Then,

$$u(x)\leqslant R_f(x)\leqslant S_f(x)\leqslant T_f(x)\leqslant U_f(x), \tag{3.10}$$

where $$f(x)=\max\ \{f_1(x)+f_3(x),\ f_2(x)\}.$$

Proof. Inequality (3.9) is equivalent to the system

$$u(x)\leqslant \sum_{i=1}^{n} a_i(x_i)+\sum_{s=0}^{x-1}[f_1(s)\,u(s)+f_2(s)\,v(s)],\quad v(x)=\sum_{s=0}^{x-1} f_3(s)\,u(s). \tag{3.11}$$

Thus, we find $u(x)+v(x)\leqslant \sum_{i=1}^{n} a_i(x_i)+\sum_{s=0}^{x-1} f(s)\,(u(s)+v(s))$ and (3.10) follows as in theorem 3.1, corollary 3.2 and remark 1.

Remark 2. Let in inequality (3.9), $f_1(x)=f_2(x)$ then $f(x)=f_1(x)+f_3(x)$ and (3.10) reduces to

$$u(x)\leqslant R_{f_1+f_3}(x)\leqslant S_{f_1+f_3}(x)\leqslant T_{f_1+f_3}(x)\leqslant U_{f_1+f_3}(x). \tag{3.12}$$

Next, using (3.12) in (3.11), we obtain

$$u(x)-\sum_{i=1}^{n} a_i(x_i)\leqslant \sum_{s=0}^{x-1} f_1(s)\,R_{f_1+f_3}(s)\leqslant \sum_{s=0}^{x-1} f_1(s)\,S_{f_1+f_3}(s)\leqslant$$

$$\leqslant \sum_{s=0}^{x-1} f_1(s)\,T_{f_1+f_3}(s)\leqslant \sum_{s=0}^{x-1} f_1(s)\,U_{f_1+f_3}(s). \tag{3.13}$$

For $n=2$ and 3 the last part of the inequality (3.13) was obtained by Pachpatte [4—6] using different methods.

Remark 3. The inequality $\Delta_x^n u(x)\leqslant \sum_{i=1}^{n} a_i(x_i)+\sum_{s=0}^{x-1} c(s)\,(u(s)+\Delta_s^n u(s))$ where $u(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)=0$, $1\leqslant i\leqslant n$ is same as $\Delta_x^n u(x)\leqslant \sum_{i=1}^{n} a_i(x_i)+$ $+\sum_{s=0}^{x-1}[c(s)\,\Delta_s^n u(s)+c(s)\sum_{p=0}^{s-1}\Delta_p^n u(p)]$. Thus, as in remark 2 for $\Delta_x^n u(x)$ we obtain the chain of estimates (3.12), (3.13) with $f_1=c$, $f_3=1$. The last part of these chains ($n=2$) is also obtained in [5].

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SOME STRUCTURES INDUCED ON THE TANGENT BUNDLE OF AN ALMOST CONTACT MANIFOLD

BY

CORNELIA-LIVIA BEJAN

Introduction. In this paper, the almost complex structures induced on the tangent bundle of an almost contact manifold with the help of the complete and horizontal lifts [6], [1], will be generalized. Also, the pseudo-riemannian metrics on the tangent bundle of a Riemannian manifold, constructed with the help of the complete and horizontal lift and the Sasaki Riemannian metric on the tangent bundle of a Riemannian manifold will be generalized and we shall give the conditions under which we obtain some almost Hermitian structures, almost Kählerian structures and Kählerian structures on the tangent bundle.

In §1 we recall some general notions about an almost contact manifold. In §2 we give the properties of a generalized almost complex structure on the tangent bundle constructed with the help of the complete lift of an almost contact structure on the base manifold and in §3 we shall give the properties of a generalized almost complex structure on the tangent bundle, constructed, with the help of the horizontal lift of an almost contact structure on the base manifold.

I want to mention that in the study of these problems I was guided by prof. Vasile Oproiu, to whom I extend my thanks.

§1. Definition 1.1. *Let M be a differentiable manifold. It is said to have an almost contact structure (F, ξ, η), if it admits a tensor field F of type $(1, 1)$, a 1-form $\eta\in\mathscr{X}^*(M)$ and a vector field $\xi\in\mathscr{X}(M)$ satisfying :*

$$F^2=-I+\eta\otimes\xi \quad\text{and}\quad \eta(\xi)=1,$$

where I is unity operator.

Consequence. $F\circ\xi=0$, $\eta\circ F=0$.

Definition 1.2. *An almost contact structure on a Riemannian manifold (M, g) is an almost contact metric structure (F, ξ, η, g), if :*

$$g(FX, FY)=g(X, Y)-\eta(X)\eta(Y),\ \forall X, Y\in\mathscr{X}(M).$$

Property : $\eta(X)=g(X, \xi)$, $\forall X\in\mathscr{X}(M)$.

We suppose that the constructions of the vertical, complete and horizontal lifts from the manifold M to its tangent bundle TM are known, [5].

Using the almost contact structure (F, ξ, η) on M, some almost complex structures on its tangent bundle TM were constructed [6], [1] :

$$P=F^c+\eta^v\otimes\xi^c-\eta^c\otimes\xi^v \tag{1.1}$$
$$Q=F^h+\eta^v\otimes\xi^v-\eta^h\otimes\xi^h. \tag{1.2}$$

We shall generalize these almost complex structures considering a linear combination of the tensorial products of the lifts of η and ξ and the lifts of F.

