous case [2, 7], we relax some of the conditions needed on the functions which appear in the inequalities and obtain sharper estimates than those obtained for the particular cases in [3—6].

2. Basic Lemmas. Let $N=\{0, 1, \ldots\}$ and the product $N\times N\times\ldots\times N$ (n times) be denoted by N^n. For all $a>b\in N$ and any $f(s)$ defined on N, $\sum\limits_{s=a}^{b} f(s)=0$ and $\prod\limits_{s=a}^{b} f(s)=1$. For all s, $x\in N^n$; $0\leqslant s<x$ that is $0\leqslant s_i<x_i$ and any $f(p)$ defined on N^n, $\sum\limits_{p=s}^{x-1} f(p)$ represents the n-fold sum $\sum\limits_{p_1=s_1}^{x_1-1}\ldots\sum\limits_{p_n=s_n}^{x_n-1} f(p_1, \ldots, p_n)$ and $\sum\limits_{p=s}^{x_i-1} f(s)$ the $(n-1)$-fold sum $\sum\limits_{p_1=s_1}^{x_1-1}\ldots\sum\limits_{p_{i-1}=s_{i-1}}^{x_{i-1}-1}\ \sum\limits_{p_{i+1}=s_{i+1}}^{x_n-1}\ \sum\limits_{p_n=s_n}^{x_{n+1}-1} f(p_1, \ldots, p_n)$. The $\Delta_p^n f(p)$ denotes $\Delta_{p_1}\ldots\Delta_{p_n} f(p_1, \ldots, p_n)$ where Δ is the forward difference.

Lemma 2.1. [1] *Let $g(x)$ and $h(x)$ be defined and nonnegative on N^n and the following inequality be satisfied*

$$\Delta_x^n u(x)\leqslant g(x)\, u(x)+h(x),\quad u(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)=0,\quad 1\leqslant i\leqslant n.$$

Then, $u(x)\leqslant \sum\limits_{s=0}^{x-1} h(s)\, v(s+1, x)$ where $v(s, x)$ is the solution of

$$(-1)^n\Delta_s^n v(s, x)=g(s)\, v(s+1, x),\ v(s, x)=1 \text{ on } s_i=x_i,\ 1\leqslant i\leqslant n.$$

Lemma 2.2. [1] *Let $g(x)$ and $v(s, x)$ be same as in lemma 2.1. Then,*

$$v(s, x)\leqslant \min_{1\leqslant i\leqslant n}\left\{\prod_{p_i=s_i}^{x_i-1}\left[1+\sum_{p=s}^{x_i-1} g(p)\right]\right\}.$$

Lemma 2.3. *Let a_i, b_i ; $0\leqslant i\leqslant m$ be nonnegative constants. Then,*

$$\prod_{i=1}^{m}(1+a_i+b_i)\geqslant \prod_{i=0}^{m}(1+b_i)+\sum_{i=0}^{m} a_i\prod_{j=0}^{i-1}(1+a_j)\prod_{j=i+1}^{m}(1+b_j). \tag{2.1}$$

Proof. For $m=0$, (2.1) reduces to an identity. Let (2.1) be true for $m=k$, then

$$\prod_{i=0}^{k+1}(1+a_i+b_i)\geqslant \prod_{i=0}^{k+1}(1+b_i)+\sum_{i=0}^{k} a_i\prod_{j=0}^{i-1}(1+a_j)\prod_{j=i+1}^{k+1}(1+b_j)+$$

$$+a_{k+1}\sum_{i=0}^{k} a_i\prod_{j=0}^{i-1}(1+a_j)\prod_{j=i+1}^{k}(1+b_j)+a_{k+1}\prod_{i=0}^{k}(1+b_i)\geqslant \prod_{i=0}^{k+1}(1+b_i)+$$

$$+\sum_{i=0}^{k} a_i \prod_{j=0}^{i-1}(1+a_j)\prod_{j=i+1}^{k+1}(1+b_j)+a_{k+1}\, a_k \prod_{j=0}^{k-1}(1+a_j)+a_{k+1}.$$

Thus, by induction (2.1) is true for all m.

3. Main Results. In what follows we shall assume that the functions which appear in the inequalities are real-valued, nonnegative and defined on N^n.

