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STABILITY OF PERIODIC BILINEAR STOCHASTIC DIFFERENTIAL EQUATIONS WITH CORRELATED NOISE INPUTS

BY

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1. Introduction : We consider the following stochastic differential equation :

$$\frac{dX_t}{dt} = A(t)X_t + F(t) + \sum_{i=1}^{k} [B_i(t)X_t + G_i(t)]g_i(Z_t), \tag{1.1}$$

where $\{Z_t,\ t \geq 0\}$ is a stationary ergodic diffusion process with values in $\mathbf{R}$, $A(t)$, $F(t)$, $B_i(t)$, $G_i(t)$ are $d \times d$ matrix (resp. $\mathbf{R}^d$) valued periodic functions of time, with the same period T. În section 2, we will show, under appropriate conditions, the existence of a stable periodic (in law) solution to (1.1), provided $\{Y_t,\ t \geq 0\}$, the solution of :

$$\frac{dY_t}{dt} = A(t)Y_t + \sum_{i=1}^{k} B_i(t)Y_t g_i(Z_t) \tag{1.2}$$

is asymptotically exponentially stable.

In section 3, we will give conditions under which $\exists\ \lambda \in \mathbf{R}$ s.t. :

$$(1/t)\ \mathrm{Log}\,|\,Y_t\,| \to \lambda \text{ a.s., as } t \to \infty, \qquad \forall Y_0,$$

and give an expression for λ. The case $\lambda < 0$ is the one where we say that Y_t is asymptotically exponentially stable.

The same problems have been considered in [4] in case where the $\{g_i(Z_t),\ t \geq 0\}$, $i = 1,\dots,k$; are replaced by „white noises". The second problem has been considered in [3] and [1], in the case of constant matrices A, B_i, $i = 1,\dots,k$.

The details of proofs which are not given here, as well as the relevance of these results for studying the stability of the movement of helicopter blades, can be found in [5].

We now give the precise hypotheses which will be assumed to hold in all what follows. Suppose we are given $\{W_t^1,\dots,W_t^p\ ;\ t \geq 0\}$ p mutually independent real valued standard Wiener processes, defined on a probability space with filtration $(\Omega, \mathbf{F}, \mathbf{F}_t, P)$. Let $f_0, f_1,\dots,f_p$ be C^∞ vector fields on $\mathbf{R}^l$, s.t.

$$\dim L.\,A.\,\{f_1,\dots,f_p\}\ (z) = l, \quad \forall z \in \mathbf{R}^l. \tag{H$_1$}$$

$\{Z_t\}$ is defined as the stationary solution (which is supposed to exist) of the following stochastic differential equation (written in Stratonovich form) :

$$dZ_t = f_0(Z_t)dt + \sum_{i=1}^{p} f_i(Z_t) \circ dW_t^i.$$

It then follows from (H_1) that $\{Z_t\}$ is an ergodic process. μ will denote the unique stationary probability measure of $\{Z_t\}$.

We now assume:

(H_2) $$g_i \in L^1(\mathbf{R}^l\,;\mu), \qquad i=1,\ldots,k.$$

We finally suppose that $A(t)$, $B_i(t)$, $F(t)$ and $G_i(t)$ $(i=1,\ldots,k)$ are measurable, bounded periodic functions of t, with a common period $T>0$.

2. Existence of a unique stable periodic solution. Let us consider the fundamental solution of the equation (1.2). For $0\leqslant s<t$, we define $\Phi(t,s)$ as the $d\times d$ matrix valued process, solution of:

$$\frac{d}{dt}\Phi(t,s)=A(t)\Phi(t,s)+\sum_{i=1}^{k}B_i(t)\Phi(t,s)g_i(Z_t), \tag{2.1}$$

$$\Phi(s,s)=I.$$

We will write $\Phi(t)$ for $\Phi(t,0)$.