§2. The generalized almost complex structure constructed with the help of the complete lift. Proposition 2.1. *If K is a linear combination of F^v, F^c, $\eta^v\otimes\xi^v$, $\eta^c\otimes\xi^v$, $\eta^v\otimes\xi^c$, $\eta^c\otimes\xi^c$, then K is an almost complex structure on TM if and only if:*

$$K=\pm F^c-[(\beta\eta^2+1)/\gamma]\eta^v\otimes\xi^v+\beta\eta^v\otimes\xi^c-\beta\eta^c\otimes\xi^v+\gamma\eta^c\otimes\xi^c,$$

where $\beta,\gamma\in R$, and $\gamma\neq 0$.

Proof. By straightforward computation.

Remark. If $\beta=0$, $\gamma=-1$ and we consider F^c with positive sign, then we find the tensor field P, defined by (1.1). Thus, we see that K generalizes P.

We denote $\alpha=-(\beta^2+1)/\gamma$.

We know that if g is a Riemannian metric on M, then its complete lift g^c is a pseudometric on TM.

Proposition 2.2.

$$g^v(P\tilde{X},P\tilde{Y})=(g^v-\eta^v\otimes\eta^v+\eta^c\otimes\eta^c)(\tilde{X},\tilde{Y}),\quad \forall\tilde{X},\tilde{Y}\in\mathcal{X}(TM), \tag{2.1}$$

$$g^c(P\tilde{X},P\tilde{Y})=(g-2\eta\otimes\eta)^c(\tilde{X},\tilde{Y}),\quad \forall\tilde{X},\tilde{Y}\in\mathcal{X}(TM), \tag{2.2}$$

where P is defined by (1.1).

Proof. We have:

$$g^v(PX^v,PY^v)=\eta(X)\eta(Y)\ ;\quad g^v(PX^v,PY^c)=\eta(X)(\eta(Y))^c\ ;$$

$$g^v(PX^c,PY^c)=g(X,Y)-\eta(X)\eta(Y)+(\eta(X))^c(\eta(Y))^c,\quad \forall X,Y\in\mathcal{X}(M).$$

We have also:

$$g^c(PX^v,PY^v)=0\ :\quad g^c(PX^v,PY^c)=g(X,Y)-\eta(X)\eta(Y)\ ;$$

$$g^c(PX^c,PY^c)=(g(X,Y))^c-2(\eta(X))^c\eta(Y)-2\eta(X)(\eta(Y))^c,\ \forall X,Y\in\mathcal{X}(M).$$

From these formulas, (2.1) and (2.2) are clear.

Remark. P is not a Hermitian structure with respect to the pseudometric g^c.

We want to extend the pseudometric g^c, to obtain an almost Hermitian structure with respect to K.

Proposition 2.3. If $\bar{g}$ is a linear combination of g^v, g^c, $\eta^v\otimes\eta^v$, $\eta^v\otimes\eta^c$, $\eta^c\otimes\eta^v$, $\eta^c\otimes\eta^c$, then $\bar{g}$ is symmetric and we have: $\bar{g}(K\tilde{X},K\tilde{Y})=\bar{g}(\tilde{X},\tilde{Y})$, $\forall\tilde{X},\tilde{Y}\in\mathcal{X}(TM)$ if and only if:

$$\bar{g}=mg^c+s(\eta\otimes\eta)^c+tg^v+r\eta^v\otimes\eta^v+q\eta^c\otimes\eta^c, \tag{2.3}$$

where $m,s,r,t,q\in R$ and:

$$\gamma(t+r)+\alpha q=0,\quad \gamma(m+s)-\beta q=0.$$

Proof. We consider:

$$\bar{g}=mg^c+s\eta^v\otimes\eta^c+s'\eta^c\otimes\eta^v+tg^v+r\eta^v\otimes\eta^v+q\eta^c\otimes\eta^c,$$

where $m,s,s',t,r,q\in R$. Putting the symmetry condition, we obtain $s=s'$.

Also, putting the compatibility condition of K with respect to $\bar{g}$, we obtain (2.3).

Proposition 2.4. *If $m\,.\,q\neq 0$, then $\bar{g}$ has maximal rank.*

Proof. We have:

$$\mathrm{I}\begin{cases}\bar{g}(X^v,Y^v)=0 \text{ and}\\ \bar{g}(X^v,Y^c)=0,\ \forall Y\in\mathcal{X}(M)\end{cases}\Leftrightarrow\begin{cases}q\eta(X)\eta(Y)=0 \text{ and}\\ mg(X,Y)+s\eta(X)\eta(Y)+q(\eta(X))^c\eta(Y)=\\ =0,\ \forall Y\in\mathcal{X}(M)\end{cases}$$

Taking $Y=\xi$, since $q\neq 0$, we have $\eta(X)=0$. Thus, since $m\neq 0$, we have $g(X,Y)=0$, $\forall Y\in\mathcal{X}(M)$ and so $X=0$.

$$\mathrm{II}\qquad \bar{g}(X^c,Y^v)=0 \text{ and } \bar{g}(X^c,Y^c)=0,\quad \forall Y\in\mathcal{X}(M)\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}mg(X,Y)+s\eta(X)\eta(Y)+q(\eta(X))^c\eta(Y)=0 \text{ and}\\ m(g(X,Y))^c+s(\eta(X)\eta(Y))^c+tg(X,Y)+r\eta(X)\eta(Y)+q(\eta(X))^c(\eta(Y))^c=\\ =0,\ \forall Y\in\mathcal{X}(M)\end{cases}$$

Taking $Y=\xi$, we have: $\begin{cases}(m+s)\eta(X)+q(\eta(X))^c=0 \text{ and}\\ (m+s)(\eta(X))^c+(t+r)\eta(X)=0,\end{cases}$

we find $\eta(X)=0$ (since $q\neq 0$).

Thus, since $m\neq 0$, we have $g(X,Y)=0$, $\forall Y\in\mathcal{X}(M)$, thus $X=0$.