Theorem 3.1. *Let for all $x\in N^n$, the following inequality be satisfied*

$$u(x)\leqslant \sum_{i=1}^{n} a_i(x_i)+\sum_{s=0}^{x-1} r(s)\, u(s). \tag{3.1}$$

Then,

$$u(x)\leqslant \sum_{i=1}^{n} a_i(x_i)+\sum_{s=0}^{x-1} r(s)\,\Big(\sum_{i=1}^{n} a_i(s_i)\Big)\, v_r(s+1,\ x)=R_r(x)\ ; \tag{3.2}$$

also, if $\Delta_{x_i}(x_i)\geqslant 0$, $1\leqslant i\leqslant n$; then

$$R_r(x)\leqslant \min_{1\leqslant i\leqslant r}\ \Big\{\Big({}^{i}\sum_{j=1}^{n} a_j(x_j)+a_i(0)\Big)\, v_r(0,\ x)+\sum_{s_i=0}^{x_i-1}\Delta_{s_i}\, a_i(s_i)\, v_r(0,\ \ldots,\ 0,\ s_i+1,\ 0,\ \ldots,\ 0,\ x)\Big\}=S_r(x), \tag{3.3}$$

where $v_r(s,\ x)$ is the solution of

(3.4) $\quad (-1)^n\Delta_s^n v_r(s,\ x)=r(s)\, v_r(s+1,\ x)$, (3.5) $\ v_r(s,\ x)=1$ on $s_i=x_i$, $1\leqslant i\leqslant n$.

Proof. Let $\Phi(x)$ be defined as $\Phi(x)=\sum_{s=0}^{x-1} r(s)\, u(s)$. Then, from (3.1) it follows that $\Delta_x^n\Phi(x)\leqslant r(x)\ [\sum_{i=1}^{n} a_i(x_i)+\Phi(x)]$ and, an application of lemma 2.1 provides

$$\Phi(x)\leqslant \sum_{s=0}^{x-1} r(s)\,\Big(\sum_{i=1}^{n} a_i(s_i)\Big)\, v_r(s+1,\ x). \tag{3.6}$$

Substituting the estimate (3.6) in (3.1), the inequality (3.2) follows.

Since, $v_r(s,\ x)$ is the solution of (3.4), (3.5), we have

$$\sum_{s=0}^{x-1} r(s)\,\Big(\sum_{i=1}^{n} a_i(s_i)\Big)\, v_r(s+1,\ x)=\sum_{s=0}^{x-1}\Big(\sum_{i=1}^{n} a_i(s_i)\Big)(-1)^n\, v_r(s,\ x)=$$

$$=(-1)^n\, {}^{i}\sum_{s=0}^{x-1}\Big\{\Big[\sum_{j=1}^{n} a_j(s_j)+a_i(p_i)\Big]\times \Delta^{n-1}_{s_1\ldots s_{i-1}s_{i+1}\ldots s_n} v_r(s_1,\ \ldots,\ s_{i-1},\ p_i,\ s_{i+1},\ \ldots,\ s_n)$$

$$-\sum_{s_i=0}^{x_i-1}\Delta_{s_i} a_i(s_i)\,\Delta^{n-1}_{s_1\ldots s_{i-1}s_{i+1}\ldots s_n} v_r(s_1,\ldots,\ s_{i-1},\ \ s_i+1,\ \ s_{i+1},\ \ldots,\ s_n)\Big\}=$$

$$=(-1)^{n+1}\, {}^{i}\sum_{s=0}^{x-1}\Big[\, {}^{i}\sum_{j=0}^{n} a_j(s_j)+a_i(0)\Big]\times \Delta^{n-1}_{s_1\ldots s_{i-1}s_{i+1}\ldots s_n} v_r(s_1,\ \ldots,\ s_{i-1},\ 0,\ s_{i+1},\ \ldots,\ s_n)+$$

$$+(-1)^{n+1}\sum_{s_i=0}^{x_i-1}\Delta_{s_i} a_i(s_i)\, {}^{i}\sum_{s=0}^{x-1}\Delta^{n-1}_{s_1\ldots s_{i-1}s_{i+1}\ldots s_n} v_r(s_1,\ \ldots,\ s_{i-1},\ \ s_i+1,\ \ s_{i+1},\ \ldots,\ s_n).$$

Since $(-1)^{n-1}\Delta^{n-1}_{s_1\ldots s_{i-1}s_{i+1}\ldots s_n} v_r(s,\ x)\geqslant 0$, from the above equality it follows that

$$R_r(x)\leqslant \sum_{i=1}^{n} a_i(x_i)+(-1)^{n+1}\Big[\, {}^{i}\sum_{j=1}^{n} a_j(x_j)+a_i(0)\Big]\ (-1)^{n-1}[1-v_r(0,\ x)]+$$

$$+(-1)^{n+1}\sum_{s_i=0}^{x_i-1}\Delta_{s_i} a_i(s_i)\ (-1)^{n-1}\ [1-v_r\ (0,\ \ldots,\ 0,\ \ s_i+1,\ \ 0,\ \ldots,\ 0,\ \ x)]=$$

$$=a_i(x_i)+\Big({}^{i}\sum_{j=1}^{n} a_j(x_j)+a_i(0)\Big)\, v_r(0,\ x)-a_i(0)-(a_i(x_i)-a_i(0))+$$

$$+\sum_{s_i=0}^{x_i-1}\Delta_{s_i} a_i(s_i)\, v_r(0,\ \ldots,\ 0,\ s_i+1,\ 0,\ \ldots,\ 0,\ \ x).$$

The above inequality is true for all $1\leqslant i\leqslant n$; inequality (3.3) follows.