Lemma 2.1 : $\exists\lambda\in\mathbf{R}$ s.t.

$$(1/t)\ \mathrm{Log}\|\Phi(t)\|\to\lambda \text{ a.s., as } t\to\infty$$

Proof: It is not hard to show that $(1/t)$ Log $\|\Phi(t)\|$ behaves, as $t\to\infty$, exactly as the sequence $(1/(nT))$ Log $\|\Phi(nT)\|$, as $n\to\infty$. This fact follows from:

$$\overline{\lim_{t\to\infty}}\,(1/t)\ \mathrm{Log}\,\|\Phi(t,[t/T]T)\|\leqslant 0 \text{ a.s., and}$$

$$\overline{\lim_{t\to\infty}}\,(1/t)\ \mathrm{Log}\,\|\Phi(([t/T]+1)T,t)\|\leqslant 0 \text{ a.s.}$$

Let us check the first inequality:

$$(1/t)\ \mathrm{Log}\,\|\Phi(t,[t/T]T)\|\leqslant 1/t\int_{[t/T]T}^{t}(\|A(s)\|+\sum_{1}^{k}\|B_i(s)\|\times|g_i(Z_s)|)\mathrm{d}s.$$

The fact that the above right-hand side tends to zero as $t\to\infty$ follows from the boundedness of $\|A(s)\|$ and $\|B_i(s)\|$, (H_2) and the ergodic theorem. On the other hand,

$$\Phi(nT)=\Phi(nT,(n-1)T)\ldots\ldots\ldots\Phi(T,0),$$

and it follows from our hypotheses that $\{\Phi((k+1)T,kT),\ k\in\mathbf{N}\}$ is a stationary and ergodic sequence of random matrices, s.t.:

$$E(\mathrm{Log}^+[\Phi((k+1)T,kT)])<\infty.$$

The Lemma then follows from a well-known result in [2].

From now on, λ will denote the real constant defined in Lemma 2.1.

Theorem 2.2 : *Suppose that $\lambda<0$. Then:*

(*i*) *Equation (1.1) possesses a periodic solution, which is unique in law.*

(ii) *The difference of any two solutions to equation (1.1) tends to zero exponentially fast, as $t\to+\infty$.*

Remark: What we mean by a periodic solution is a solution $\bar{X}_t$, such that the law of $\{\bar{X}_{t+s},\ s\geqslant 0\}$ on $C(\mathbf{R}_+\,;\mathbf{R}^d)$ ia a periodic function of t, of period T.

Proof: Clearly, any solution of (1.1) can be written as:

$$X_t=\Phi(t)\,[X_0+\int_0^t\Phi(s)^{-1}(F(s)+\sum_i G_i(s)g_i(Z_s))\mathrm{d}s].$$

(ii) follows from this relation, and the hypothesis of the Theorem. The uniqueness in law of the periodic solution follows from (ii). To produce a periodic (in law) solution, we define $\{Z_t\}$ for all $t\in\mathbf{R}$, such that $\{Z_t,\ t\in\mathbf{R}\}$ be a stationary process, extend $A(t)$ and $B_i(t)$ $(i=1,\ldots,k)$ to $t<0$ by periodicity, and define $\Phi(t,s)$ for all s, $t\in\mathbf{R}$ as the solution of (2.1). Let:

$$\bar{X}_t\triangleq\Phi(t)\int_{-\infty}^{t}\Phi(s)^{-1}(F(s)+\sum_{1}^{k}G_i(s)g_i(Z_s))\mathrm{d}s.$$

Using the equality in law between $\Phi(nT)$ and $\Phi(-nT)^{-1}$, $\forall n\in\mathbf{N}$, one can show that:

$$(1/|t|)\ \mathrm{Log}\,\|\Phi(t)^{-1}\|\to\lambda \text{ a.s., as } t\to-\infty.$$