Remark. If $m\,.\,q=0$, then $\bar{g}$ has not maximal rank.

From now on, we suppose $m\,.\,q\neq 0$.

Consequence. $\bar{g}$ is a pseudometric on TM.

Proposition 2.5. *If $q>0$, then $\bar{g}$ is positively semidefinite on the vertical distribution and $\bar{g}$ is positively definite for the vertical lifts of the vector fields which are not in* Ker *η.*

Proof. $\bar{g}(X^v,X^v)=q(\eta(X))^2$, $\forall X\in\mathcal{X}(M)$.

Theorem 2.1. *K is an almost Hermitian structure with respect to $\bar{g}$.*

Proof. This theorem is a consequence of the propositions 2.3 and 2.4

Proposition 2.6.

$$[K\tilde{X}_1,K\tilde{X}_2]=[F^c\tilde{X}_1,F^c\tilde{X}_2]+\sum_{\sigma\in S_2}\varepsilon_\sigma\{\eta^v(\tilde{X}_{\sigma(1)})\{\mp\alpha[F^v\tilde{X}_{\sigma(2)},\xi^c]\mp$$
$$\mp\beta[F^c\tilde{X}_{\sigma(2)},\xi^c]-(\xi^v\eta^c(\tilde{X}_{\sigma(2)}))\xi^c-\beta(\xi\eta^c(\tilde{X}_{\sigma(2)}))(\beta\xi^v-\gamma\xi^c)\}+\eta^c(\tilde{X}_{\sigma(1)})\{\pm$$
$$\pm\beta[F^v\tilde{X}_{\sigma(2)},\xi^c]\pm\gamma[F^c\tilde{X}_{\sigma(2)},\xi^c]-(\xi^c\eta^c(\tilde{X}_{\sigma(2)}))\xi^v-\gamma(\xi^c\eta^c(\tilde{X}_{\sigma(2)}))(\beta\xi^v-\gamma\xi^c)\}\pm$$
$$\pm F^c\tilde{X}_{\sigma(1)}\eta^v(\tilde{X}_{\sigma(2)})(\alpha\xi^v+\beta\xi^c)\mp F^c\tilde{X}_{\sigma(1)}\eta^c(\tilde{X}_{\sigma(2)})(\beta\xi^v-\gamma\xi^c)\},\ \forall\tilde{X}_1,\tilde{X}_2\in\mathcal{X}(TM). \tag{2.4}$$

Proof. We find it by a straightforward calculation.

Let φ be the two-form associated to F: $\varphi(X,Y)=g(X,FY)$, $\forall X,Y\in\mathcal{X}(M)$. Φ is the two-form associated to K and defined by: $\Phi(\tilde{X},\tilde{Y})=\bar{g}(\tilde{X},K\tilde{Y})$, $\forall\tilde{X},\tilde{Y}\in\mathcal{X}(TM)$.

Lemma 2.1. *If ω is a closed form, then Φ is a closed form if and only if φ is closed.*

Proof. Using the following formulas:

$$\Phi(X^v, Y^v)=0\ ;\quad \Phi(X^v, Y^c)=\pm m\varphi(X, Y)-(q/\gamma)\eta(X)\eta(Y)\ ;$$

$$\Phi(X^c, Y^c)=\pm m(\varphi(X, Y))^c\pm t\varphi(X, Y)-(q/\gamma)(\eta(\in X))^c\eta(Y)+(q/\gamma)\eta(X)(\eta(Y))^c$$

we have:

$$d\Phi(X^v, Y^v, Z^v)=0\ ;\quad d\Phi(X^v, Y^v, Z^c)=0\ ;$$

$$d\Phi(X^v, Y^c, Z^c)=\pm md\varphi(X, Y, Z)+(q/\gamma)\{\eta(X)d\eta(Y, Z)+\eta(Y)d\eta(X, Z)+$$
$$+\eta(Z)d\eta(Y, X)\}\ ;$$

$$d\Phi(X^c, Y^c, Z^c)=\pm m(d\varphi(X, Y, Z))^c\pm td\varphi(X, Y, Z)+$$
$$+(q/\gamma)\sum_{cicl}\{\eta(X)d\eta(Z,Y)+(\eta(X))^c d\eta(Y, Z)\}.$$

If $d\eta=0$ the assertion is proved.

Theorem 2.2. *If η is a closed form, then K is an almost Kählerian structure with respect to the pseudometric $\tilde{g}$ if and only if φ is closed.*

Proof. This theorem is a consequence of the lemma 2.1.

Remark. Calculating the interior product of the vertical and complete lifts of the vector field ξ with the two-form Φ, we have:

$$i_{\xi^v}\Phi=-(q/\gamma)\eta^v \text{ and } i_{\xi^c}\Phi=(q/\gamma)\eta^c.$$

We want to obtain the conditions under which the integrability of K holds.

We shall use the following notations:

$$N^{(1)}(X, Y)=N_F(X, Y)-d\eta(X, Y)\xi\ ;\quad N^{(2)}(X, Y)=(L_{FX}\eta)(Y)-(L_{FY}\eta)(X)$$

$$N^{(3)}(X)=(L_\xi F)(X)\ ;\quad N^{(4)}(X)=(L_\xi\eta)(X),\quad \forall X, Y\in\mathcal{X}(M),$$

where:

$$N_F(X, Y)=[FX, FY]-F[FX, Y]-F[X, FY]+F^2[X, Y]$$

is the Nijenhuis tensor field of F.

$(L_\xi F)(X)=[\xi, FX]-F[\xi, X]$ is the Lie derivative of the tensor field F with respect to ξ.

$(L_\xi\eta)(X)=\xi\eta(X)-\eta([\xi, X])$ is the Lie derivative of the Pfaff form η with respect to ξ.