Corollary 3.2. *Let for all $x\in N^n$, the inequality (3.1) be satisfied and $\Delta_{x_i} a_i(x_i)\geqslant 0$, $1\leqslant i\leqslant n$. Then,*

$$u(x)\leqslant\Big[\sum_{j=2}^{n} a_i(x_i)+a_1(0)\Big]\prod_{s_1=0}^{x_1-1}\Big[1+\, {}^{1}\sum_{s=0}^{x-1} r(s)\Big]+$$

$$+\sum_{s_1=0}^{x_1-1}\Delta_{s_1} a_1(s_1)\prod_{p_1=s_1+1}^{x_1-1}\Big[1+\, {}^{1}\sum_{s=0}^{x-1} r(p_1,\ s_2,\ \ldots,\ s_n)\Big]=T_r(x). \tag{3.7}$$

Proof. The inequality (3.7) follows from (3.3) and lemma 2.2.

Remark 1. The estimate (3.7) is sharper than

$$u(x)\leqslant\Big[\sum_{j=2}^{n} a_i(x_i)+a_1(0)\Big]\prod_{s_1=0}^{x_1-1}\Bigg[1+\frac{\Delta_{s_1} a_1(s_1)}{a_1(s_1)+a_2(0)+\sum_{i=3}^{n} a_i(x_i)}+\, {}^{1}\sum_{s=0}^{x-1} r(s)\Bigg]=U_r(x)$$

obtained in [3] (also for several particular cases in [4—6]) with an additional condition $a_i(x_i)>0$, $1\leqslant i\leqslant n$.

To show $T_r(x)\leqslant U_r(x)$, we use lemma 2.3, and obtain

$$U_r(x)\geqslant\Big[\sum_{j=2}^{n} a_i(x_i)+a_1(0)\Big]\Big\{\prod_{s_1=0}^{x_1-1}\Big[1+\, {}^{1}\sum_{s=0}^{x-1} r(s)\Big]+$$

$$+\sum_{s_1=0}^{x_1-1} A(s_1)\prod_{p_1=s_1+1}^{x_1-1}\Big[1+\sum_{s=0}^{x_1-1} r(p_1,\ s_2,\ \ldots,\ s_n)\Big]\Big\} \tag{3.8}$$

where

$$A(s_1)=\frac{\Delta_{s_1} a_1(s_1)}{a_1(s_1)+a_2(0)+\sum_{i=3}^{n} a_i(x_i)}\prod_{p_1=0}^{s_1-1}\Bigg[1+\frac{\Delta_{p_1} a_1(p_1)}{a_1(p_1)+a_2(0)+\sum_{i=3}^{n} a_i(x_i)}\Bigg].$$

Next, an easy computation provides

$$A(s_1)=\Delta_{s_1} a_1(s_1)/[a_1(0)+a_2(0)+\sum_{i=3}^{n} a_i(x_i)]$$

and $A(s_1)\geqslant\Delta_{s_1} a_1(s_1)$ follows from the fact $\Delta_{x_2} a_2(x_2)\geqslant 0$. Substituting this in (3.8), $U_r(x)\geqslant T_r(x)$ follows.

Theorem 3.3. *Let for all $x\in N^n$, the following inequality be satisfied*

$$u(x)\leqslant \sum_{i=1}^{n} a_i(x_i)+\sum_{s=0}^{x-1}\Big[f_1(s)\, u(s)+f_2(s)\sum_{p=0}^{s-1} f_3(p)\, u(p)\Big] \tag{3.9}$$

where $a_i(x_i)$, $1\leqslant i\leqslant n$ are same as in theorem 3.1 and remark 1.

Then,

$$u(x)\leqslant R_f(x)\leqslant S_f(x)\leqslant T_f(x)\leqslant U_f(x), \tag{3.10}$$

where $f(x)=\max\ \{f_1(x)+f_3(x),\ f_2(x)\}$.

Proof. Inequality (3.9) is equivalent to the system

$$u(x)\leqslant \sum_{i=1}^{n} a_i(x_i)+\sum_{s=0}^{x-1}[f_1(s)\,u(s)+f_2(s)\,v(s)],\quad v(x)=\sum_{s=0}^{x-1} f_3(s)\,u(s). \tag{3.11}$$

Thus, we find $u(x)+v(x)\leqslant \sum_{i=1}^{n} a_i(x_i)+\sum_{s=0}^{x-1} f(s)\,(u(s)+v(s))$ and (3.10) follows as in theorem 3.1, corollary 3.2 and remark 1.