It then follows that the above integral is well defined. The periodicity of the law of $\{\bar{X}_s,\ s\geqslant t\}$ on $C(\mathbf{R}_+\,;\mathbf{R}^d)$ as a function of t, follows from that of

$$\{(\Phi(t)\Phi(t+s)^{-1},\ Z_{t+s}),\ s\in\mathbf{R}\}.$$

3. Uniqueness and charaterization of the Lyapounov exponent of $\{Y_t\}$. For any $y\in\mathbf{R}^d-\{0\}$, let us define:

$$\lambda(y)\triangleq\overline{\lim_{t\to+\infty}}\,(1/t)\ \mathrm{Log}\,|Y_t(y)|$$

where $Y_t(y)=\Phi(t)y$ is the solution of (1.2) starting at $Y_0=y$. Obviously, $\lambda(\alpha y)=\lambda(y)$ a.s., $\forall\alpha\in\mathbf{R}-\{0\}$, and $\lambda(y)\leqslant\lambda$ a.s., $\forall y\in\mathbf{R}^d-\{0\}$. We will now state conditions under which $\lambda(y)=\lambda$ a.s., $\forall y\in\mathbf{R}^d-\{0\}$.

We first assume:

(H_3) $$A(t),\ B_i(t),\ i=1,\ldots,k, \text{ are } C^\infty \text{ functions of } t.$$

We define the process $R_t\triangleq t$ (mod. T) with values in $S^1\simeq[0,T[$, and the process $U_t\triangleq|Y_t|^{-1}Y_t$ with values in S^{d-1}. It follows readily from (1.2) that:

$$\frac{dR_t}{dt}=1,$$

$$\frac{dU_t}{dt}=h(A(R_t),U_t)+\sum_{i=1}^{k}h(B_i(R_t),U_t)g_i(Z_t), \tag{3.1}$$

where $h(C,u)=Cu-(Cu,u)u$.

The process (R_t,U_t,Z_t) is then a homogeneous Markov process, s.t. (R_t,U_t)takes values in the compact set $S^1\times S^{d-1}$, and Z_t is itself a positive recurrent homogeneous Markov process. It then follows (see [5]) that (R_t,U_t,Z_t) possess at least one invariant probability measure ν, which is of the form $T^{-1}\nu_r(du,dz)dr$, where $\{\nu_t(dr,dt),\ t\in[0,T[\}$ is a periodic law of the process (U_t,Z_t). Define the following vector fields over $S^1\times S^{d-1}$:

$$\hat{h}_0(r, u) = \begin{pmatrix} 1 \\ h(A(r), u) \end{pmatrix}, \quad \hat{h}_i(r, u) = \begin{pmatrix} 0 \\ h(B_i(r), u) \end{pmatrix} \quad i = 1, \ldots, r$$

and consider the following Lie algebra of vector fields over $S^1 \times S^{d-1}$:

$$\mathcal{A} = \text{L.A.}\ \{\hat{h}_0 + \sum_1^k \hat{h}_i g_i(z),\ z \in \mathbf{R}^l\}.$$

We can now formulate the main hypothesis of this section:

$$(\mathrm{H}_4) \qquad \operatorname{rank} \mathcal{A}\ (r, u) = d, \qquad \forall r \in S^1, \qquad u \in S^{d-1}.$$

It is easily seen that $(\mathrm{H}_1)+(\mathrm{H}_4)$ imply that, if $(L_t, t \geq 0)$ denotes the infinitesimal generator of the Markov process (U_t, Z_t), $\partial/\partial t + L_t$ is hypoelliptic (in other words, the infinitesimal generator of the process (R_t, U_t, Z_t) is hypoelliptic).

With the same arguments as those in the proof of Lemma 2.1., the behaviour of $(1/t)$ Log $|Y_t|$, as $t \to \infty$, is the same as that of $(1/nT)$ Log $|Y_{nT}|$, as $n \to \infty (n \in \mathbf{N})$. Now

$$Y_{nT} = \Phi(nT,\ (n-1)T) \ldots \Phi(T, 0) Y_0$$

and the sequence $\{\Phi((k+1)T, kT), k \in \mathbf{N})\}$ forms a stationnary and ergodic sequence of random matrices, to which the multiplicative ergodic theorem of Oseledec applies:

Proposition 3.1. (See e.g. [6])
$\exists r \in N - \{0\}$, $\lambda_1 < \lambda_2 < \ldots < \lambda_r$, *and a strictly increasing sequence of ω-depending subspaces of* $\mathbf{R}^d$:

$$V_0 = \{0\} \subset V_1(\omega) \subset \ldots \subset V_{r-1}(\omega) \subset V_r = \mathbf{R}^d,$$

and $N \in \mathcal{F}$ *with* $P(N) = 0$ s.t. $\forall \omega \notin N$, $\forall y \in V_i(\omega) - V_{i-1}(\omega)$,

$$\frac{1}{nT} \text{Log}\, |\Phi(nT, \omega) y| \to \lambda_i, \textit{ as } n \to \infty.$$

Clearly, $\lambda_r = \lambda$. We can now prove:
Theorem 3.2. : *Under the hypotheses* (H_1) *to* (H_4),

$$(i)\ \lambda(y) = \lim_{t \to \infty} \frac{1}{t} \text{Log}\, |Y_t(y)| \quad a.s., \quad \forall y \in \mathbf{R}^d - \{0\}$$

$$(ii)\ \lambda(y) = \lambda \quad a.s., \quad \forall y \in \mathbf{R}^d - \{0\}$$

$$(iii)\ \lambda = \int_{S^1 \times S^{d-1} \times R^l} q(r, u, z) \nu(dr, du, dz),$$

$\forall \nu$ *invariant measure of* (R_t, U_t, Z_t), *where*:

$$q(r, u, z) = (A(r)u, u) + \sum_{i=1}^k (B_i(r)u, u) g_i(z).$$

Proof: a) Let Y_0 be any random vector with values in $\mathbf{R}^d - \{0\}$, independent of $\{W_t, t \geq 0\}$.

$$P\left(\frac{1}{nT} \text{Log}\, |\Phi(nT) Y_0| \to \lambda\right) = P\left(\frac{1}{nT} \text{Log}\, |\Phi(nT, T) Y_T| \to \lambda\right) =$$

$$= P\left(\frac{1}{nT} \text{Log}\, |\Phi((n+1)T, T) Y_T| \to \lambda\right) = P\left(\frac{1}{nT} \text{Log}\, |\Phi'(nT) Y_T| \to \lambda\right),$$

with $\Phi'(nT) = \Phi((n+1)T, T)$.
But from Proposition 3.1,

$$P\left(\frac{1}{nT} \text{Log}\, |\Phi'(nT) Y_T| \to \lambda\right) = P(Y_T \in V'_{r-1}),$$

and V'_{r-1} is $\sigma(Z_{t+T}, t \geq 0) = \mathcal{G}$ adapted

$$P(Y_T \in V'_{r-1}) = E[P(U_T \in V'_{r-1} \mathcal{G})] = E[\theta(V'_{r-1})]$$

where θ is the conditional law of U_T, given $\mathcal{G}$, i.e. the conditional law of U_T, given Z_T. But the latter possess a density with respect to Lebesgue measure, since, from $(\mathrm{H}_1)+(\mathrm{H}_4)$, the law of (U_T, Z_T) possess a density. Then:

$$\theta(V'_{r-1}) = 0 \quad \text{a.s.}$$

We have proved that:

$$(1/t) \text{Log}\, |\Phi(t) Y_0| \to \lambda \quad \text{a.s.}$$

b) (i) and (ii) follow immediately from a), by choosing $Y_0 = y$.
Let $\nu(dr, du, dz) = T^{-1} \nu_r(du, dz)\, dr$ be an invariant probability measure of $\{(R_t, U_t, Z_t), t \geq 0\}$.

Choose (Y_0, Z_0) independent of $\{W_t, t \geq 0\}$, the law of (Y_0, Z_0) being $\nu_0(du, dz)$ (considered as a measure on $\mathbf{R}^d \times \mathbf{R}^l$). But:

$$|Y_t| = |Y_0| \exp \int_0^t q(R_s, U_s, Z_s) ds;$$

on the other hand, $\left\{\int_{nT}^{(n+1)T} q(R_s, U_s, Z_s) ds, n \in \mathbf{N}\right\}$ is a stationary random sequence. The result then follows from the additive ergodic theorem of Birkhoff-Khinchine, and the fact that, from a), $(1/t)$ Log $|Y_t|$ converges a.s., as $t \to \infty$, to a constant.

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