Remark. $d\eta(\xi, X)=(L_\xi\eta)(X)$, $\forall X\in\mathcal{X}(M)$.

Recall that if $N^{(1)}=0$, then $N^{(2)}=0$, $N^{(3)}=0$, $N^{(4)}=0$ and in this case the almost contact structure (F, ξ, η) is said to be normal.

Proposition 2.7. *The Nijenhuis tensor field of K has the expression:*

$$N_K=(N^{(1)})^c+\eta^v\wedge\{\pm\alpha(N^{(3)})^v\pm\beta(N^{(3)})^c-(N^{(4)})^c\otimes\xi^v-$$
$$-\beta(N^{(4)})^v\otimes(\beta\xi^v-\gamma\xi^c)\}-\eta^c\wedge\{\pm\beta(N^{(3)})^v\mp\gamma(N^{(3)})^c+(N^{(4)})^v\otimes\xi^v+$$
$$+\gamma(N^{(4)})^c\otimes(\beta\xi^v-\gamma\xi^c)\}\pm(N^{(2)})^v(\alpha\xi^v+\beta\xi^c)\mp(N^{(2)})^c(\beta\xi^v-\gamma\xi^c)$$

Proof. We obtain this result, expressing:

$$N_K(X^v, Y^v)=\gamma\{\pm\eta^c\wedge(N^{(3)})^c+\eta^c\wedge(N^{(4)})^c(\gamma\xi^c-\beta\xi^v)\}(X^v, Y^v)$$

$$N_K(X^v, Y^c)=\{(N^{(1)})^c\pm\beta\eta^v\wedge(N^{(3)})^c\mp\beta\eta^c\wedge(N^{(3)})^v\pm\gamma\eta^c\wedge(N^{(3)})^c-(\eta^c\wedge(N^{(4)})^c)\otimes$$
$$\otimes\xi^c-\beta^2(\eta^v\wedge(N^{(4)})^c)\wedge\xi^v+\gamma^2(\eta^c\wedge(N^{(4)})^c)\otimes\xi^c-\beta\gamma(\eta^c\wedge(N^{(4)})^c\otimes\xi^v+$$
$$+\beta\gamma(\eta^v\wedge(N^{(4)})^c)\otimes\xi^c\pm(N^{(2)})^v(\gamma\xi^c-\beta\xi^v)\}(X^v, Y^c),$$

$$N_K(X^c, Y^c)=(N^{(1)}(X, Y))^c+\eta^v\wedge\{\pm\alpha(N^{(3)})^c\pm\beta(N^{(3)})^c-(N^{(4)})^v\otimes\xi^v-\beta^2(N^{(4)})^c\otimes$$
$$\otimes\xi^v+\beta\gamma(N^{(4)})^c\otimes\xi^c\}(X^c, Y^c)-\eta^c\wedge\{\pm\beta(N^{(3)})^v\mp\gamma(N^{(3)})^c+\beta\gamma(N^{(4)})^c\otimes$$
$$\otimes\xi^v-\gamma^2(N^{(4)})^c\otimes\xi^c+(N^{(4)})^c\otimes\xi^v\}(X^c, Y^c)\pm N^{(2)}(X, Y)\,(\alpha\xi^v+\beta\xi^c)\pm$$
$$\pm(N^{(2)}(X, Y))^c(\gamma\xi^c-\beta\xi^v).$$

Theorem 2.3. *The almost complex structure K is integrable if and only if the almost contact structure (F, ξ, η) is normal.*

Proof. The integrability condition is equivalent to the vanishing of the Nijenhuis tensor field. If the almost contact structure (F, ξ, η) is normal, then $N^{(1)}$, $N^{(2)}$, $N^{(3)}$, $N^{(4)}$ vanish. Using the Proposition 2.7, it is easily seen that $N_K=0$.

Conversely, if $N_K=0$, then $N_K(X^v, Y^v)=0, N_K(X^c, Y^c)=0$, $\forall X, Y\in\mathcal{X}(M)$. But $N_K(X^v, Y^v)=0\Leftrightarrow\gamma\eta^c\wedge\{\pm(N^{(3)})^c-(N^{(4)})^c(\beta\xi^v-\gamma\xi^c)\}(X^v, Y^v)=0$. Thus $N^{(4)}=0$ and $N^{(3)}=0$.

In the expression of $N_K(X^c, Y^c)$ we consider $N^4=0$ and $N^3=0$. Then, $N_K(X^c, Y^c)=0$, $\forall X, Y\in\mathcal{X}(M)$ implies $N^{(1)}(X, Y)=0$, $\forall X, Y\in\mathcal{X}(M)$, and the proof is finished.

Let ∇^c be the complete lift of a connection ∇ on M, [5].

Proposition 2.8. $\nabla^c_{\tilde{X}}K=0$, $\forall\tilde{X}\in\mathcal{X}(TM)$ if and only if F and ξ are parallel with respect to the linear connection ∇ on M.

Remark. We know that if F and ξ are parallel with respect to ∇, then η is also parallel with respect to ∇.