Remark 2. Let in inequality (3.9), $f_1(x)=f_2(x)$ then $f(x)=f_1(x)+f_3(x)$ and (3.10) reduces to

$$u(x)\leqslant R_{f_1+f_3}(x)\leqslant S_{f_1+f_3}(x)\leqslant T_{f_1+f_3}(x)\leqslant U_{f_1+f_3}(x). \tag{3.12}$$

Next, using (3.12) in (3.11), we obtain

$$\begin{aligned} u(x)-\sum_{i=1}^{n} a_i(x_i)&\leqslant \sum_{s=0}^{x-1} f_1(s)\,R_{f_1+f_3}(s)\leqslant \sum_{s=0}^{x-1} f_1(s)\,S_{f_1+f_3}(s)\leqslant\\ &\leqslant \sum_{s=0}^{x-1} f_1(s)\,T_{f_1+f_3}(s)\leqslant \sum_{s=0}^{x-1} f_1(s)\,U_{f_1+f_3}(s). \end{aligned} \tag{3.13}$$

For $n=2$ and 3 the last part of the inequality (3.13) was obtained by Pachpatte [4—6] using different methods.

Remark 3. The inequality $\Delta_x^n u(x)\leqslant \sum_{i=1}^{n} a_i(x_i)+\sum_{s=0}^{x-1} c(s)\,(u(s)+\Delta_s^n u(s))$ where $u(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n)=0$, $1\leqslant i\leqslant n$ is same as $\Delta_x^n u(x)\leqslant \sum_{i=1}^{n} a_i(x_i)+\sum_{s=0}^{x-1}[c(s)\,\Delta_s^n u(s)+c(s)\sum_{p=0}^{s-1}\Delta_p^n u(p)]$. Thus, as in remark 2 for $\Delta_x^n u(x)$ we obtain the chain of estimates (3.12), (3.13) with $f_1=c$, $f_3=1$. The last part of these chains ($n=2$) is also obtained in [5].

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SOME STRUCTURES INDUCED ON THE TANGENT BUNDLE OF AN ALMOST CONTACT MANIFOLD

BY

CORNELIA-LIVIA BEJAN

Introduction. In this paper, the almost complex structures induced on the tangent bundle of an almost contact manifold with the help of the complete and horizontal lifts [6], [1], will be generalized. Also, the pseudo-riemannian metrics on the tangent bundle of a Riemannian manifold, constructed with the help of the complete and horizontal lift and the Sasaki Riemannian metric on the tangent bundle of a Riemannian manifold will be generalized and we shall give the conditions under which we obtain some almost Hermitian structures, almost Kählerian structures and Kählerian structures on the tangent bundle.

In §1 we recall some general notions about an almost contact manifold. In §2 we give the properties of a generalized almost complex structure on the tangent bundle constructed with the help of the complete lift of an almost contact structure on the base manifold and in §3 we shall give the properties of a generalized almost complex structure on the tangent bundle, constructed, with the help of the horizontal lift of an almost contact structure on the base manifold.

I want to mention that in the study of these problems I was guided by prof. Vasile Oproiu, to whom I extend my thanks.

§1. Definition 1.1. *Let M be a differentiable manifold. It is said to have an almost contact structure (F,ξ,η), if it admits a tensor field F of type (1, 1), a 1-form $\eta\in\mathcal{X}^*(M)$ and a vector field $\xi\in\mathcal{X}(M)$ satisfying:*

$$F^2=-I+\eta\otimes\xi \quad\text{and}\quad \eta(\xi)=1,$$

where I is unity operator.

Consequence. $F\circ\xi=0$, $\eta\circ F=0$.

Definition 1.2. *An almost contact structure on a Riemannian manifold (M,g) is an almost contact metric structure (F,ξ,η,g), if:*

$$g(FX,FY)=g(X,Y)-\eta(X)\eta(Y),\ \forall X,Y\in\mathcal{X}(M).$$

Property: $\eta(X)=g(X,\xi)$, $\forall X\in\mathcal{X}(M)$.

We suppose that the constructions of the vertical, complete and horizontal lifts from the manifold M to its tangent bundle TM are known, [5].

Using the almost contact structure (F,ξ,η) on M, some almost complex structures on its tangent bundle TM were constructed [6], [1]:

$$P=F^c+\eta^v\otimes\xi^c-\eta^c\otimes\xi^v \tag{1.1}$$

$$Q=F^h+\eta^v\otimes\xi^v-\eta^h\otimes\xi^h. \tag{1.2}$$