Proof.

$$(\nabla^c{}_{X^v}K)Y^v=\gamma\eta(Y)\nabla_X\xi\ ;$$

$$(\nabla^c{}_{X^v}K)Y^c=\pm(\nabla_XF)Y-\beta(\nabla_X\eta)\,(Y)\xi^v+\gamma(\nabla_X\eta)\,(Y)\xi^c+\beta\eta(Y)\nabla_X\xi+$$
$$+\gamma(\eta(Y))^c\nabla_X\xi\ ;$$

$$(\nabla^c{}_{X^c}K)Y^v=\pm(\nabla_XF)Y-\beta(\nabla_X\eta)\,(Y)\xi^c+\gamma(\nabla_X\eta)\,(Y)\xi^c-\beta\eta(Y)\nabla_X\xi+$$
$$+\gamma\eta(Y)(\nabla_X\xi)^c\ ;$$

$$(\nabla^c{}_{X^c}K)Y^c=\pm((\nabla_XF)\,(Y))^c+\alpha(\nabla_X\eta)(Y)\xi^v+\alpha\eta(Y)\nabla_X\xi+\beta(\nabla_X\eta)\,(Y)\xi^c+$$
$$+\beta\eta(Y)\,(\nabla_X\xi)^c-\beta((\nabla_X\eta)\,(Y))^c\xi^v-\beta(\eta(Y))^c\nabla_X\xi)+\gamma((\nabla_X\eta)\,(Y))^c\xi^c+$$
$$+\gamma(\eta(Y))^c(\nabla_X\xi)^c,\quad \forall X,Y\in\mathcal{X}(M).$$

It easily seen that if F and ξ are parallel with respect to ∇, then

$$(\nabla^c{}_{\tilde{X}}K)\,(\tilde{Y})=0,\quad \forall\tilde{X}, \tilde{Y}\in\mathcal{X}(TM).$$

Conversely, if $(\nabla^c{}_{\tilde{X}}K)\,(\tilde{Y})=0$, $\forall\tilde{X}, \tilde{Y}\in\mathcal{X}(TM)$, then $(\nabla^c{}_{X^v}K)Y^v=0$, $(\nabla^c{}_{X^v}K)Y^c=0$, $(\nabla^c{}_{X^c}K)Y^v=0$, $\forall X, Y\in\mathcal{X}(M)$. Thus, $\nabla_X\xi=0$, $\forall X\in\mathcal{X}(M)$, since $(\nabla^c{}_{X^c}K)Y^c=0$, $\forall X, Y\in\mathcal{X}(M)$. Taking $\nabla_X\xi=0$, $\forall X\in\mathcal{X}(M)$ in the expression : $(\nabla^c{}_{X^c}K)Y^c=0$, $\forall X, Y\in\mathcal{X}(M)$, we obtain :

$$\pm(\nabla_X F)Y-\beta(\nabla_X\eta)(Y)\xi^v+\gamma(\nabla_X\eta)(Y)\xi^c=0,\quad \forall X, Y\in\mathcal{X}(M).$$

Thus

$$\nabla_X\eta=0,\ \forall X\in\mathcal{X}(M)\ \text{and}\ \nabla_X F=0,\ \forall X\in\mathcal{X}(M).$$

§3. The generalized almost complex structure constructed with the help of the horizontal lift.

We consider the Levi-Civita connection ∇ on the almost contact metric manifold M. We shall study the generalization of the almost complex structure Q, defined by (1.2).

Definition 3.1. *Let (M, g) be a Riemannian manifold. Than the Sasaki metric on TM is given by :*

$$G(X^v, Y^v)=g(X, Y)\ ;\ G(X^v, Y^h)=G(X^h, Y^v)=0\ ;\ G(X^h, Y^h)=$$
$$=g(X, Y),\quad \forall X, Y\in\mathcal{X}(M).$$

Proposition 3.1. *Q is a Hermitian structure with respect to the Sasaki metric G. Thus :*

$$G(Q\tilde{X}, Q\tilde{Y})=G(\tilde{X}, \tilde{Y}),\quad \forall\tilde{X}, \tilde{Y}\in\mathcal{X}(TM). \tag{3.1}$$

Proof.

$$QX^v=FX-\eta(X)\xi^h\ ;\ QX^h=(FX)^h+\eta(X)\xi^v,\quad \forall X\in\mathcal{X}(M)$$

and then :

$$G(QX^v, QY^v)=g(X, Y)\ ;\ G(QX^v, QY^h)=0\ ;\ G(QX^h, QY^h)=g(X, Y),\ \forall X, Y\in$$
$$\in\mathcal{X}(M).$$

We know that if g is a Riemannian metric on M, then its horizontal lift g^h is a pseudometric on TM.

Proposition 3.2.

$$g^v(Q\tilde{X}, Q\tilde{Y})=(g^v-\eta^v\otimes\eta^v+\eta^h\otimes\eta^h)(\tilde{X}, \tilde{Y}),\quad \forall\tilde{X}, \tilde{Y}\in\mathcal{X}(TM), \tag{3.2}$$

$$g^h(Q\tilde{X}, Q\tilde{Y})=(g-2\eta\otimes\eta)^h(\tilde{X}, \tilde{Y}),\quad \forall\tilde{X}, \tilde{Y}\in\mathcal{X}(TM), \tag{3.3}$$

where Q is defined by (1.2).

Proof. We have :

$$g^v(QX^v, QY^v)=\eta(X)\eta(Y)\ ;\ g^v(QX^v, QY^h)=0\ ;\ g^v(QX^h, QY^h)=g(X,Y)-$$
$$-\eta(X)\eta(Y),\quad \forall X, Y\in\mathcal{X}(M).$$

We have also :

$$g^h(QX^v, QY^v)=0;\ g^h(QX^v, QY^h)=g(X, Y);\ g^h(QX^h, QY^h)=0, \forall X, Y\in\mathcal{X}(M).$$

Remark. Q is not a Hermitian structure with respect to th· pseudometric g^h.

Now, we shall extend the tensor field Q.

Proposition 3.3. *If L is a linear combination of F^c, F^h, $\eta^v\otimes\xi^v$, $\eta^h\otimes\xi^v$ $\eta^v\otimes\xi^h$, $\eta^H\otimes\xi^h$, then L is an almost complex structure if and only if :*

$$L=\pm F^h-[(\beta^2+1)/\gamma]\,\eta^v\otimes\xi^v+\beta\eta^h\otimes\xi^v-\beta\eta^v\otimes\xi^h+\gamma\eta^h\otimes\xi^h,$$

where $\beta, \gamma\in R$ and $\gamma\neq 0$.

Remark. If $\beta=0$, $\gamma=-1$ and we consider F^h with the positive sign, then we find the tensor field Q.

Denote $\alpha=-(\beta^2+1)/\gamma$. We want to find a metric on TM, to obtain an almost Hermitian structure related to L.

Proposition 3.4. *If $\tilde{g}$ is a linear combination of g^v, g^h, G, $\eta^v\otimes\eta^v$, $\eta^v\otimes\eta^h$, $\eta^h\otimes\eta^v$, $\eta^h\otimes\eta^h$, then $\tilde{g}$ is symmetric and we have :*

$$\tilde{g}(L\tilde{X}, L\tilde{Y})=\tilde{g}(\tilde{X}, \tilde{Y}),\quad \forall\tilde{X}, \tilde{Y}\in\mathcal{X}(TM)$$

if and only if :

$$\tilde{g}=mG+s(\eta\otimes\eta)^h+tg^v+pg^h+r\eta^v\otimes\eta^v+q\eta^h\otimes\eta H, \tag{3.4}$$

where m, s, t, p, r, $q\in R$, and :

$$\beta(m+q)+\gamma(s+p)=0,\quad \gamma(m+r+t)+\alpha(m+q)=0.$$

Proof. Let :

$$\tilde{g}=mG+s\eta^v\otimes\eta^h+s'\eta^h\otimes\eta^v+tg^v+pg^h+r\eta^v\otimes\eta^v+q\eta^h\otimes\eta^h,$$

where m, s, s', p, r, $q\in R$.

Putting the symmetry condition, we obtain $s=s'$. Also, putting the compatibility condition of L with respect to $\tilde{g}$, we obtain (3.4).

Proposition 3.5. *If $(m+q)(\alpha s-\beta r)(m\beta+p\gamma)\neq 0$, then $\tilde{g}$ has maximal rank.*

Proof. We have :

$$\mathrm{I}\begin{cases}\tilde{g}(X^v, Y^v)=0\ \text{and}\\ \tilde{g}(X^v, Y^h)=0,\quad \forall Y\in\mathcal{X}(M)\end{cases}\Leftrightarrow\begin{cases}mg(X, Y)+q\eta(X)\eta(Y)=0\ \text{and}\\ pg(X, Y)+s\eta(X)\eta(Y)=0,\quad \forall Y\in\mathcal{X}(M)\end{cases}\Rightarrow$$

$$\Rightarrow\begin{cases}\dfrac{m+q}{\gamma}(m\beta+p\gamma)g(X, Y)=0\ \text{and}\\[2ex] \dfrac{m+q}{\gamma}(m\beta+p\gamma)\eta(X)\eta(Y)=0,\quad \forall Y\in\mathcal{X}(M)\end{cases}\Rightarrow$$

$$\Rightarrow(\text{since }(m+q)(m\beta+p\gamma)\neq 0)\ g(X, Y)=0,\quad \forall Y\in\mathcal{X}(M)\Rightarrow X=0.$$

$$II\begin{cases}\tilde{g}(X^h, Y^v)=0\ \text{and}\\ \tilde{g}(X^h, Y^H)=0,\quad \forall Y\in\mathcal{X}(M)\end{cases}\Leftrightarrow\begin{cases}pg(X, Y)+s\eta(X)\eta(Y)=0\ \text{and}\\ (m+t)g(X, Y)+r\eta(X)\eta(Y)=0, \forall Y\in\mathcal{X}(M)\end{cases}\Rightarrow$$

$$\Rightarrow\begin{cases}\dfrac{m+q}{\gamma}(\alpha s-\beta r)g(X, Y)\ \text{and}\\[2ex] \dfrac{m+q}{\gamma}(\alpha s-\beta r)\eta(X)\eta(Y)=0,\quad \forall Y\in\mathcal{X}(M)\end{cases}\Rightarrow$$

$\Rightarrow$(since $(m+q)(\alpha s-\beta r)\neq 0)g(X, Y)=0,\quad \forall Y\in\mathcal{X}(M)\Rightarrow X=0.$

Remark. If $(m+q)(m\beta+p\gamma)(\alpha s-\beta r)=0$, then $\tilde{g}$ has not maximal rank.

From now on, we suppose $(m+q)(m\beta+p\gamma)(\alpha s-\beta r)\neq 0$.

Consequence. $\tilde{g}$ is a pseudometric on TM.

Proposition 3.6. *If $m>0$, $q>0$, $m+t>0$, $r>0$, then $\tilde{g}$ is positively definite.*

Proof. We have:

$$\begin{cases} \tilde{g}(X^v, X^v)=mg(X, X)+q(\eta(X))^2, \\ \tilde{g}(X^h, X^h)=(m+t)g(X, X)+r(\eta(X))^2, \quad \forall X\in\mathcal{X}(M). \end{cases}$$

Thus, it is easily seen that this proposition is true.

Consequence. If $m>0$, $q>0$, $m+t>0$, $r>0$, then the pseudometric $\tilde{g}$ is a Riemannian metric on TM.

Now, we want to obtain an almost Kählerian structure. Let $\tilde{\Phi}$ be the two-form associated to L, defined by:

$$\tilde{\Phi}(\tilde{X}, \tilde{Y})=\tilde{g}(\tilde{X}, L\tilde{Y}),\quad \forall\tilde{X}, \tilde{Y}\in\mathcal{X}(TM).$$

Proposition 3.7. *If the curvature tensor field R vanishes and ξ is parallel with respect to ∇, then L is an almost Kählerian structure with respect to $\tilde{g}$ with $m\neq 0$, if and only if F is parallel with respest to ∇ and η and φ are closed.*

Proof.

$$\tilde{\Phi}(X^v, Y^v)=\pm m\varphi(X, Y)\,;\ \tilde{\Phi}(X^v, Y^h)=\pm p\varphi(X, Y)-[(m+q)/\gamma]\eta(X)\eta(Y);$$

$$\tilde{\Phi}(X^h, Y^h)=\pm(m+t)\varphi(X, Y).$$

Thus, we have:

$$d\tilde{\Phi}(X^v, Y^v, Z^v)=0\,;\ d\tilde{\Phi}(X^v, Y^v, Z^h)=\pm mg((X, (\nabla_Z F)Y)\,;$$

$$d\tilde{\Phi}(X^v, Y^h, Z^h)=\pm p\{(\nabla_Z F)Y-(\nabla_Y F)Z\}+[(m+q)/\gamma]\{\eta(Z)g(X, \nabla_Y\xi)-$$
$$-\eta(Y)g(X, \nabla_Z\xi)+\eta(X)d\eta(Y, Z)\}\,;$$

$$d\tilde{\Phi}(X^h, Y^h, Z^h)=\pm(m+t)d\varphi(X, Y, Z)+\sum_{cicl}\{\pm p\varphi(vR(X, Y), Z)-$$
$$-[(m+q)/\gamma]\eta(vR(X, Y))\eta(Z)\},$$

where v is an operator well known [5].

If $R=0$, $\nabla_X\xi=0$, $\nabla_X F=0$, $\forall X\in\mathcal{X}(M)$, $d\eta=0$ and $d\varphi=0$, then $d\tilde{\Phi}=0$.

Conversely, if $d\tilde{\Phi}=0$, then $d\tilde{\Phi}(X^v, Y^v, Z^h)=0$, $\forall X, Y\in\mathcal{X}(M)$. Thus, $\nabla_Z F=0$, $\forall Z\in\mathcal{X}(M)$, since $m\neq 0$.

In the expression of $d\tilde{\Phi}(X^v, Y^h, Z^h)$, we consider $\nabla_Z F=0$, $\nabla_Z\xi=0$, $\forall Z\in\mathcal{X}(M)$. Thus, $d\tilde{\Phi}(X^v, Y^h, Z^h)=0$, $\forall X, Y, Z\in\mathcal{X}(M)$, implies $d\omega=0$. In the expression of $d\tilde{\Phi}(X^h, Y^h, Z^h)$, we consider $R=0$ and the proof is finished.

Proposition 3.8. *If $R=0$ and ξ is parallel with respect to ∇, then L is an almost Kählerian structure with respect to $\tilde{g}$, with $m\neq 0$ and $m+t=0$, if and only if F is parallel with respect to ∇ and η is closed.*

Proof. Similar with the proof of the proposition 3.7.

Remark. Calculating the interior product of the vertical and horizontal lifts of the vector field ξ with the two-form Φ, we have:

$$i_{\xi^v}\tilde{\Phi}=-[(m+q)/\gamma]\eta^v\,;\ i_{\xi^h}\tilde{\Phi}=[(m+q)/\gamma]\eta^h.$$

Proposition 3.9.

$$[L\tilde{X}_1, L\tilde{X}_2]=[F^h\tilde{X}_1, F^h\tilde{X}_2]\pm\alpha(\eta^v\vee U^v)(\tilde{X}_1, \tilde{X}_2)\pm(\eta^v\wedge\eta^h)(\tilde{X}_1,$$
$$\tilde{X}_2)\nabla_\xi\xi+\sum_{\sigma\in P_2}\varepsilon_\sigma\{\mp\beta\eta^v(\tilde{X}_{\sigma(1)})[\xi H, F^h\tilde{X}_{\sigma(2)}]\pm\gamma\eta^h(\tilde{X}_{\sigma(1)})[\xi^h, F^h\tilde{X}_{\sigma(2)}]\pm$$
$$\pm F^v\tilde{X}_{\sigma(1)}\eta^v(\tilde{X}_{\sigma(2)})(\alpha\xi^v-\beta\xi^h)\pm F^h\tilde{X}_{\sigma(1)}\eta^h(\tilde{X}_{\sigma(2)})(\beta\xi^v+\gamma\xi^h)-$$
$$-\beta\eta^v(\tilde{X}_{\sigma(1)})(\xi^H\eta^v(\tilde{X}_{\sigma(2)}))(\alpha\xi^v-\beta\xi^h)+\gamma\eta^h(\tilde{X}_{\sigma(1)})(\xi^h\eta^v(\tilde{X}_{\sigma(2)}))(\alpha\xi^v-\beta\xi^h)-$$
$$-\beta\eta^v(\tilde{X}_{\sigma(1)})(\xi^h\eta^h(\tilde{X}_{\sigma(2)}))(\beta\xi^v+\gamma\xi^h)+\gamma\eta^h(\tilde{X}_{\sigma(1)})(\xi^h\eta^h(\tilde{X}_{\sigma(2)}))(\beta\xi^v+\gamma\xi^h)\},$$
$$\forall\tilde{X}_1, \tilde{X}_2\in\mathcal{X}(TM), \tag{3.5}$$

where $U(X)=\nabla_{FX}\xi$, $\forall X\in\mathcal{X}(M)$.

Proof. We find it by a straightforward calculation.

Proposition 3.10. *If ξ is parallel with respect to ∇, then L is parallel with respect to ∇^h if and only if F is parallel with respect to ∇.*

Remark. If F and ξ are para lel with respect to ∇, then η is parallel with respect to ∇.

Proof.

$$(\nabla^h_{X^v}L)Y^v=0\,;\ (\nabla^h_{X^v}L)Y^h=0\,;$$

$$(\nabla^h_{X^h}L)Y^v=\pm(\nabla_X F)(Y)+\beta(\nabla_X\eta)(Y)\xi^v+\beta\eta(Y)\nabla_X\xi+\gamma(\nabla_X\eta)(Y)\xi^h+$$
$$+\gamma\eta(Y)(\nabla_X\xi)^h\,;$$

$$(\nabla^h_{X^h}L)Y^h=\pm((\nabla_X F)(Y))^h+\alpha(\nabla_X\eta)(Y)\xi^v+\alpha\eta(Y)\nabla_X\xi-\beta(\nabla_X\eta)(Y)\xi^h-$$
$$-\beta\eta(Y)(\nabla_X\xi)^h.$$

It is easily seen that if $\nabla_X F=0$, $\nabla_X\eta=0$, $\forall X\in\mathcal{X}(M)$, then $\nabla^h_{\tilde{X}}L=0$, $\forall\tilde{X}\in\mathcal{X}(TM)$.

Conversely, if $(\nabla^h_{\tilde{X}}L)\tilde{Y}=0$, $\forall\tilde{X}, \tilde{Y}\in\mathcal{X}(TM)$, then $(\nabla^h_{X^h}L)Y^v=0$; $(\nabla^h_{X^h}L)Y^h=0$, $\forall X, Y\in\mathcal{X}(M)$. Thus:

$$(\nabla_X\eta)(Y)\xi^v+\eta(Y)\nabla_X\xi=0,\quad \forall X, Y\in\mathcal{X}(M),\ \text{since}\ (\nabla^h_{X^h}L)Y^h=0, \tag{3.6}$$

$\forall X, Y\in\mathcal{X}(M)$ and $\alpha\neq 0$. Making (3.6) in the expression $(\nabla^h_{X^h}L)Y^v=0$, $\forall X, Y\in\mathcal{X}(M)$, we obtain: $\pm(\nabla_X F)(Y)+\gamma(\nabla_X\eta)(Y)\xi^h+\gamma\eta(Y)(\nabla_X\xi)^h=0$, $\forall X, Y\in\mathcal{X}(M)$. Thus, $\nabla_X F=0$, $\forall X\in\mathcal{X}(M)$.

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CHAMPS PRESQUE ξ-PRINCIPAUX DANS L'ALGÈBRE ASSOCIÉE À UN CHAMP TENSORIEL DE TYPE (1,2)

PAR

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On introduit les champs presque ξ-principaux et les courbes presque ξ-principales dans l'algèbre associée à un champ tensoriel de type (1,2) sur une variété différentiable. On obtient une extension du théorème de K. Yano concernant les transformations sous-projectives d'une connexion linéaire. Sont présentées des applications liées aux algèbres de Tzitzéica et de Weyl.

Dans ce qui suit, M est une variété C^∞-différentiable à n dimensions. On note par $\mathcal{F}(M)$ l'anneau des fonctions réeles de classe C^∞ sur M et par $\mathcal{T}^r_s(M)$ le $\mathcal{F}(M)$-module des champs de tenseurs de type (r, s) sur M. Pour $\mathcal{T}^1_0(M)$ on utilise aussi la notation $\mathcal{X}(M)$.

Soit $A \in \mathcal{T}^1_2(M)$. Alors $\mathcal{X}(M)$ devient une $\mathcal{F}(M)$-algèbre douée du produit $X \circ Y = A(X, Y)$. On va la noter $\mathcal{U}(M, A)$ et l'appeler algèbre associée à A. Pour $\overset{1}{\nabla}$ et $\overset{2}{\nabla}$ deux connexions linéaires sur M et pour $A = \overset{2}{\nabla} - \overset{1}{\nabla} \in \mathcal{T}^1_2(M)$, l'algèbre $\mathcal{U}(M, A)$ s'appèlle l'algèbre de déformation du couple $(\overset{1}{\nabla}, \overset{2}{\nabla})$.

§1. Champs presque ξ-principaux. On considère l'algèbre $\mathcal{U}(M, A)$ et soit $\xi \in \mathcal{X}(M)$.

Définition 1.1. *Un élément $X \in \mathcal{X}(M)$ s'appelle champ presque ξ-principal dans l'algèbre $\mathcal{U}(M, A)$ s'il existe $f \in \mathcal{F}(M)$ et ω, $\theta \in \mathcal{T}^0_1(M)$ t.q.*

$$A(Z, X) = f \,.\, Z + \omega(Z)X + \theta(Z) \,.\, \xi, \quad \forall Z \in \mathcal{X}(M). \tag{1.1}$$

Remarque 1.2. 1.2.1. Si $\theta = 0$, alors (1.1) définit un champ X presque principal dans l'algèbre $\mathcal{U}(M, A)$ [8].

1.2.2. Quand $f = 0$, X est un champ ξ-principal.

1.2.3. Si $\omega = 0$, alors (1.1) montre que X est un champ presque ξ-spécial [13].

1.2.4. Si $\omega = 0$ et $f = 0$, alors on retrouve les champs ξ-spéciaux [13].

1.2.5. Si $f = 0$ et $\theta = 0$, alors (1.1) montre que X est un champ principal [9].

1.2.6. Pour $\omega = 0$ et $\theta = 0$ dans (1.1) on retrouve le cas des champs presque spéciaux [12].

1.2.7. Si $f = 0$, $\omega = 0$, $\theta = 0$, alors X est un champ spécial [10].

Remarque 1.3. Soient $\omega_0 \in \mathcal{T}^0_1(M)$ et $\xi \in \mathcal{X}(M)$. Notons par $\mathcal{P}(M, \xi)$ l'ensemble des champs presque ξ-principaux dans l'algèbre $\mathcal{U}(M, A)$. Alors